The Energy Triple-Product Correlation

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Work in progress

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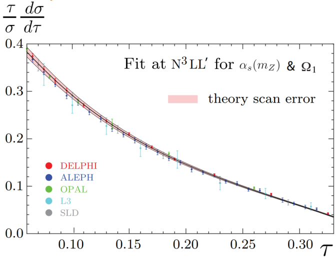
Event Shapes in the Dijet limit see also Larkoski's talk on Monday

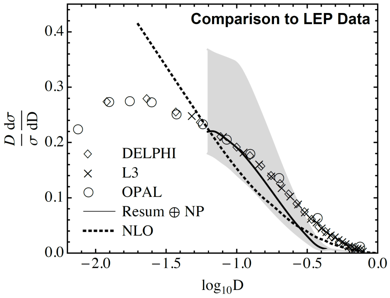
- Event shape observables in the dijet limit at lepton colliders are well studied, to NNLO and N^{3}LL accuracy [Ridder et al, 2007; Weinzierl, 2008, 2009; Becher et al, 2008; Abbate et al, 2011; Chien et al, 2010; Hoang et al, 2014].
- Thrust T [Farhi, 1977], C-parameter [Parisi, 1978; Donoghue et al, 1979; Ellis, 1981], broadening B [Rakow et al 1981], the energy-energy correlations (EEC) [Basham et al, 1978]



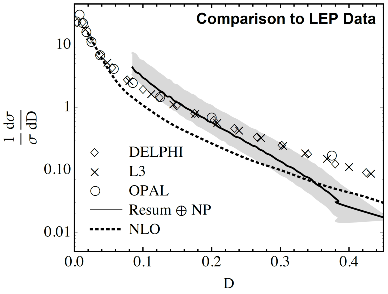
[Abbate, et al, 2010]

Observables in the Trijet Limit see also Larkoski's talk on Monday

- Thrust minor T_m [Banfi et al, 2001]
- D-parameter: the three-jet coplanar region was studied[Banfi et al, 2001]; recently all regions of $D \to 0$ were studied to perform the full resummation[Larkoski and Procita, 2018]



(a)



(b)

[Larkoski and Procita, 2018]

Review of the EEC

The Energy-Energy Correlation (EEC) is defined as [Basham et al, 1978]

$$\mathrm{EEC} = \sum_{ij} \int d\sigma \frac{E_i E_j}{Q^2 \sigma_{\mathrm{tot}}} \delta\left(\cos\theta_{ij} - \cos\chi\right),$$

which measures the correlations of energy deposited in two detectors with angle χ

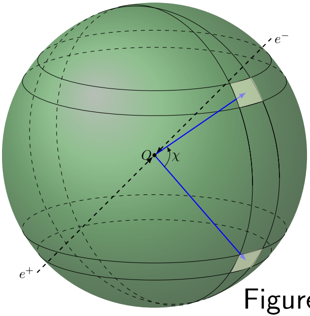


Figure from [Moult and Zhu, 2018]

From CMB to Colliders

- In the research of the Cosmic Microwave Background (CMB), two- and three-point correlation functions were studied
- At the Lepton Collider, instead of only one universe at the CMB, we have numerous events at colliders
- Color evolutions of multiple Wilson Lines; More differencial structures
- The general form of three-point corrections is a multivariable function
- *We seek for a function with one variable as first step*

$\Rightarrow$ *The Energy Triple-Product Correlation*

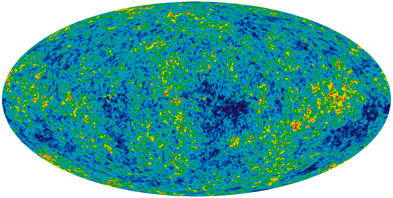


Figure: CMB [Wikipedia]

