Energy-energy correlations at next-to-next-to leading order

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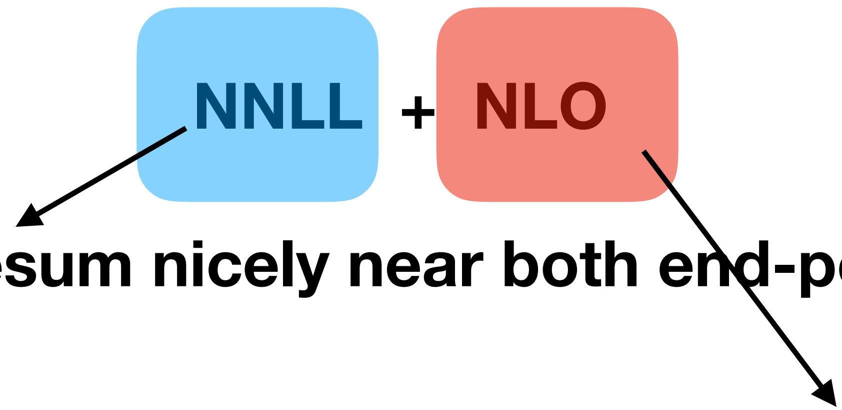
Energy-energy correlator (EEC)

- an infrared-safe event-shape observable that lies at the frontier of precision QCD



factorize and resum nicely near both end-points

fixed order correction known analytically (can we go beyond?)

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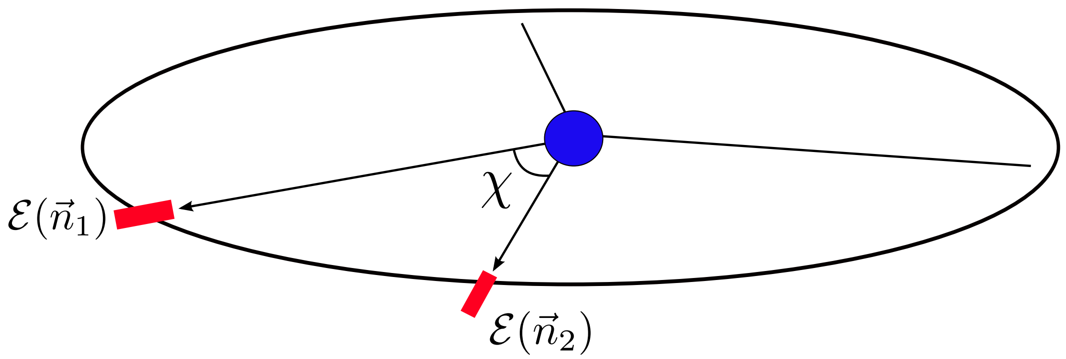


- defined through correlation function :

$$\text{EEC}(\chi; q^2) \sim \int d^4x\, e^{iqx} \langle O(x)\, \mathcal{E}(\vec{n})\, \mathcal{E}(\vec{n}')\, O(0)\rangle$$

-a bridge between conformal field theory and collider physics.

-conformal symmetry provides new insights that allows to calculate EEC in a way that bypasses infrared divergences.

Extremely simple formula for EEC in N=4 sYM:

J. M. Henn, E. Sokatchev, K. Yan, A. Zhibodoev, arXiv: 1903.05314v2

- **IR finite, two-fold integral representation**

$$\mathrm{EEC}(\zeta) = \int_D du dv \, \frac{\mathrm{TDisc}[\mathcal{G}(u,v)]}{\sqrt{(\zeta + (1-\zeta)u + \zeta v)^2 - 4\zeta(1-\zeta)uv}}$$

TDisc: triple discontinuity

G (u,v) : 4-point correlator

(u, v) : conformal cross ratios

We obtain analytically EEC @ NNLO in N=4 sYM;

the space of functions;

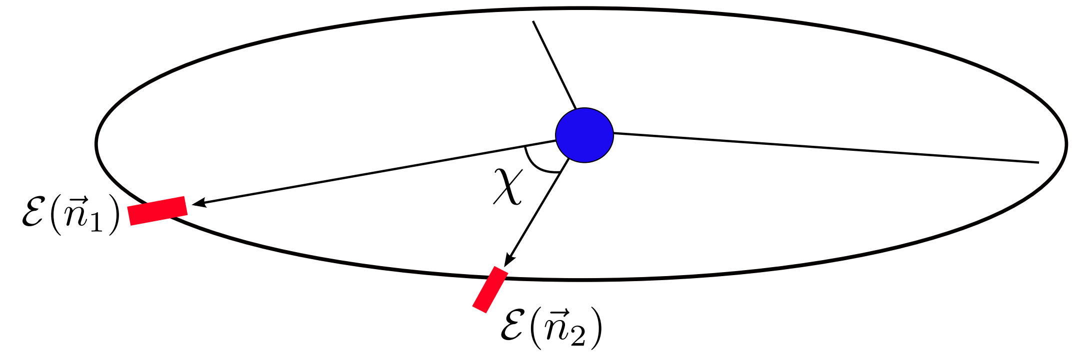
end-point asymptotics;

possible future development

EEC in conformal field theory

$$\zeta = \frac{q^2 (n\cdot n')}{2(q\cdot n)(q\cdot n')}$$

$$\mathrm{EEC}(\zeta) \sim \int d^4x\, e^{iqx} \langle O(x)\, \mathcal{E}(\vec{n})\, \mathcal{E}(\vec{n}')\, O(0)\rangle$$



