General Numerical Resummation

SCET 2019, San Diego

Work in collaboration with
Pier Monni

At SCET 2018, showed how the thrust cumulant can be related to a simpler observable using a transfer function

Can find a simple observable to have multiplicative factorization theorem

$$\Sigma_{\max}(\tau) = H(\mu)\, \Sigma_{J_n}^{\max}(\tau_n, \mu)\, \Sigma_{J_{\bar{n}}}^{\max}(\tau_{\bar{n}}, \mu)\, \Sigma_{S}^{\max}(\tau_s, \mu)$$

Thrust cumulant $\Sigma(\tau)$ related to simpler observable $\Sigma_{\max}(\tau)$ via transfer function $F_F(\tau)$

$$\Sigma(\tau) = \Sigma_{\max}(\tau)\, \mathcal{F}(\tau)$$

Transfer function can be computed numerically within SCET

At SCET 2018, showed how the thrust cumulant can be related to a simpler observable using a transfer function

$$\Sigma(\tau) = \Sigma_{\max}(\tau)\, \mathcal{F}(\tau)$$

Thrust cumulant in SCET can be factorized as

$$\Sigma(\tau) = H(\mu) \int \mathrm{d}\tau_n\, \Sigma'_{J_n}(\tau_n, \mu) \int \mathrm{d}\tau_{\bar{n}}\, \Sigma'_{J_{\bar{n}}}(\tau_{\bar{n}}, \mu) \int \mathrm{d}\tau_s\, \Sigma'_S(\tau_s, \mu)\; \Theta[\tau > \tau_n + \tau_{\bar{n}} + \tau_s]$$

Simple observable has multiplicative factorization

$$\Sigma_{\max}(\tau) = H(\mu)\, \Sigma_{J_n}^{\max}(\tau_n, \mu)\, \Sigma_{J_{\bar{n}}}^{\max}(\tau_{\bar{n}}, \mu)\, \Sigma_S^{\max}(\tau_s, \mu)$$