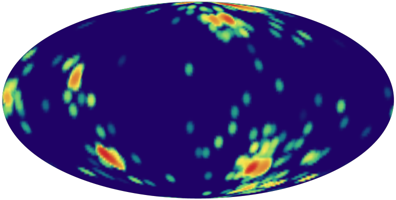


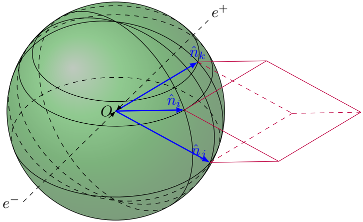
Figure: One Event at the Lepton Collider

Definition of the ETPC

- We define a *NEW* observable, the Energy Triple-Product Correlation (ETPC)

$$ETPC = \frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma}{d\tau_p} = \sum_{ijk}\int d\sigma \frac{E_i E_j E_k}{Q^3 \sigma_{\text{tot}}}\delta\left(\tau_p - \tau_{ijk}\right)$$

- i, j, k run over all the different final state particles
- $\tau_{ijk} = |(\hat{n}_i \times \hat{n}_j) \cdot \hat{n}_k|$ *is the volume of the parallelepiped formed by the unit vectors* $\hat{n}_i$, $\hat{n}_j$ *and* $\hat{n}_k$
- $\tau_{ijk} \to 0$ corresponds to the limit that i, j, k are coplanar



The Relation between the ETPC and the D-Parameter

- The D-parameter is product of three eigenvalues of the spherocity tensor $\Theta_{\alpha\beta} = \frac{1}{Q}\sum_i \frac{p_{i\alpha}p_{i\beta}}{E_i}$ [Parisi, 1978; Donoghue et al, 1979]
- If we assume all the final particles are massless

$$
\begin{aligned}
D &= 27\lambda_1\lambda_2\lambda_3 \\
&= \frac{27}{6}\left\{(\lambda_1+\lambda_2+\lambda_3)\left[(\lambda_1+\lambda_2+\lambda_3)^2 - 3\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\right] + 2\left(\lambda_1^3+\lambda_2^3+\lambda_3^3\right)\right\} \\
&= \frac{27}{6}\left\{Tr\Theta\left[(Tr\Theta)^2 - 3Tr\Theta^2\right] + 2Tr\Theta^3\right\} \\
&= \frac{27}{Q^3}\sum_{i<j<k}\frac{\left|(\vec{p}_i\times\vec{p}_j)\cdot\vec{p}_k\right|^2}{E_iE_jE_k} \\
&= \frac{27}{Q^3}\sum_{i<j<k}E_iE_jE_k\tau_{ijk}^2
\end{aligned}
$$

- *The average of the D-parameter is the third moment of the ETPC*

$$
\langle D\rangle = \frac{9}{2}\int d\tau_p\;\tau_p^2\;ETPC\left(\tau_p\right)
$$

Trijet Coplanar limit

- For the D-parameter, the trijet coplanar region corresponds to

$$D \ll C^2 \sim 1\,,$$

- In the work of [Banfi et al, 2001], the trijet resolution variable y_3 is required to be large

$$y_3 > y_{\text{cut}}$$

- y_3 is defined according to the k_T (Durham) algorithm [Catani, 1991]

$$y_{hh'} = 2\left(1 - \cos\theta_{hh'}\right)\min\left(E_h^2, E_{h'}^2\right)/Q^2$$

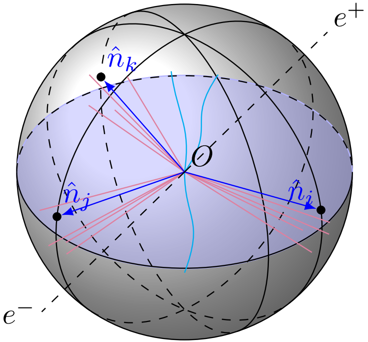


Figure: Trijet Coplanar Limit

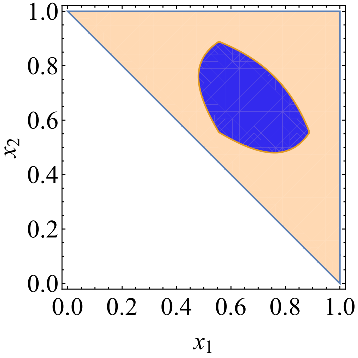
Trijet Coplanar Limit for the ETPC

For the ETPC, we apply two ways to approach the trijet coplanar limit

1 In each event, choose the three particles set $\{i, j, k\}$ such that

$$\sin\theta_{ij} > a_{\text{cut}}\,,\;\; \sin\theta_{jk} > a_{\text{cut}}\,,\;\; \sin\theta_{ki} > a_{\text{cut}}$$