$$\mathcal{E}(\vec{n}) \equiv \lim_{r\to\infty} r^2 \int dx_- n^j T_{0,j}(t = x_- + r, r\vec{n})$$

x_ : retarded time

" conformal collider" : detectors sitting at null infinity

N=4 super conformal symmetry relates EEC to all-scalar 4-pt correction functions

$$\mathrm{EEC}(\zeta) \sim \int d^4x\, e^{iq\cdot x} \underline{\underline{\int_{-\infty}^{\infty} dx_{2-} dx_{3-}}} \lim_{x_{2+,3+}\to\infty} x_{2+}^2 x_{3+}^2 \underline{\underline{\langle 0| O^\dagger(x) \mathcal{O}(x_2) \mathcal{O}(x_3) O(0)|0\rangle}}$$

detector time integrals

Wightman correlation function

Euclidean correlation functions

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \frac{\mathcal{G}(u,v)}{(x_{14}^2 x_{23}^2)^{\Delta}}$$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.$$

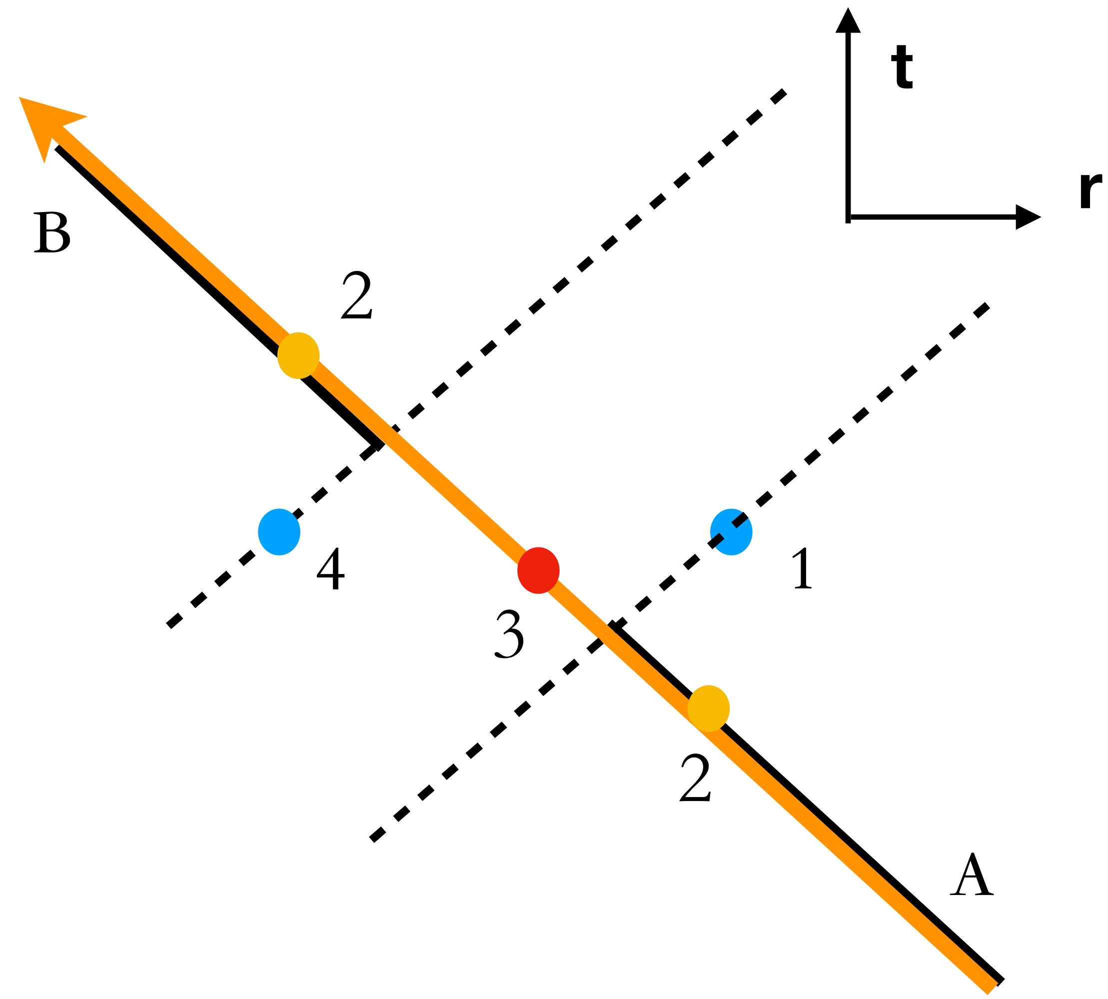
How to obtain Wightman correlation function through analytic continuation:

Wightman ordering :

$$\langle \cdots O_L \cdots O_R \rangle \qquad \text{Im}\, t_L < \text{Im}\, t_R$$

$$x_{LR}^2 \to \hat{x}_{LR}^2 = -x_{LR}^2 + i\epsilon x_{LR}^0$$

$\mathcal{G}_W(u,v)$ **is multi-valued function when detectors are time-like separated from the source/sink**



Causal relationships between the scattering points.

Detectors are integrated along a null line.