This gives

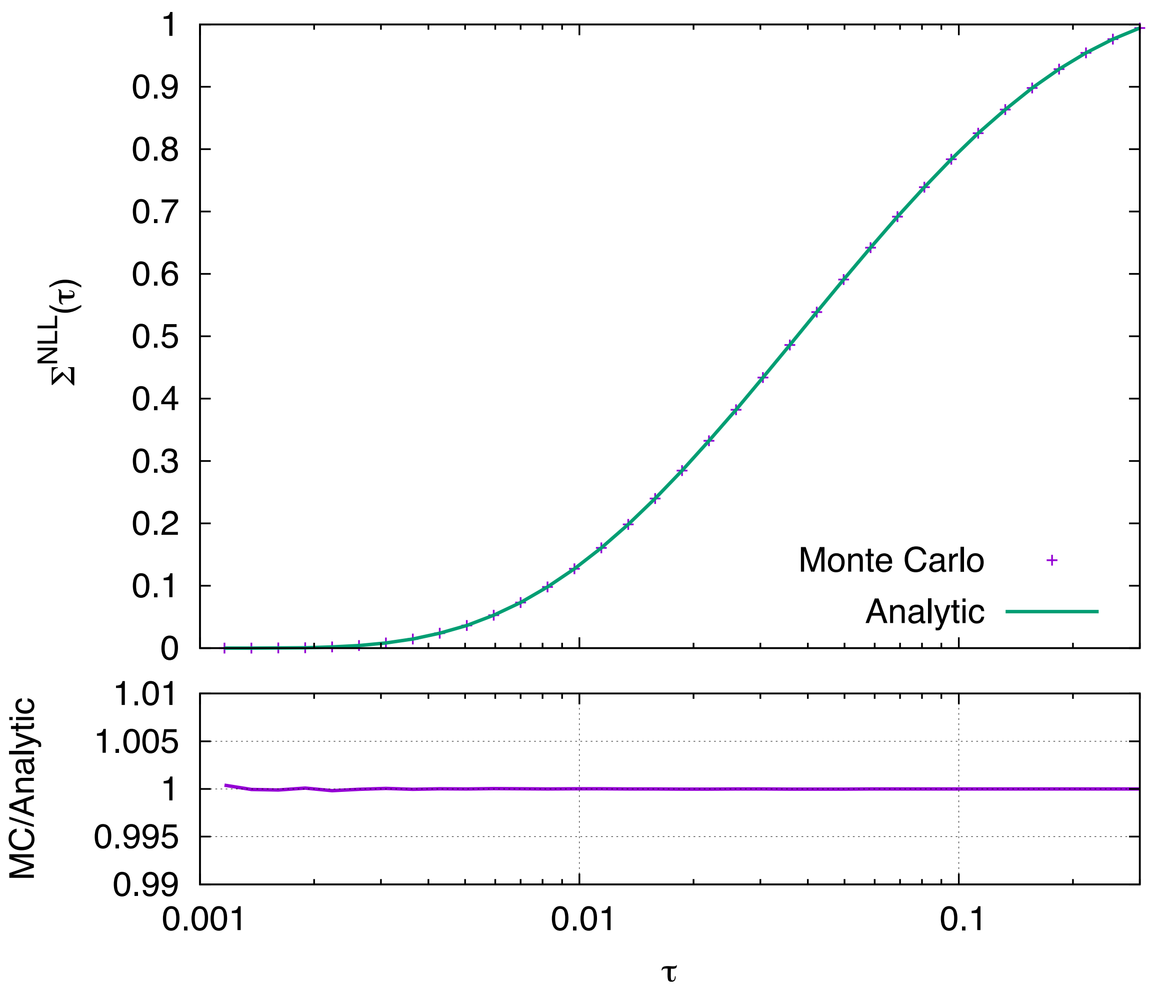
$$\Sigma(\tau) = \Sigma_{\max}(\tau) \int \mathrm{d}\tau_n\, \mathcal{F}'_{J_n}(\tau_n, \mu) \int \mathrm{d}\tau_{\bar{n}}\, \mathcal{F}'_{J_{\bar{n}}}(\tau_{\bar{n}}, \mu) \int \mathrm{d}\tau_s\, \mathcal{F}'_S(\tau_s, \mu)\; \Theta[\tau - \tau_n - \tau_{\bar{n}} - \tau_s]$$