where $\sqrt{3}/2 < a_{\text{cut}} < 1$ is a parameter that control the size of the allowed phase space.



The phase space of three particles, with $a_{\text{cut}} = 0.6$
Here $x_i = 2E_i/Q$

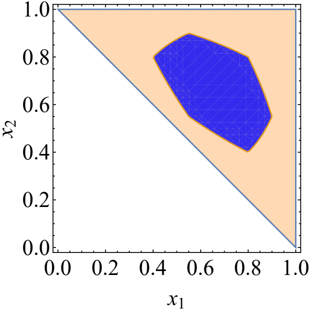
Trijet Coplanar Limit for the ETPC

For the ETPC, we applied two ways to approach the trijet coplanar limit

2 Use the k_T algorithm to find three jets, only if $y_3 > y_{\text{cut}}$ we keep the event, and modify the definition of the ETPC to

$$\sum_{\substack{i\in J_1\\ j\in J_2\\ k\in J_3}} \int d\sigma \frac{E_i E_j E_k}{E_{J_1} E_{J_2} E_{J_3} \sigma_{\text{tot}}} \delta\left(\tau_p - \tau_{ijk}\right)$$

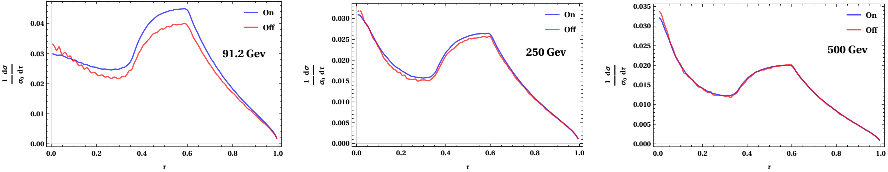
where J_1, J_2, J_3 denote three jets.



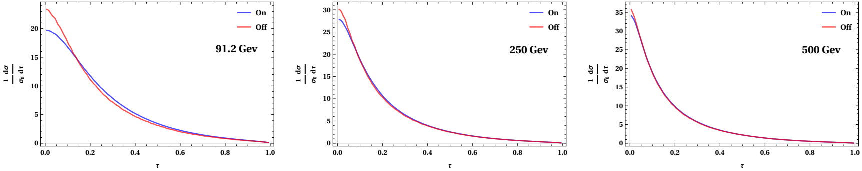
The phase space of three particles, with $y_{\text{cut}} = 0.1$
Here $x_i = 2E_i/Q$

Using PYTHIA8.2 *Preliminary!*

- Used PYTHIA8.2 [Sjöstrand et al, 2015] to generate events
- Turn on/off hadronization
- The hadronization effect is $\frac{1}{Q}$-dependent
- $a_{\text{cut}} = 0.6$

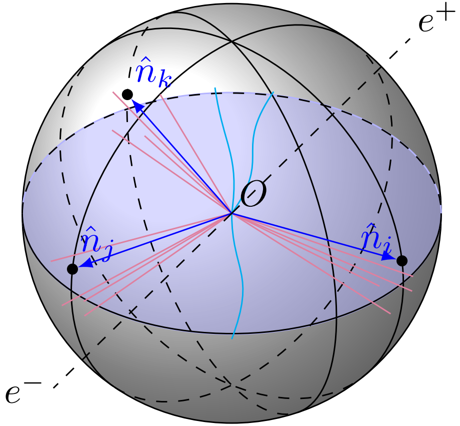


- $y_{\text{cut}} = 0.1$



Factorization: Kinematics

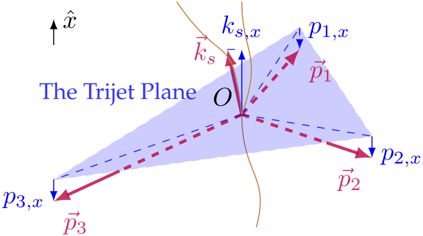
In the trijet coplanar limit, soft radiations and collinear fragmentations dominate $\frac{d\sigma}{d\tau_p}$