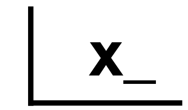
Region A : $x_{12}^2 \to e^{i\pi}|x_{12}|^2$

$u \to u, \; v \to e^{i\pi} v$

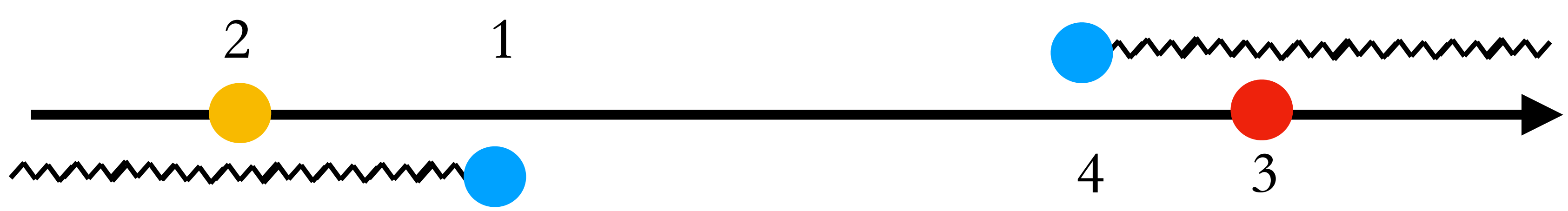
Region B : $x_{24}^2 \to e^{i\pi}|x_{24}|^2$

$u \to e^{-i\pi} u, \; v \to e^{-i\pi} v$

- **Our approach: integrated discontinuities**



$$x_1 = x, x_4 = 0, x_{2+} = x'_{3+} \to \infty$$



Detector-time integration

time-integration contour specified by Wightman ordering encoded in ie prescriptions

$$\int_{-\infty}^{\infty} dx_{2-} dx'_{3-}\, \mathcal{G}(u,v)$$

no end-point singularity; branch cuts starting at

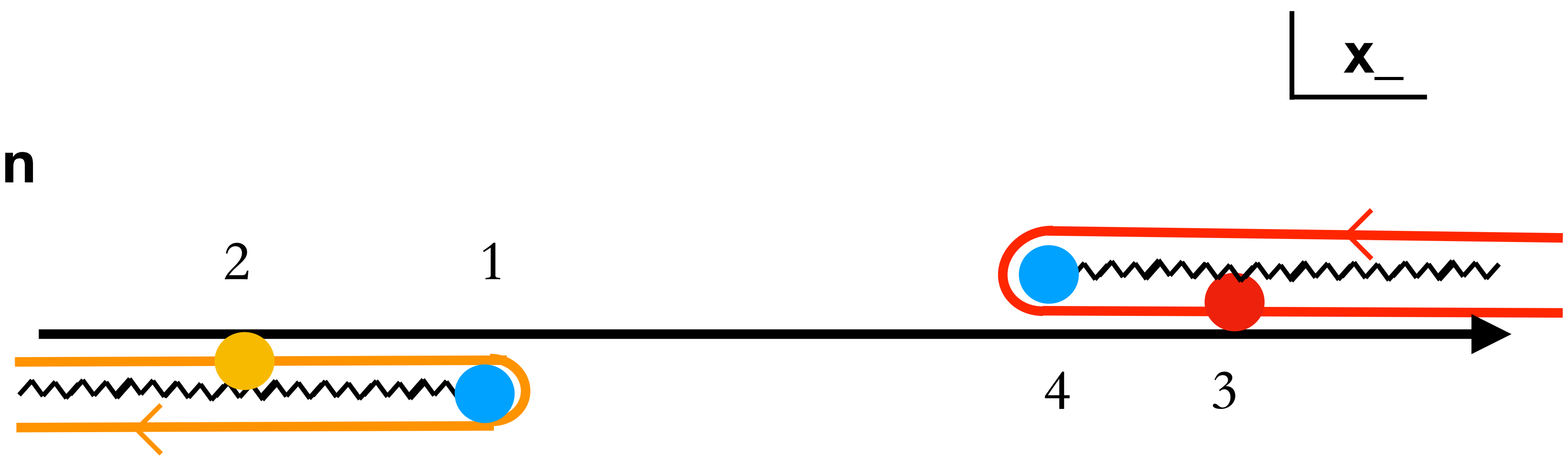
$$x_{2-} = x_-, x'_{3-} = 0$$

cross ratio after sending detectors to null infinity :

$$u = \frac{\hat{x}^2\,(n \cdot n')}{2(-x'_- + x'_{3-} + i\epsilon)(-x_{2-} + i\epsilon)},$$

$$v = \frac{(-x_- + x_{2-} + i\epsilon)(-x'_{3-} + i\epsilon)}{(-x'_- + x'_{3-} + i\epsilon)(-x_{2-} + i\epsilon)}$$

Contour deformation



Detector-time integration

integrated double discontinuity:

$$\int_{\textcolor{orange}{C_2}} dx_{2-} \int_{\textcolor{red}{C_3}} dx'_{3-} \; \boxed{\mathrm{disc}_{x_{2-}=x_-}\, \mathrm{disc}_{x'_{3-}=0}[\mathcal{G}(u,v)]}$$

$$v = \frac{(-x_- + x_{2-} + i\epsilon)(-x'_{3-} + i\epsilon)}{(-x'_- + x'_{3-} + i\epsilon)(-x_{2-} + i\epsilon)}$$