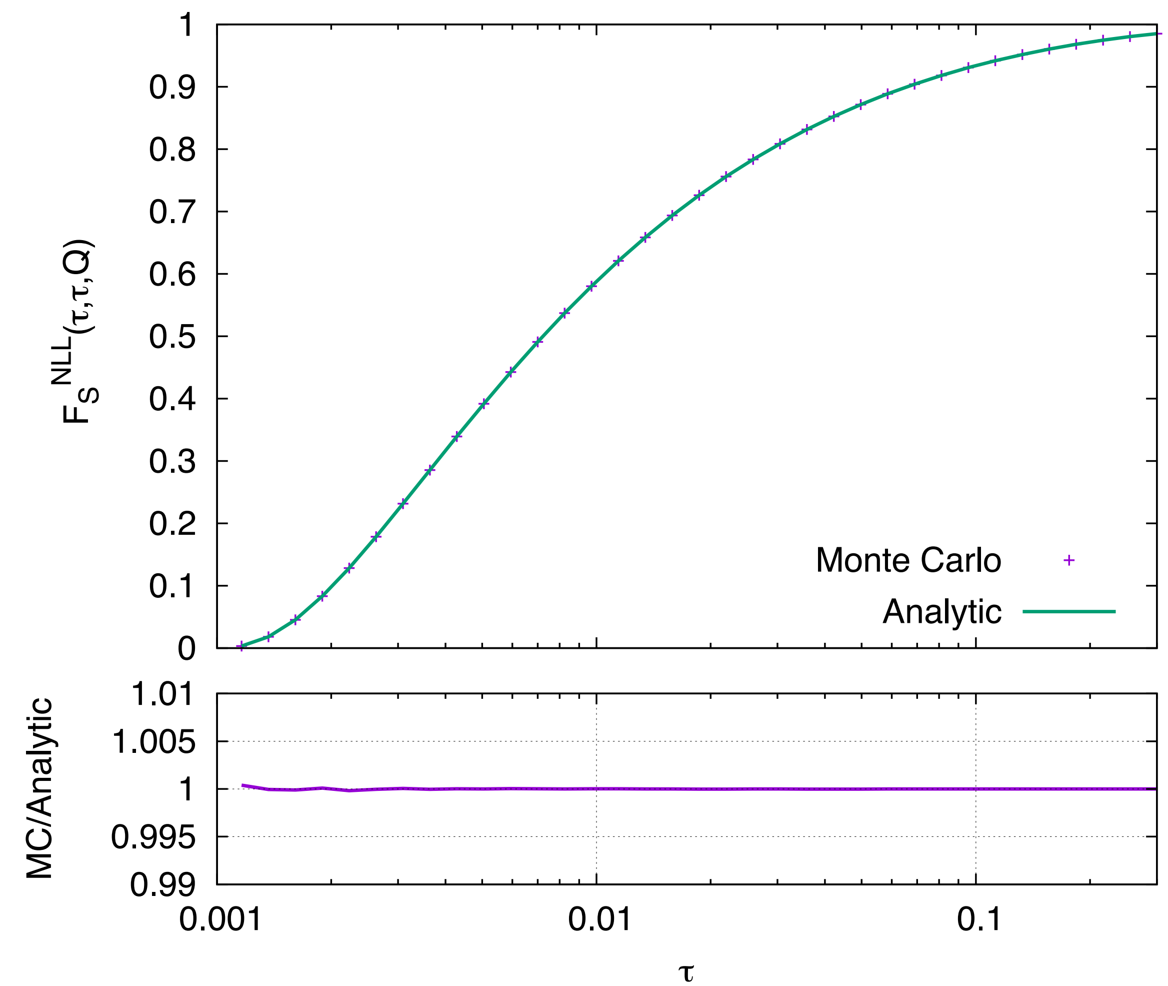
$$\mathcal{F}_F(\tau_F, \tau, \mu) = \frac{\Sigma_F^{\max}(\delta\tau, \mu)}{\Sigma_F^{\max}(\tau, \mu)}\, \frac{\Sigma_F(\tau_F, \mu)}{\Sigma_F^{\max}(\delta\tau, \mu)}$$

At SCET 2018, showed how the thrust cumulant can be related to a simpler observable using a transfer function

Putting all information together, we obtained

$$\Sigma^{\rm NLL}(\tau) = \Sigma_{\rm max}(\tau)\mathcal{F}_S^{\rm NLL}(\tau,\tau,Q)$$





How can this be generalized to any observable of interest?

General expression for numerical resummation is obtained using 6 basic steps

1. Identify DOF required for given observable to LL
2. Obtain fully differential factorization theorem
3. Define simplified observable using same DOF
4. Perform resummation for simplified observable using SCET
5. Define transfer function for each of the DOF
6. Find efficient way to perform computation of transfer function

General expression for numerical resummation is obtained using 5 basic steps

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Steps 1 and 4 same as in any SCET calculation

Not discussed further

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Steps 2, 3, 5 and 6 deserve further discussion

Rest of the talk

General expression for numerical resummation is obtained using 5 basic steps

2. Obtain fully differential factorization theorem

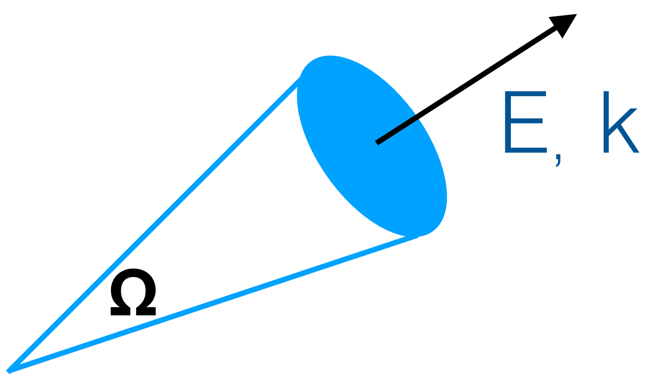
Obtaining a fully differential factorization theorem follows old SCET ideas to prove general factorization theorem