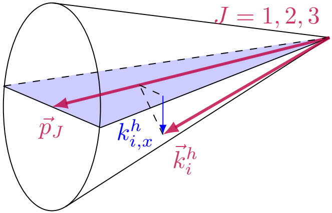
Factorization: Kinematics

In the trijet coplanar limit, soft radiations and collinear fragmentations dominate $\frac{d\sigma}{d\tau_p}$

- The recoil effect of soft radiations.



- Collinear fragmentation

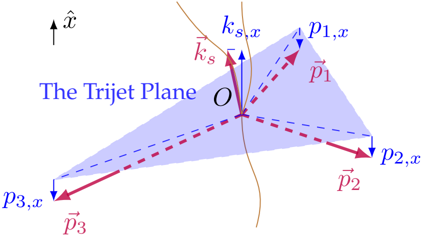


- $\vec{p}_J$, $\vec{k}_s$, and $\vec{k}^h_i$, are the momenta of three jets, the soft radiations, final-state hadrons; z^h_i, the longitudinal fragment of $\vec{k}^h_i$ to p_J; $k_{s,x}$ and $p_{J,x}$, $k^h_{i,x}$, momentum components perpendicular to the trijet plane

$$\Rightarrow \quad \tau_{ijk} = \frac{|\vec{p}_1 \times \vec{p}_2|}{E_1 E_2 E_3} \left| \frac{k^h_{i,x}}{z^h_i} + \frac{k^h_{j,x}}{z^h_j} + \frac{k^h_{k,x}}{z^h_k} - k_{s,x} \right| + \mathcal{O}\left(\tau_p^2\right)$$

In the trijet coplanar limit, soft radiations and collinear fragmentations dominate $\frac{d\sigma}{d\tau_p}$

- The recoil effect of soft radiations.



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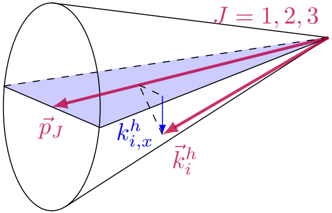

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$$\Rightarrow \quad \tau_{ijk} = \underbrace{\boxed{\frac{|\vec{p}_1 \times \vec{p}_2|}{E_1 E_2 E_3}}}_{\xi} \left| \frac{k^h_{i,x}}{z^h_i} + \frac{k^h_{j,x}}{z^h_j} + \frac{k^h_{k,x}}{z^h_k} - k_{s,x} \right| + \mathcal{O}\left(\tau_p^2\right)$$

The Factorization Formula

$$\frac{1}{\sigma_b}\frac{d\sigma}{d\tau_p} = \int_D dx_1 dx_2\; H(x_1, x_2, \mu) \sum_{ijk} \int dk_{i,x}^h \int dk_{j,x}^h \int dk_{k,x}^h \int dk_{s,x}$$

$$\times \int dz_i^h dz_j^h dz_k^h\; z_i^h z_j^h z_k^h S(k_{s,x}, \mu, \nu)\; \delta\left(\tau_p - \xi \left| \frac{k_{i,x}^h}{z_i^h} + \frac{k_{j,x}^h}{z_j^h} + \frac{k_{k,x}^h}{z_k^h} - k_{s,x} \right|\right)$$

$$\times F_{1\to i}(k_{i,x}^h, z_i^h, \mu, \nu) F_{2\to j}(k_{j,x}^h, z_j^h, \mu, \nu) F_{3\to k}(k_{k,x}^h, z_k^h, \mu, \nu) + \text{power corrections}\,,$$