$\circlearrowleft$ / $\circlearrowright$ **brings v to adjacent Riemann sheets**

$$\boxed{\mathrm{disc}_{x_{2-}=x_-} = \mathrm{disc}_v^{\circlearrowright} \qquad \mathrm{disc}_{x'_{3-}=0} = \mathrm{disc}_v^{\circlearrowleft}}$$

$$\mathrm{dDisc}_v[\mathcal{G}(u,v)]$$

$$\mathrm{dDisc}_v g \equiv 2\, \mathrm{disc}_v^{\circlearrowleft} \mathrm{disc}_v^{\circlearrowright} g$$
$$= g(v) - \frac{1}{2} g(v^{\circlearrowleft}) - \frac{1}{2} g(v^{\circlearrowright})$$

$$\mathrm{EEC}(\zeta) \sim \int d^4x \frac{e^{iq\cdot x}}{x^2 - ix^0\epsilon} \int_{-\infty}^{\infty} dx_{2-} dx'_{3-}\, \mathcal{G}(u,v)$$

set $x_- = x'_- = 1$

integration contour along the branch cuts parametrized by (t, tb)

$$t \equiv \frac{x_{2-} - 1}{x_{2-}}, \quad \bar{t} \equiv \frac{x'_{3-}}{x'_{3-} + 1}$$

$$\int_0^1 \frac{dt}{t^2} \frac{d\bar{t}}{\bar{t}^2} \mathrm{dDisc}_v \left[\mathcal{G}(u = \frac{1}{\gamma} t\bar{t}, v = (1-t)(1-\bar{t}))\right]$$

double discontinuity at v =0

$$\gamma = \frac{2(n\cdot x)(n'\cdot x)}{x^2(n\cdot n')},$$

$$-\frac{1}{2}\langle [O(x_1), O(x_2)][O(x_3), O(x_4)]\rangle \geq 0$$ 1703.00278

$$\mathrm{EEC}(\zeta) \sim \int d^4x \frac{e^{iq\cdot x}}{x^2 - ix^0\epsilon} \left[\int_0^1 \frac{dt}{t^2}\frac{d\bar{t}}{\bar{t}^2} \mathrm{dDisc}_v \left[\mathcal{G}(u = \frac{1}{\gamma} t\bar{t}, v = (1-t)(1-\bar{t})) \right] \right]$$

$$\zeta = \frac{q^2(n\cdot n')}{2(q\cdot n)(q\cdot n')}$$

$$\gamma = \frac{2(n\cdot x)(n'\cdot x)}{x^2(n\cdot n')},$$

Fourier transform generates the third discontinuity

$$\int \frac{d^4x\, e^{iq\cdot x}}{x^2 - ix^0\epsilon} G(\gamma) = \frac{1}{q^2} \mathrm{disc}_\gamma G(\gamma)\Big|_{\gamma = 1 - \frac{1}{\zeta}}$$

$$\int \frac{d^4x\, e^{iq\cdot x}}{x^2 - ix^0\epsilon} \mathcal{G}(u = \frac{1}{\gamma} t\bar{t}, v = (1-t)(1-\bar{t})) = \frac{1}{q^2} \mathrm{disc}_u \mathcal{G}(u = \frac{\zeta t\bar{t}}{(\zeta - 1)}, v = (1-t)(1-\bar{t}))$$

standard discontinuity across u <0 branch, real-valued

Analyticity + Conformal symmetry :

$\mathrm{Disc}_u \mathrm{dDisc}_v$

$$\mathrm{EEC}(\zeta) = \int_D dudv \, \frac{\mathrm{TDisc}[\mathcal{G}(u,v)]}{\sqrt{(\zeta + (1-\zeta)u + \zeta v)^2 - 4\zeta(1-\zeta)uv}}$$

$(u,v) \in [\frac{\zeta}{\zeta - 1}, 0] \times [0,1], R(u,v,\zeta) > 0$

$(t, \bar{t}) \to (u,v)$ **: Jacobian** $\sqrt{R(u,v,\zeta)}$

two steps toward NNLO analytic:

- **Taking triple discontinuities**
- **Automating the two-fold integration**

How to take TDisc

Correlation functions are given in terms of multiple poly-logarithms in z, zb

$$u \equiv z\bar{z}\,, \quad v \equiv (1-z)(1-\bar{z})\,. \qquad \mathrm{Disc}_u \mathrm{dDisc}_v = \mathrm{Disc}_{z=0}\mathrm{dDisc}_{\bar{z}=1}\,.$$

triple disc acts on terms singular at z = 0 and zb = 1.

Example: EEC @ LO

$$\int_D \frac{dz d\bar{z}}{\sqrt{R}} \,\mathrm{TDisc}\left\{\frac{z\bar{z}}{(1-z)(1-\bar{z})}\left[2\mathrm{Li}_2(\bar{z}) - 2\mathrm{Li}_2(z) + \ln\frac{1-\bar{z}}{1-z}\ln(z\bar{z})\right]\right\}$$

↓

Extract logarithms at z = 0: $$\mathrm{dDisc}_{\bar{z}=1}\left\{\frac{z\bar{z}}{(1-z)(1-\bar{z})}\ln\frac{1-\bar{z}}{1-z}\,\mathrm{Disc}_{z=0}[\ln z]\right\}$$

How to take TDisc

Correlation functions are given in terms of multiple poly-logarithms in z, zb

$$u \equiv z\bar{z}\,, \quad v \equiv (1-z)(1-\bar{z})\,. \qquad \mathrm{Disc}_u \mathrm{dDisc}_v = \mathrm{Disc}_{z=0}\mathrm{dDisc}_{\bar{z}=1}\,.$$

triple disc acts on terms singular at z = 0 and zb = 1.