General SCET factorization theorems from energy density formalism

CWB, Fleming, Lee, Sterman ('08)
CWB, Hornig, Tackmann ('08)

Define energy and momentum density

$$\omega_X(\Omega) = \sum_{i=1}^{N} E_i\, \delta(\Omega - \Omega_i)\,, \qquad \vec{k}_X(\Omega) = \sum_{i=1}^{N} \vec{k}_i\, \delta(\Omega - \Omega_i)$$



Phase space at given solid angle given by

$$[\mathrm{d}k]_{\Omega_k} = \frac{[\mathrm{d}\omega]_{\Omega_k}[\mathrm{d}^3k]_{\Omega_k}}{(2\pi)^3}\,\delta[\omega(\Omega_k)^2 - k^2(\Omega_k)] = \frac{\omega_{\Omega_k}[\mathrm{d}\omega]_{\Omega_k}}{2(2\pi)^3}$$

Allows to write phase space as

$$\mathcal{D}\omega^{(N)} = S(N)\prod_{i=1}^{N}\mathrm{d}\Omega_i[\mathrm{d}k]_{\Omega_i} \equiv S(N)\prod_{i=1}^{N}[\mathrm{d}k_i]$$

Obtaining a fully differential factorization theorem follows old SCET ideas to prove general factorization theorem

General SCET factorization theorems from energy density formalism

CWB, Fleming, Lee, Sterman ('08)
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Write general observable as

$$\Sigma(\Phi_B; v) \equiv \int \mathcal{D}\omega \, \frac{\delta\sigma}{\mathrm{d}\Phi_B \delta\omega} \, \Theta\left(V[\omega] < v\right)$$

$$\equiv \sum_N \int \mathcal{D}\omega^{(N)} \, \frac{\delta\sigma}{\mathrm{d}\Phi_B \delta\omega^{(N)}} \, \Theta\left(V[\omega^{(N)}] < v\right)$$

Define energy flow operator

$$\mathcal{E}^0(\Omega)|X\rangle = \omega_X(\Omega)|X\rangle$$

$$\mathcal{E}^0(\Omega) = \lim_{R\to\infty} R^2 \int_0^\infty \mathrm{d}t \, \vec{u}^i \, T^{0i}(t, R\,\vec{u})$$

Fully diff cross section given by

$$\frac{\delta\sigma}{\delta\omega^{(N)}} \equiv \frac{1}{2F} \left|M(q_n, q_{\bar{n}}; k_1, \ldots, k_N)\right|^2$$

Linearity of EM tensor gives

$$\mathcal{E}^0(\Omega) = \sum_\ell \mathcal{E}^0_{n_\ell}(\Omega) + \mathcal{E}^0_s(\Omega)$$

Gives general factorization theorem

$$\frac{\delta\sigma}{\mathrm{d}\Phi_B \delta\omega} = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \, \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \, \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \, \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} \delta\left[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\right]$$

Obtaining a fully differential factorization theorem follows old SCET ideas to prove general factorization theorem

General SCET factorization theorems from energy density formalism

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Gives general factorization theorem

$$\frac{\delta\sigma}{\mathrm{d}\Phi_B\delta\omega} = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \frac{\delta\sigma_n}{\mathrm{d}\Phi_B\delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B\delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \frac{\delta\sigma_S}{\mathrm{d}\Phi_B\delta\omega_s} \delta\left[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\right]$$

From this any observable can be constructed

$$\Sigma(\Phi_B; v) = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \frac{\delta\sigma_n}{\mathrm{d}\Phi_B\delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B\delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \frac{\delta\sigma_S}{\mathrm{d}\Phi_B\delta\omega_s} \times \Theta(V[\omega] < v)\, \delta\left[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\right],$$

Standard factorization theorem if observable factorizes

General expression for numerical resummation is obtained using 5 basic steps

3. Define simplified observable using same DOF

Define a simplified observable that gives multiplicative factorization

$$\Sigma(\Phi_B; v) = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} \times \Theta(V[\omega] < v)\, \delta\left[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\right],$$