- $x_1 = 2E_q/Q$, $x_2 = 2E_{\bar{q}}/Q$, $x_3 = 2E_g/Q$; i, j, k belong to the three different jets; σ_b is the born cross section for $e^+e^- \to q\bar{q}$

D The domain of the integrals, constrained by the phase space cuts

H The hard function

S The soft function

F TMD fragmentation Functions

$$F_{q\to h}\left(b, z_h\right) = \frac{1}{4 z_h N_c} \sum_X \int \frac{d\xi^+}{2\pi} e^{-ip_h^- \xi^+/z_h} \left\langle 0 \left| \overline{\chi}_n(\xi) \right| X, h \right\rangle \not{\bar{n}} \left\langle X, h \right| \chi(0) \left| 0 \right\rangle$$

$$\xi = \left(\xi^+, ib_0/\nu, b, 0\right)$$

Simplicity of the Factorization Formula

The TMDFF factorize into the standard FF and matching coefficients

$$F_{i\to h}(k_x^h, z_h) = \sum_j \int \frac{dz}{z^2} f_{h/j}(z,\mu)\mathcal{I}_{ji}\left(\frac{k_x^h}{z}, \frac{z_h}{z}\right)\left[1 + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{(k_x^h)^2}\right)\right] .$$

Changing variables to

$$\zeta_i = \frac{z_i}{x_i}, \quad dz_i dx_i = x_i d\zeta_i dx_i ,$$

using the momentum-conservation sum rule,

$$\sum_h \int dx \; x \; f_{h/i}(x,\mu) = 1 ,$$

and after defining the jet functions as

$$J_i(b) = \sum_j \int_0^1 d\zeta \; \zeta \; \mathcal{I}_{ij}\left(\frac{b}{\zeta}, \zeta\right) ,$$

we simplify our formula to

$$\frac{1}{\hat{\sigma}_0}\frac{d\sigma}{d\tau_p} = \int_D dx_1 dx_2 H(x_1,x_2,\mu)\int_{-\infty}^{\infty}\frac{db}{2\pi\xi}\cos\left(b\tau_p/\xi\right) S(b,\mu,\nu) J_q\left(b,\mu,\nu\right) J_{\bar{q}}\left(b,\mu,\nu\right) J_g\left(b,\mu,\nu\right)$$

Jet Functions

The quark and gluon jet functions are the same as for the EEC [Moult, Zhu], can be calculated to two loops [Ming-xing Luo et al]. To one loop, they are

$$J_q(b,\mu,\nu) = J_{\bar{q}}(b,\mu,\nu) = 1 + \left(\frac{\alpha_s}{4\pi}\right) C_F \left(-2L_b L_Q + 3L_b + 4 - 8\zeta_2\right) + \mathcal{O}\left(\alpha_s^2\right)$$

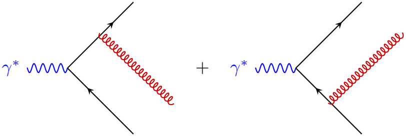
$$J_g(b,\mu,\nu) = 1 + \left(\frac{\alpha_s}{4\pi}\right) \left[-2C_A L_b L_Q + \beta_0 L_b + \left(\frac{65}{18} - 8\zeta_2\right) C_A - \frac{5}{18} n_f\right] + \mathcal{O}\left(\alpha_s^2\right)$$

where $L_b = \ln\left(b^2\mu^2/b_0^2\right)$, $L_Q = \ln(2p^0)^2/\nu^2$, with p^0 the energy of the jet.