Example: EEC @ LO

$$\int_D \frac{dz d\bar{z}}{\sqrt{R}}\, \mathrm{dDisc}_{\bar{z}=1}\left\{ \frac{z\bar{z}}{(1-z)(1-\bar{z})} \ln\frac{1-\bar{z}}{1-z} \boxed{\mathrm{Disc}_{z=0}[\ln z]} \right\}$$

$$\mathrm{Disc}_{z=0}[\ln z] = \pi$$

$\downarrow$

Integrate over z producing

$$A \ln(1-\bar{z}) + B(\bar{z})$$

B's: analytic at zb=1

$$\int_0^1 d\bar{z}\ \mathrm{dDisc}_{\bar{z}=1}\left\{ \frac{\pi\zeta\,\bar{z}}{(1-\bar{z})} \ln[(1-\zeta)(1-\bar{z})] \right\} \ln(1-\bar{z})$$
$$+\mathrm{dDisc}_{\bar{z}=1}\left\{ \frac{\pi\,\bar{z}}{(1-\bar{z})} \left[\ln(1-\bar{z}) B_1(\bar{z}) + B_2(\bar{z}) \right] \right\}$$

Extract logarithmic and polar singularities at zb = 1 :

$$\longrightarrow \int_0^1 d\bar{z} \left(\begin{array}{r} \pi\zeta\, \mathrm{dDisc}_{\bar{z}=1}\left[\dfrac{\ln(1-\bar{z})}{1-\bar{z}}\right]\ln(1-\bar{z}) \\ +\pi\zeta\, \ln(1-\zeta)\, \mathrm{dDisc}_{\bar{z}=1}\left[\dfrac{1}{1-\bar{z}}\right]\ln(1-\bar{z}) \\ +\pi B_1(1)\, \mathrm{dDisc}_{\bar{z}=1}\left[\dfrac{\ln(1-\bar{z})}{1-\bar{z}}\right] \\ +\pi B_2(1)\, \mathrm{dDisc}_{\bar{z}=1}\left[\dfrac{1}{1-\bar{z}}\right] \end{array} \right.$$

dDisc[Pole]*Log can be derived with the help of an analytic regulator

$$\mathrm{dDisc}[w^{-1+\epsilon}] = 2\sin^2(\pi\epsilon)\, w^{-1+\epsilon} \longrightarrow \left(\frac{1}{\epsilon}\delta(w) + w_+^{-1} + \cdots\right)$$

singular distributions

$$\mathrm{dDisc}[w^{-1+\epsilon}\ln^m w]\ln^n w = \partial_\epsilon^m \left[2\sin^2(\pi\epsilon)\,\partial_\epsilon^n\left(\frac{\delta(w)}{\epsilon} + \sum_{k=0}\frac{\epsilon^k}{k!}\left[w^{-1}\ln^k w\right]_+\right)\right].$$

Convert double disc into distributions:

$$\int_0^1 d\bar{z} \left(\begin{array}{l} \pi\zeta \, \mathrm{dDisc}_{\bar{z}=1}\left[\frac{\ln(1-\bar{z})}{1-\bar{z}}\right]\ln(1-\bar{z}) \longrightarrow 0 \\ +\pi\zeta\,\ln(1-\zeta)\,\mathrm{dDisc}_{\bar{z}=1}\left[\frac{1}{1-\bar{z}}\right]\ln(1-\bar{z}) \longrightarrow -2\pi^2\delta(1-\bar{z}) \\ +\pi B_1(1)\,\mathrm{dDisc}_{\bar{z}=1}\left[\frac{\ln(1-\bar{z})}{1-\bar{z}}\right] \longrightarrow 2\pi^2\delta(1-\bar{z}) \\ +\pi B_2(1)\,\mathrm{dDisc}_{\bar{z}=1}\left[\frac{1}{1-\bar{z}}\right] \longrightarrow 0 \end{array} \right)$$

$$\downarrow$$

$$F_{\mathrm{LO}}(\zeta) = -\ln(1-\zeta)\,.$$ 1311.6800

$$\mathrm{EEC}(\zeta;a) \equiv \frac{F(\zeta;a)}{4\zeta^2(1-\zeta)}$$

At higher loop orders,

$$\int_D \frac{dz d\bar{z}}{\sqrt{R}} \, \mathrm{TDisc}\{\cdots\} = \int_0^1 d\bar{z} \, \{\text{contact terms} + \text{plus-distributions}\} \, .$$

this procedure can be done in a highly automated way.

Algorithm (e.g. for ladder diagrams):

- **Log extraction.**

  Implement shuffle algebra with the help of HPL **package to extract powers of logarithms at z=0 and zb=1.**

- **Taking triple disc: transcendental weight -= 3.**

- **Rationalization.**

  So long as integrands are linearly reducible (factorize linearly in certain integration variables), integrals can be handled by program HyperInt

Finding variables that rationalizes the square root:

$$z\bar{z} \equiv \frac{\zeta}{\zeta-1}\, t\bar{t}\,, \quad (1-z)(1-\bar{z}) \equiv (1-t)(1-\bar{t})\,.$$ **switching back to** $(t,\bar{z})$ **,**

$$\text{EEC}(\zeta) = \frac{1}{4\pi^3\zeta^2}\int_0^1 d\bar{z}\int_0^{\bar{z}} dt\,\frac{1}{t(\zeta-\bar{z})+(1-\zeta)\bar{z}} \times \underbrace{\text{dDisc}_{\bar{z}=1}\text{Disc}_{z=0}\left[(z-\bar{z})\mathcal{G}(z,\bar{z})\right]}_{\text{HPL in } z,\bar{z}}$$

$$\boxed{z = \frac{\zeta t(t-\bar{z})}{t(\zeta-\bar{z})+(1-\zeta)\bar{z}}\,.}$$

- **Integration and subtraction.**

  Carrying out the integral algorithmically with a subtraction procedure.