Simplified observable defined such that observable has structure

$$V_{\max}[\omega] = \max\{V[\omega_n], V[\omega_{\bar{n}}], V[\omega_s]\}$$

Theta function therefore implies as

$$\Theta[\max[v_n, v_{\bar{n}}, v_s] < v] = \Theta[v_n < v]\,\Theta[v_{\bar{n}} < v]\,\Theta[v_s < v]$$

which gives multiplicative factorization

$$\Sigma_{\max}(\Phi_B; v) = |C(\Phi_B)|^2 \Sigma_n^{\max}(\Phi_B; v) \Sigma_{\bar{n}}^{\max}(\Phi_B; v) \Sigma_s^{\max}(\Phi_B; v)$$

Simplified observable multiplicative with same LL dependence as desired observable.

Many observable have same simplified observable version

General expression for numerical resummation is obtained using 5 basic steps

5. Define transfer function for each of the DOF

Gives a general expression for the transfer function of any observable

$$\Sigma(\Phi_B; v) = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \, \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \, \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \, \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s}$$
$$\times \, \Theta(V[\omega] < v) \, \delta\Big[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\Big] \, ,$$

$$\Sigma_{\max}(\Phi_B; v) = |C(\Phi_B)|^2 \Sigma_n^{\max}(\Phi_B; v) \Sigma_{\bar{n}}^{\max}(\Phi_B; v) \Sigma_s^{\max}(\Phi_B; v)$$

Gives a general expression for the transfer function of any observable

$$\Sigma(\Phi_B; v) = |C(\Phi_B)|^2 \int \mathcal{D}\omega_n \, \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \, \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \, \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s}$$
$$\times \Theta(V[\omega] < v) \, \delta\left[\omega - \omega_s - \omega_n - \omega_{\bar{n}}\right] ,$$

$$\Sigma_{\max}(\Phi_B; v) = |C(\Phi_B)|^2 \Sigma_n^{\max}(\Phi_B; v) \Sigma_{\bar{n}}^{\max}(\Phi_B; v) \Sigma_s^{\max}(\Phi_B; v)$$

This allows to write the expression

$$\Sigma(\Phi_B; v) = \Sigma_{\max}(\Phi_B; v) \mathcal{F}(\Phi_B; v)$$

$$\mathcal{F}(\Phi_B; v) = \int \mathcal{D}\omega_n \, \mathcal{F}'_n(\Phi_B; \omega_n, v) \int \mathcal{D}\omega_{\bar{n}} \, \mathcal{F}'_{\bar{n}}(\Phi_B; \omega_{\bar{n}}, v) \int \mathcal{D}\omega_s \, \mathcal{F}'_S(\Phi_B; \omega_s, v)$$
$$\times \Theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

$$\mathcal{F}'_F(\Phi_B; \omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_F^{\max}(\Phi_B; v)}$$

Gives a general expression for the transfer function of any observable

$$\Sigma(\Phi_B; v) = \Sigma_{\max}(\Phi_B; v)\,\mathcal{F}(\Phi_B; v)$$

$$\mathcal{F}'_F(\Phi_B; \omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_F^{\max}(\Phi_B; v)}$$

$$\mathcal{F}(\Phi_B; v) = \int \mathcal{D}\omega_n\, \mathcal{F}'_n(\Phi_B; \omega_n, v) \int \mathcal{D}\omega_{\bar{n}}\, \mathcal{F}'_{\bar{n}}(\Phi_B; \omega_{\bar{n}}, v) \int \mathcal{D}\omega_s\, \mathcal{F}'_S(\Phi_B; \omega_s, v) \times \Theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

Simple observable with multiplicative factorization

Transfer function corrects to result of desired observable

Transfer function from individual pieces in each SCET sector

Transfer function computed through fully differential transfer functions, convoluted against desired (in general non-factorizing) observable

General expression for numerical resummation is obtained using 5 basic steps

6. Find efficient way to perform computation of transfer function

UV divergences that are present in individual pieces, and only cancel in complete result, can be regulated