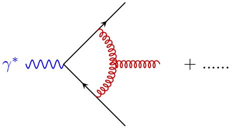
Hard Function

Our hard function incorporates virtual correlations for $e^+e^- \to 3$ Jets

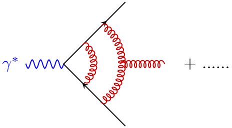
- LO



- NLO



- NNLO (for future work) [Garland, Gehrmann, Glover, Koukoutsakis, Remiddi]*year*



The soft function

$$S\left(n_q, n_{\overline{q}}, n_g, b_x\right) = \mathrm{tr}\langle 0|\mathrm{T}\left[Y_{n_q} Y_{n_{\overline{q}}} Y_g\left(0, b_x, 0, 0\right)\right]\overline{\mathrm{T}}\left[Y^\dagger_{n_q} Y^\dagger_{n_{\overline{q}}} Y^\dagger_g(0,0,0,0)\right]|0\rangle$$

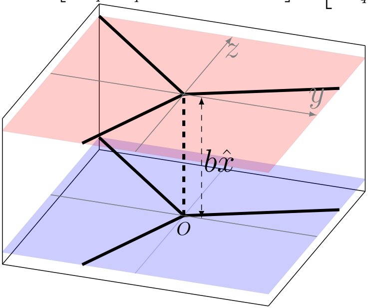


Figure: The spatial structure of the ETPC soft function. Each set of Wilson lines lies in the trijet plane, and their relative displacement is perpendicular to the plane

- Used the exponential regulator to deal with the rapidity divergences [Li et al, 2016]

$$\int d^dk\theta\left(k^0\right)\delta\left(k^2\right) \to \int d^dk\theta\left(k^0\right)\delta\left(k^2\right)e^{-2k^0\tau e^{-\gamma_E}}\,, \quad \nu = \frac{1}{\tau}$$

Soft Function: Factorization

The crucial reason why our formula is amenable to analytic higher order calculations is that the soft function factorize into three dipole soft functions

- Impossible to construct scaling invariant from three light-like vectors.
- Double real [Catani and Grazzini, 1999]
- Real-virtual [Catani and Grazzini, 2000], the one-loop current is

$$J_a^{\mu(1)}(q,\epsilon) \propto i f_{abc} \sum_{i\neq j} T_i^b T_j^c \left(\frac{p_i^\mu}{p_i \cdot q} - \frac{p_j^\mu}{p_j \cdot q} \right) \left[\frac{p_i \cdot p_j}{(p_i \cdot q)\,(p_j \cdot q)} \right]^\epsilon$$

for the ETPC (also for the transverse EEC), q is perpendicular to the plane of Wilson lines

$\Rightarrow$ $[\cdots\cdots] \propto \frac{1}{q_\perp^2}$ does not depend on Wilson lines p_i

- Multiply the tree-level soft-gluon current $\sum_k T_k \frac{p_k^\mu}{p_k \cdot q}$

$\Rightarrow$ There are no interference among three different Wilson lines. The tripole soft function is the product of three dipole soft function (at least to two loops),

$$S(n_q, n_{\bar{q}}, n_g, b_x, \mu, \nu) = \hat{S}_{q\bar{q}}(b_x, \mu, \nu, n_q, n_{\bar{q}})\, \hat{S}_{qg}(b_x, \mu, \nu, n_q, n_g)\, \hat{S}_{\bar{q}g}(b_x, \mu, \nu, n_{\bar{q}}, n_g)$$

Dipole Soft Function

$\hat{S}_{ij}\left(b_x,\mu,\nu,n_i,n_j\right)=S_{ij}\left(b_x,\mu,\nu\sqrt{n_i\cdot n_j/2}\right)$

S_{ij} *The back-to-back dipole soft function, to three loops* [Li and Zhu, 2017]

- In the lightcone coordinate, $k^\mu = n_j^\mu \frac{k\cdot n_i}{n_i\cdot n_j} + n_i^\mu \frac{k\cdot n_j}{n_i\cdot n_j} + k_\perp$. Let $n_{ij}=n_i\cdot n_j$, $k^\pm = k\cdot n_{i,j}$, $v^\mu=(1,0,0,0)$, then $k^0 = k\cdot v = \frac{k^+ + k^-}{n_{ij}} + k_\perp\cdot v_\perp$.
- The integral for soft functions (using the rapidity regulator [Li et al, 2016])