At L-loop order, highest transcendental weight = 2 L - 3 +2

- **ladder diagrams : uniform transcendental weight**
- **other types of diagrams : complications due to their leading singularities and symbol alphabet.**

EEC @ NLO:

HPL of transcendental weight 2 and 3.

alphabet : $\left\{\zeta, 1-\zeta, \frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}\right\}$

squared one-loop box diagrams:

$$-4\sqrt{\zeta}\, H_{+,0}(\sqrt{\zeta}) + (1+2\zeta)\, H_{+,+,0}(\sqrt{\zeta}) + \cdots$$

In terms of classical polylogarithms:

$$F_{\text{NLO}}(\zeta) = (1-\zeta)F_2(\zeta) + F_3(\zeta) \quad 1311.6800$$

EEC @ NNLO :

HPL + explicit two-fold finite integral

$$F_{\mathrm{NNLO}}(\zeta) = f_{\mathrm{HPL}}(\zeta) + \int_0^1 d\bar{z} \int_0^{\bar{z}} dt \, \frac{\zeta - 1}{t(\zeta - \bar{z}) + (1-\zeta)\bar{z}} \times [R_1(z,\bar{z})P_1(z,\bar{z}) + R_2(z,\bar{z})P_2(z,\bar{z})]$$

2 <= weight <=5 , **alphabet :** $\left\{\zeta, 1-\zeta, \frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}\right\}$

Rational function:

$$R_1 = \frac{z\bar{z}}{1-z-\bar{z}}\,, \quad R_2 = \frac{z^2\bar{z}}{(1-z)^2(1-z\bar{z})}$$

algebraic prefactors of $\mathrm{easy}(\mathrm{E})$ **and** $\mathrm{hard}(\mathrm{H})$ **integrals in certain orientations**

Polylogarithmic function:

$$P_1 = \mathrm{TDisc}[E(1/z) + E(1/(1-z)) + H^b(1/z)],$$

$$P_2 = \mathrm{TDisc}[E(1-z) + E(1-1/z) + H^b(1-z)]$$

1303.6909

R1, R2 are not linearly reducible.

Each of them contains an irreducible integration kernel

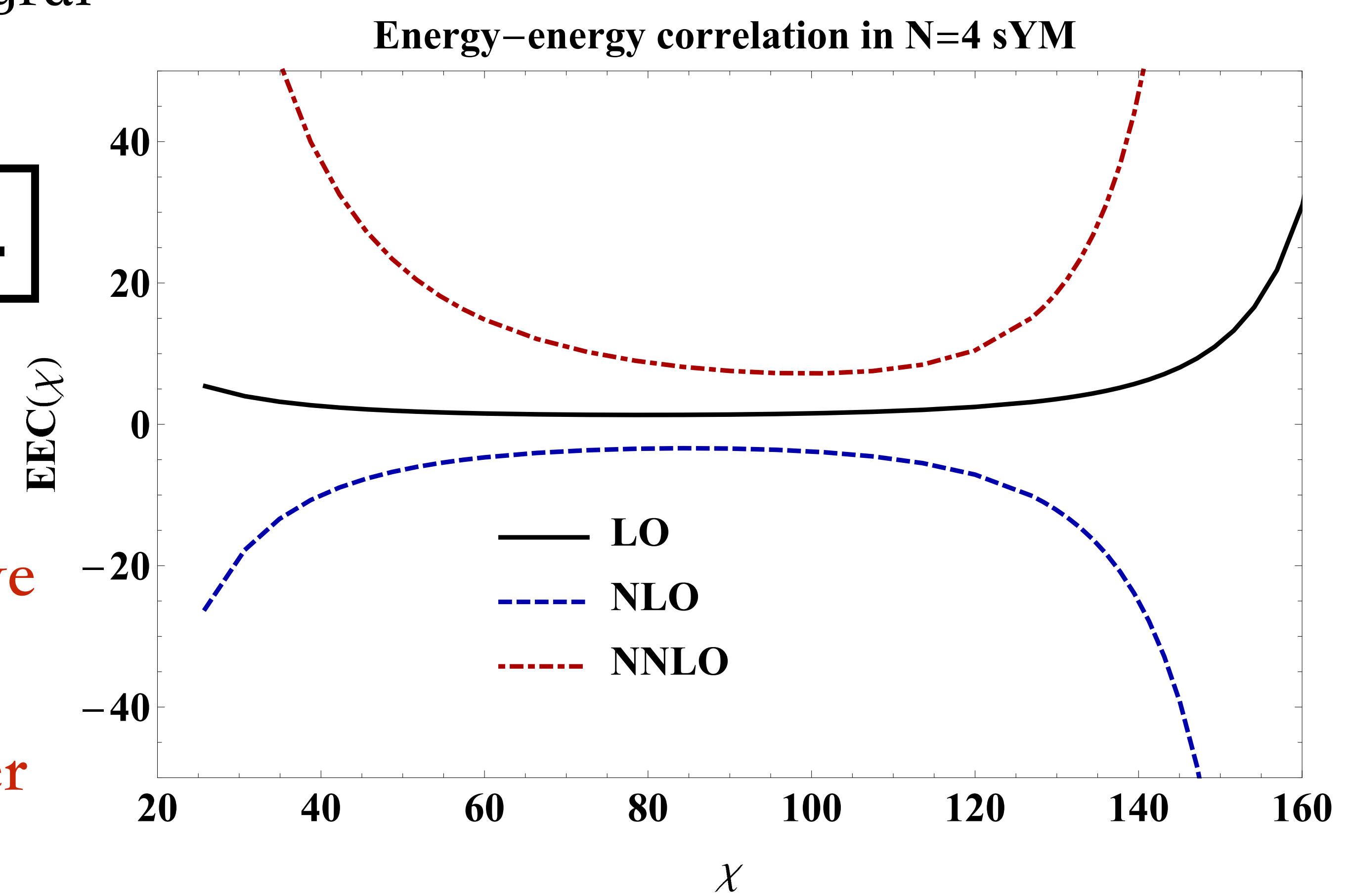
$$K_1 = \frac{t(1-t)\,\zeta(1-\zeta)}{(1-t)(1-\bar{z})\bar{z} + \zeta\,(t-\bar{z})(1-t-\bar{z})}\,, \quad K_2 = \frac{1}{(1-t)^2(1-\bar{z})^2} K_1\left(\frac{t}{t-1}, \frac{\bar{z}}{\bar{z}-1}\right)\,.$$