For the transfer function, need to compute terms like

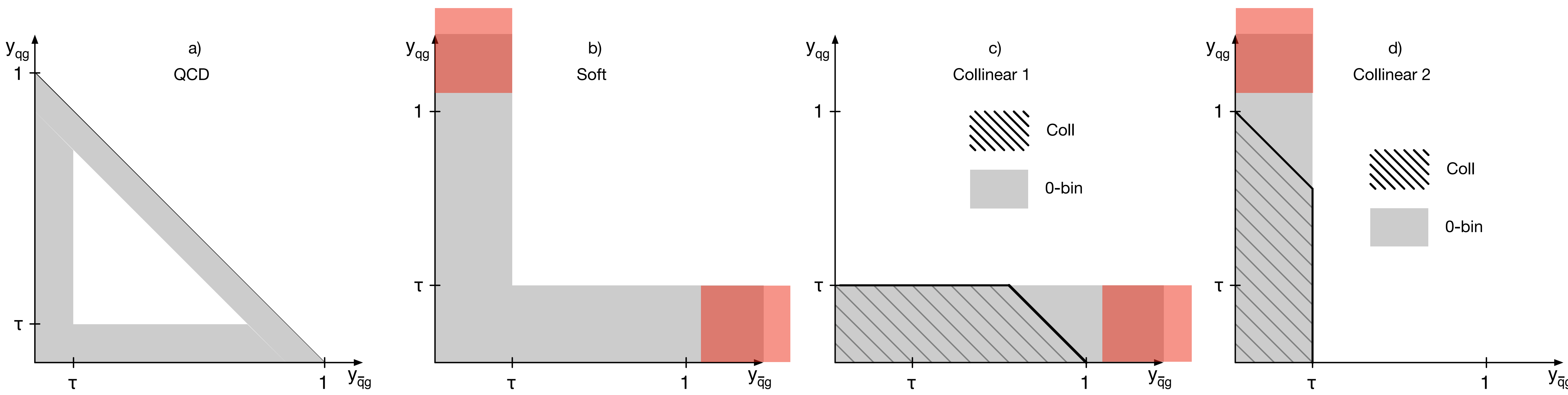
$$\mathcal{F}'_F(\Phi_B; \omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_F^{\max}(\Phi_B; v)}$$

Numerically, can only compute finite quantities. What happens to divergences present?

- All virtual contributions (independent of observable) cancel
- IR divergences cancel in the ratio defining the differential transfer function (more later)
- However, in standard SCET soft and jet functions contain UV divergences in real radiation

UV divergences that are present in individual pieces, and only cancel in complete result, can be regulated

UV divergence arises from regions of large rapidity

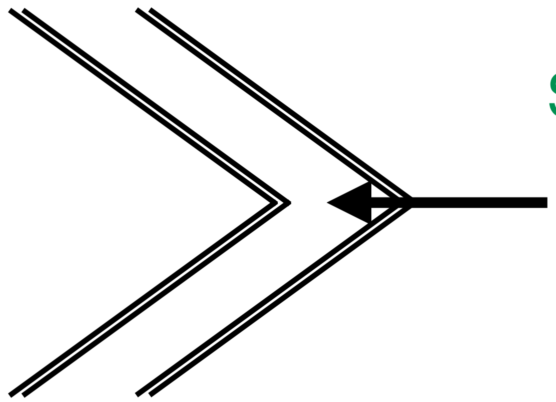


UV divergence arises from regions of large rapidity. Suggests that they can be regulated by rapidity regulator

UV divergences that are present in individual pieces, and only cancel in complete result, can be regulated

Convenient choice for regulator is exponential regulator

Li, Neill, Zhu ('16)