$$I\left(b,\tau,n_i,n_j\right)=\int d^dk\ \delta^+\left(k^2\right)\exp\left(-2k^0\tau e^{-\gamma_E}+ib_\perp\cdot k\right)\cdots\cdots$$

$$=\int \frac{dk^+dk^-}{n_{ij}/2}d^{d-2}k_\perp\ \delta^+\left(\frac{k^+k^-}{n_{ij}/2}+k_\perp^2\right)$$

$$\cdot\exp\left(-\frac{k^+ + k^-}{n_{ij}/2}\tau e^{-\gamma_E}+\left(-2v_\perp e^{-\gamma_E}\tau+ib_\perp\right)\cdot k_\perp\right)\cdots\cdots$$

- The $\cdots\cdots$ denotes integrands
- *Since we will take $\tau_p\to 0$, while keeping $b_\perp$ finite, we ignore $\tau v_\perp$*
- $k^\pm \to q^\pm = k^\pm/\sqrt{n_{ij}/2}$, the integral is the same as the back-to-back dipole soft function except we replace $\nu(=1/\tau)$ with $\nu\sqrt{n_{ij}/2}$

$$I=\int dq^+dq^-d^{d-2}k_\perp\delta^+\left(q^+q^- + k_\perp^2\right)\exp\left[-\left(q^+ + q^-\right)\frac{\tau}{\sqrt{n_{ij}/2}}e^{-\gamma_E}+ib_\perp\cdot k_\perp\right]\cdots\cdots$$

Renormalization Group Equations

$$\frac{dH}{d\ln\mu^2} = \left[\frac{C_A + 2C_F}{2}\gamma_{\rm cusp}\left(\alpha_s\right)\ln\frac{Q^2}{\mu^2} + \gamma_H\left(y,z,\alpha_s\right)\right]H\,,$$

$$\begin{aligned}\frac{d\ln S}{d\ln\mu^2} &= \left[\frac{2C_F + C_A}{2}\left(\gamma_{\rm cusp}\left[\alpha_s\right]\ln\frac{\mu^2}{\nu^2} - \gamma_s\left[\alpha_s\right]\right) + \frac{C_A}{2}\gamma_{\rm cusp}\left[\alpha_s\right]\ln\frac{x_3^2(1-x_3)}{(1-x_1)(1-x_2)}\right. \\ &\quad \left. + C_F\gamma_{\rm cusp}\left[\alpha_s\right]\ln\frac{x_1 x_2}{1-x_3}\right]\,,\end{aligned}$$

$$\frac{d\ln S}{d\ln\nu^2} = \frac{2C_F + C_A}{2}\left(\int_{\mu^2}^{b_0^2/b^2}\frac{d\overline{\mu}^2}{\overline{\mu}^2}\gamma_{\rm cusp}\left(\alpha_s[\overline{\mu}]\right) + \gamma_r\left(\alpha_s\left[b_0/b\right]\right)\right)\,,$$

$$\frac{dJ_i}{d\ln\mu^2} = \left(-\frac{1}{2}c_i\gamma_{\rm cusp}\ln\frac{4\left(p_i^0\right)^2}{\nu^2} + \gamma_{J,i}\right)J_i\,,$$

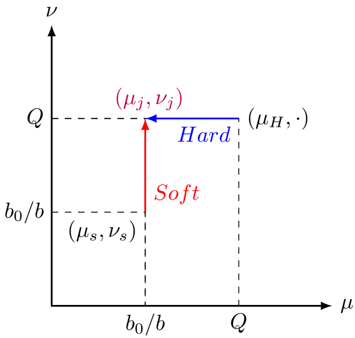
$$\frac{dJ_i}{d\ln\nu^2} = \frac{c_i}{2}\left(\int_{b_0^2/b^2}\frac{d\overline{\mu}^2}{\overline{\mu}^2}\gamma_{\rm cusp}\left[\alpha_s(\overline{\mu})\right] - \gamma_r\left[\alpha_s\left(b_0/b\right)\right]\right)J_i\,.$$