$$\zeta \to \frac{S}{m^2 - S}$$

Symanzik polynomial of equal-mass Sunrise Integral

EEC @ NNLO contains elliptic functions.

Fixed-order perturbative corrections displayed separately by loop order



End-point asymptotic behavior

can be understood from soft and collinear physics, analogous to QCD.

- **Collinear limit: jet calculus**

$$F(\zeta) \overset{\zeta \to 0}{\sim} a\,\zeta\,\left(Q^2/S_{ab}\right)^{-\gamma_T(3)}.$$

twist-two time-like anomalous dimension

- **Back-to-back limit: sudakov factorization**

$$F(\zeta = 1 - y) \sim \frac{1}{2} H(a) \int_0^\infty db\, b J_0(b) \exp\left\{-\frac{1}{2}\Gamma_{\text{cusp}}(a) L^2 - \Gamma(a) L\right\}.$$

Hard emission, known at NLO

Collinear, known to N^2LO

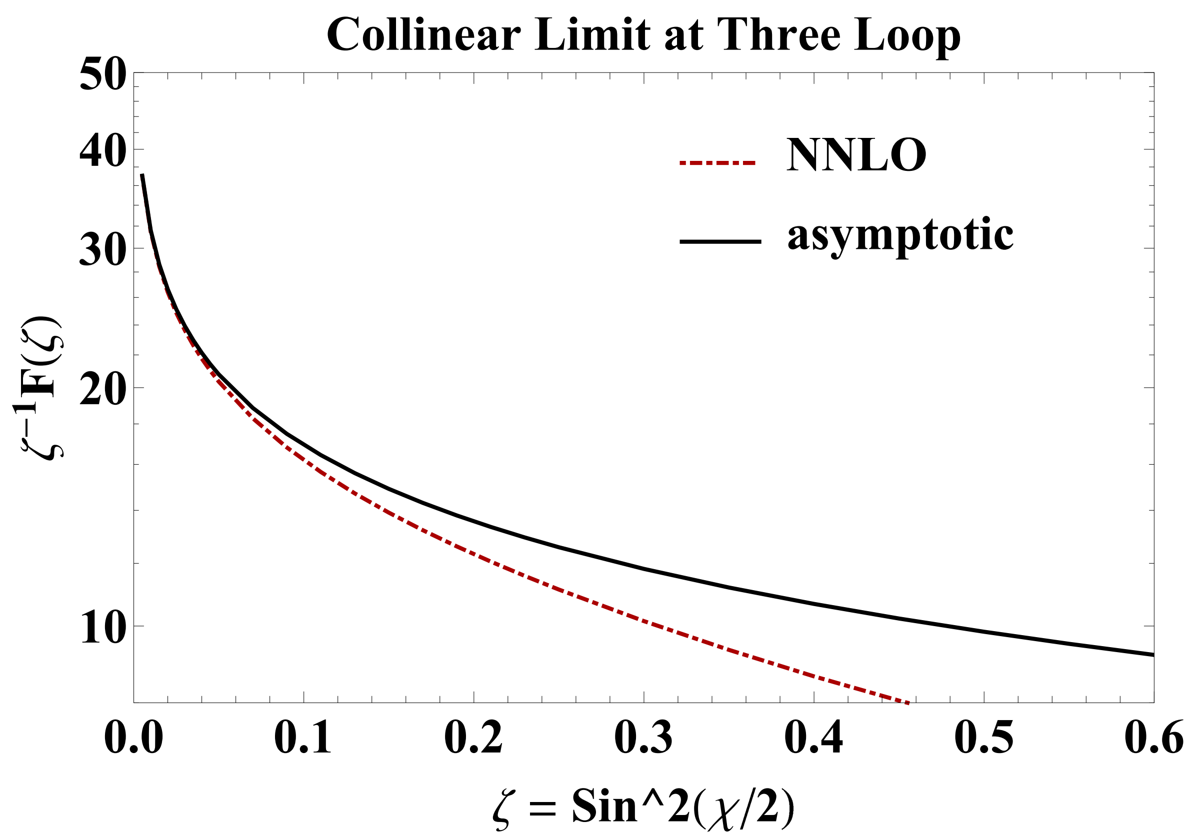
Our integral formula contains useful data for leading and sub-leading power N^3LL resummations of large logarithms in both regimes.