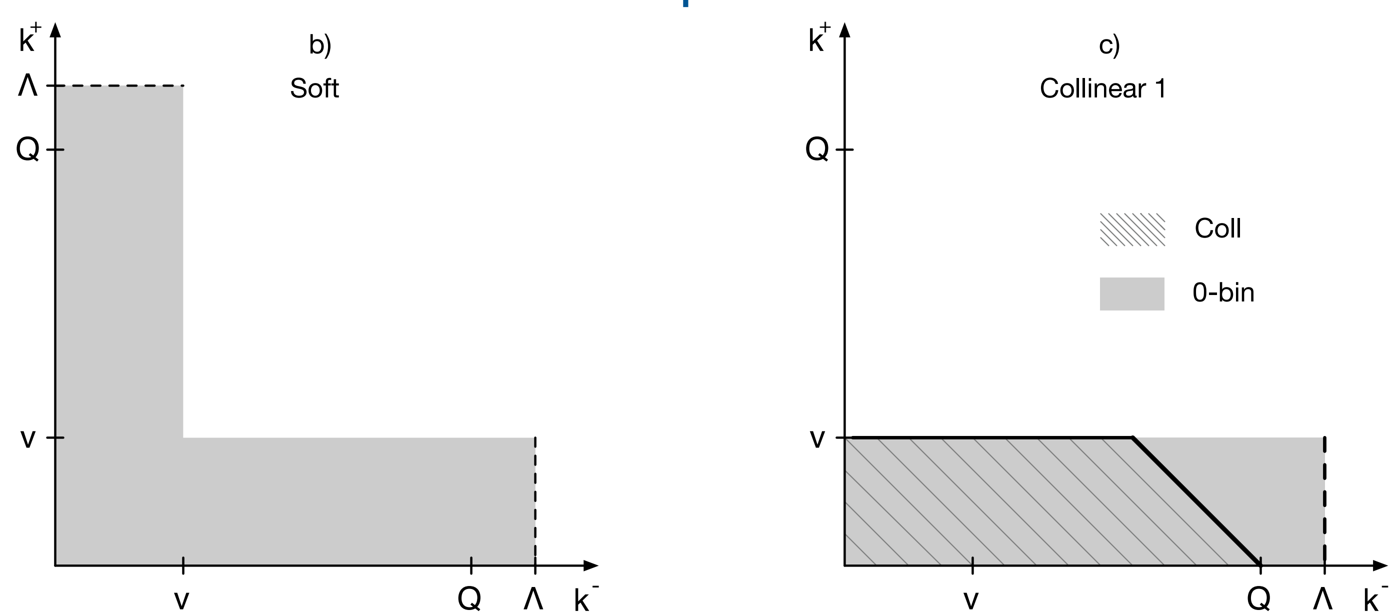
separated Wilson lines by small x^0

$\tau \sim 1/\Lambda$

Introduces exponential in k running across cut

$\exp\left[-k_0/\Lambda\right]$

Drop power corrections in $1/\Lambda$. replace exponential by step-function



Just a cutoff on the real radiation

Now that everything is finite, how do we compute transfer function at given logarithmic accuracy?

Main results so far:

Resummed expression desired related to simpler resummed expression via transfer function

$$\Sigma(\Phi_B; v) = \Sigma_{\max}(\Phi_B; v)\mathcal{F}(\Phi_B; v)$$

Now that everything is finite, how do we compute transfer function at given logarithmic accuracy?

Main results so far:

Resummed expression desired related to simpler resummed expression via transfer function

$$\Sigma(\Phi_B; v) = \Sigma_{\max}(\Phi_B; v)\mathcal{F}(\Phi_B; v)$$

Transfer function

$$\mathcal{F}(\Phi_B; v) = \int \mathcal{D}\omega_n \, \mathcal{F}'_n(\Phi_B; \omega_n, v) \int \mathcal{D}\omega_{\bar{n}} \, \mathcal{F}'_{\bar{n}}(\Phi_B; \omega_{\bar{n}}, v) \int \mathcal{D}\omega_s \, \mathcal{F}'_S(\Phi_B; \omega_s, v)$$
$$\times \, \Theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

Each term in transfer function can be computed numerically

Now that everything is finite, how do we compute transfer function at given logarithmic accuracy?

$$\mathcal{F}(\Phi_B; v) = \int \mathcal{D}\omega_n \, \mathcal{F}'_n(\Phi_B; \omega_n, v) \int \mathcal{D}\omega_{\bar{n}} \, \mathcal{F}'_{\bar{n}}(\Phi_B; \omega_{\bar{n}}, v) \int \mathcal{D}\omega_s \, \mathcal{F}'_S(\Phi_B; \omega_s, v)$$

$$\times \Theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

Write amplitude at given multiplicity through correlated as

$$|M_F(k_1)|^2 \equiv \tilde{M}_F^2(k_1)$$

$$|M_F(k_1, k_2)|^2 = \tilde{M}_F^2(k_1)\tilde{M}_F^2(k_2) + \tilde{M}_F^2(k_1, k_2)$$