- All the anomalous dimensions are known to at least three loops
- RG invariant condition

$$\gamma_H - \frac{C_A + 2C_F}{2}\gamma_s - 2\gamma_{J,q} - \gamma_{J,g} = 0$$

Renormalization Group Evolution

Setting $\mu = \mu_j = b_0/b$, $\nu = \nu_j = Q$,



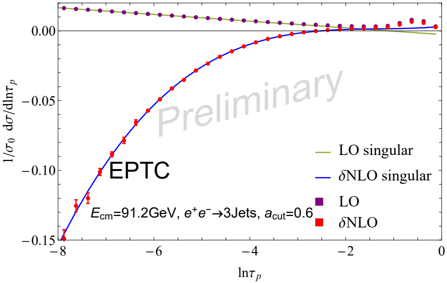
There is rapidity evolution for the soft function

Numerical Implementation

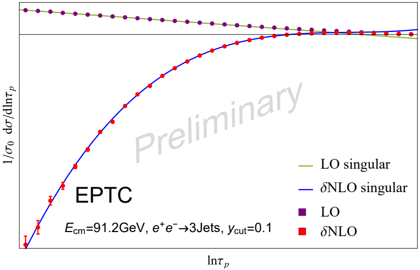
- Apply the NLOJet++ code [Nagy] to calculate the fixed-order ETPC: 4-jet LO + (5-jet real + 4-jet virtual) NLO
- Use these two different settings:
  1. $a_{\text{cut}} = 0.6$ $(\sin\theta_{ij} > a_{\text{cut}}\,,\ \sin\theta_{jk} > a_{\text{cut}}\,,\ \sin\theta_{ki} > a_{\text{cut}})$
  2. $y_{\text{cut}} = 0.1$ $(y_3 > y_{\text{cut}})$
- Verify our factorization formula by comparing the predicted singular fixed-order results with NLOJet++
- Resummation + power corrections

Validation from NLOJet++ *Preliminary!*

- Expanding the factorization formula with $\frac{\alpha_s}{4\pi}$, the b integral can be calculated numerically for each order.
- The LO of $\frac{d\sigma}{d\ln\tau_p}$ is linear in $\ln\tau_p$, and the NLO is a cubic polynomial.
- The formula-predicted of $\frac{d\sigma}{d\ln\tau_p}$ approaches the full fixed-order results given by NLOJet++, in the $\tau \to 0$ limit.



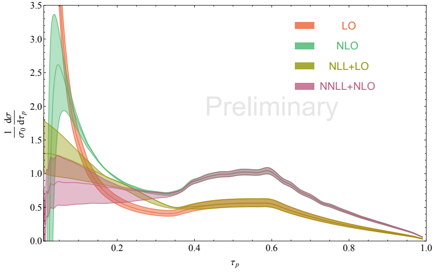
$a_{\text{cut}} = 0.6$



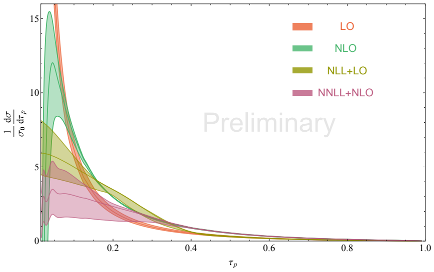
$y_{\text{cut}} = 0.1$

Our factorization formula proves to be right, comparing with the NLOJet++ fixed-order results in the $\tau_p \to 0$ limit

 Preliminary!



$a_{\text{cut}} = 0.6$



$y_{\text{cut}} = 0.1$

- The fixed-order results are unreasonable at the $\tau_p \to 0$ limit; resummation is necessary
- The reduction of scale uncertainties from NLO to NNLL+NLO
- The perturbative corrections from NLL+LO to NNLL+NLO are large
- NLO LO do not overlap; NLL NNLL overlap, converge

Conclusions

- We initiated the study of a new event shape observable called the Energy Triple-Product Correlation
- Hadronization Effects
- Derived a factorization formula for the ETPC in the coplanar limit
- Presented the results of NNLL+NLO

Thank You!