Collinear limit

Extract leading-power asymptotic of the elliptic integrals

Power-expand in ζ at the integrand level and integrating the leading term:

$$F_{\rm NNLO}(\zeta) \overset{\zeta\to 0}{\sim} \zeta \left[\frac{1}{2}\ln^2\zeta + \left(-5+\frac{\pi^2}{3}-\zeta_3\right)\ln\zeta + 17 - \frac{4\pi^2}{3} - \zeta_3 + \frac{5\pi^4}{144} + \frac{3}{2}\zeta_5\right].$$



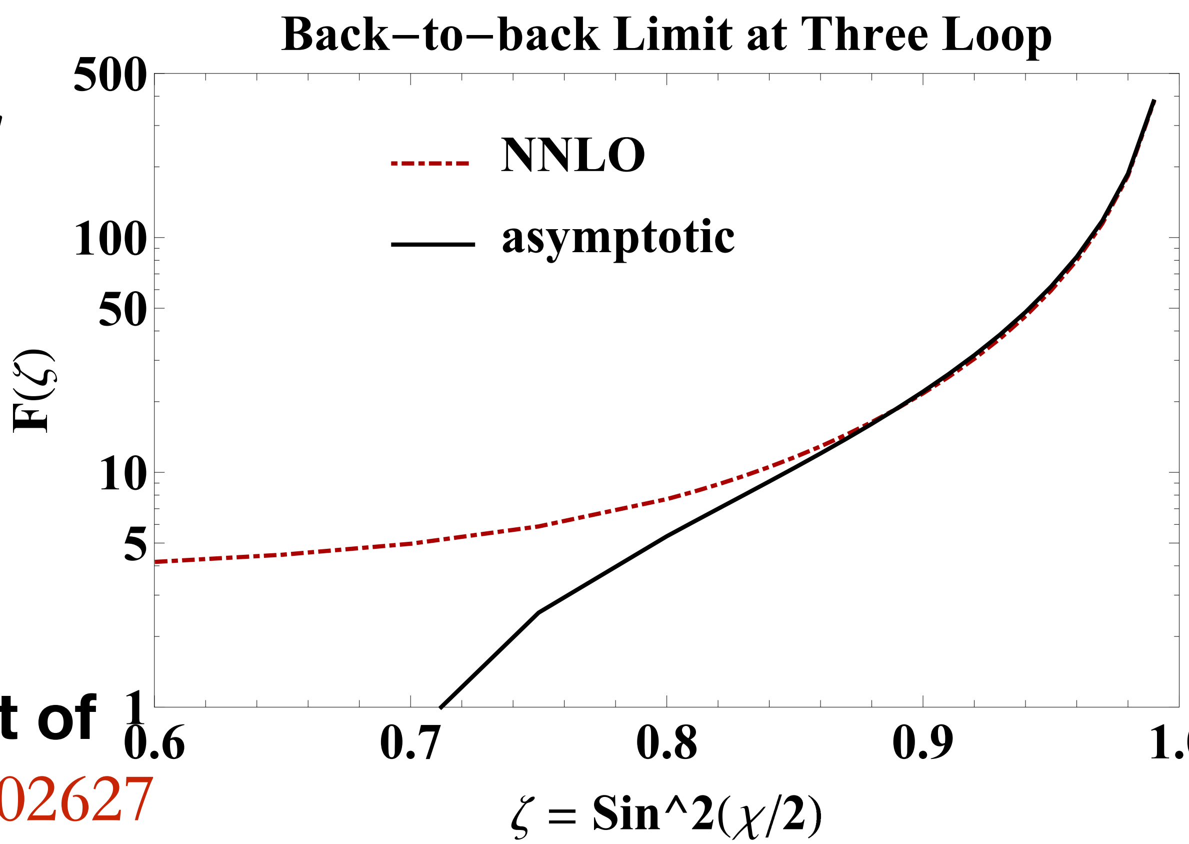
Agrees with CFT result obtained from light-ray OPE

A.Z. el al, to appear.

Back-to-back limit

$y \equiv 1 - \zeta \to 0$ **the elliptic integrals are power-suppressed.**

$$F_{\text{NNLO}}(\zeta) \overset{\zeta\to 1}{\sim} -\frac{1}{8}\ln^5 y - \frac{\pi^2}{6}\ln^3 y - \frac{11}{4}\zeta_3 \ln^2 y - \frac{61}{720}\pi^4 \ln y - \frac{\pi^2}{2}\zeta_3 - \frac{7}{2}\zeta_5 .$$



consistent with the resummation formula expanded at three-loop order.

coincides with maximal transcendental part of QCD, setting CF-> CA= Nc. 1801.02627

The coefficient of the single logarithm determines H @NNLO

$$H(a) = 1 - \zeta_2\, a + 5\zeta_4\, a^2$$

see also Korchemsky, to appear

Future directions

We obtain for the first time analytic result for EEC at next-to-next-to leading order, from a new and efficient approach.

- **further study on the elliptic functions, might also appear in QCD.**
- **further development of our method**

  -in generic QFT, where scalar operators -> conserved currents

  -in QCD (?)

- **analogous study on other observables**

  e.g. multiple detector correlations, gaining better understanding of event-shape from Quantum Field Theory .

Thanks !