To given logarithmic accuracy, only low order correlations are needed

	soft sector		collinear sector	
	$n_{\text{loops}} + n_{\text{particles}}$	times	$n_{\text{loops}} + n_{\text{particles}}$	times
	$\mathcal{F}'_S$		$\mathcal{F}'_{n,\bar{n}}$	
NLL	1	∞	–	–
	$\delta\mathcal{F}'_S$		$\delta\mathcal{F}'_{n,\bar{n}}$	
NNLL	2	1	1	1

NLL accuracy:

Infinite number of tree level, single soft emission terms

NNLL accuracy:

Need to add 1) one loop single soft, 2) tree level double soft, 3) tree level single collinear

As example, at NLL, can compute transfer function using a simple algorithm

$$\mathcal{F}^{\rm NLL}(\Phi_B; v) = \int \mathcal{D}\omega_s \, \mathcal{F}_S^{\prime \rm NLL}(\Phi_B; \omega_s, v) \, \Theta(V[\omega_s + \omega[q_n] + \omega[q_{\bar{n}}]] < v)$$

Choose a particular phase space parametrization to write

$$\Delta_S^{\rm LL}(\Phi_B; v, \delta v) \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \sum_{\ell} \int_{\delta v} \frac{{\rm d}v_i}{v_i} \int {\rm d}\chi_i \, P_\ell^{\rm LL}(v; \chi_i) \int \frac{{\rm d}\phi_i}{2\pi}\right] \Theta(V[q_n, q_{\bar{n}}; k_1, \ldots, k_n] < v)$$

Can be implemented in straightforward algorithm

Algorithm 2: Computing the NLL transfer function

Set weight $W = 1$, $W_{\rm Sq} = 1$;
for $i = 1 \ldots N$ **do**
  Generate a set of soft-collinear emissions $\{k_i\}$ with weight w using Algorithm 1;
  if $V[q_n, q_{\bar{n}}; \{k_i\}] < v$ **then**
    Increase W by w;
    Increase $W_{\rm Sq}$ by w^2;
  end
end
Compute $\mathcal{F}^{\rm NLL} \pm \Delta\mathcal{F}^{\rm NLL}$ from the average value of W and its standard deviation;

From these emission, one can then compute the transfer function for a given observable

Soft momenta generated via MCMC algorithm

Algorithm 1: Generating the soft-collinear emissions

Set weight $w = 1$;
Start with $i = 0$ and $v_0 = v$;
while true **do**
  Increase i by 1;
  Generate a random number $r \in [0, 1]$;
  Determine v_i by solving $\Delta_S^{\mathrm{LL}}(\Phi_B; v_{i-1}, v_i) = r$;
  if $v_i < \delta v$ **then**
    break;
  end
  Choose the leg ℓ randomly from a flat distribution;
  Generate $\chi_i \in [0, 1]$ and $\phi_i \in [0, 2\pi]$ from a flat distribution;
  Multiply the event weight w by $P_\ell^{\mathrm{LL}}(v, \chi_i) / \left[\sum_\ell P_\ell^{\mathrm{LL}}(v)\right] \times n_{\mathrm{legs}}$;
  Determine $k_i = k(v_i, \chi_i, \phi_i)$ and add to the list of emissions;
end
Return the list of momenta $\{k_i\}$ and associate weight w;

This works for any observable

Higher order resummation is obtained by a systematic expansion in SCET to higher orders

Everything is defined within effective theory, so going to higher orders just requires computing things systematically to higher order.

Simplified observable resummed to higher logarithmic order using normal SCET counting of anomalous dimensions

Transfer function computed to higher logarithmic order by systematically computing higher correlated matrix elements numerically

In summary, one can obtain resummed expressions for any observable numerically using systematic SCET expansion

Questions?