Operator Bases and the HQET Lagrangian

Andrew Kobach
Los Alamos National Laboratory

SCET 2019
La Jolla, CA
March 27, 2019

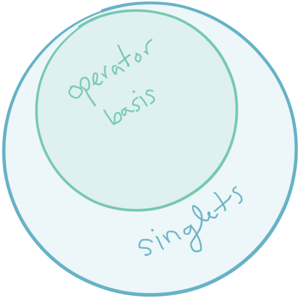
$\mathcal{L}_{\text{EFT}} = ?$

If you're fortunate to know enough about an theory, one way to construct the set of operators for an EFT is to:

- identify the degrees of freedom at the low scale,
- identify how they transform under given symmetries,
- and write down all possible singlets.

This procedure will over-count the number of operators that should appear in the Lagrangian.

Operator Bases



Effective operators $\mathcal{O}_a$ and $\mathcal{O}_b$ give the same S-matrix elements if:

- $\mathcal{O}_a = \mathcal{O}_b + \partial\mathcal{O}$ (integration by parts)
- $\mathcal{O}_a = \mathcal{O}_b + \mathcal{O}\frac{\delta S}{\delta\phi}$ (equations of motion)

Eliminate *either* $\mathcal{O}_a$ or $\mathcal{O}_b$. This choice is called an operator basis.

Easy, right?

[Politzer, (1980)], [Georgi, (1991)]

An Example

Real singlet scalar s in 3+1 spacetime dimensions.

The singlet operators at mass dimension $d = 5$ and 6 are:

$$s^5 \qquad s(\partial_\mu s)(\partial^\mu s) \qquad s^6 \qquad s^3\partial^2 s \qquad (\partial^2 s)^2$$

An Example

Real singlet scalar s in 3+1 spacetime dimensions.

The singlet operators at dimension $d = 5$ and 6 are:

$$s^5 \qquad s(\partial_\mu s)(\partial^\mu s) \qquad s^6 \qquad \boxed{s^3\partial^2 s} \qquad \boxed{(\partial^2 s)^2}$$

If the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu s)^2 - \frac{1}{2}m^2 s^2 - \frac{1}{3!}\mu s^3 - \frac{1}{4!}\lambda s^4 \cdots$$

The equation of motion for s can be written as:

$$-\partial^2 s = m^2 s + \frac{1}{2}\mu s^2 + \frac{1}{3!}\lambda s^3 + \cdots$$

An instance of $\partial^2 s$ can be replaced with other operators, yielding operators already in the basis.

Best to think of this as a field redefinition.

An Example

Real singlet scalar s in 3+1 spacetime dimensions.

The singlet operators at dimension $d = 5$ and 6 are:

$$s^5 \qquad s(\partial_\mu s)(\partial^\mu s) \qquad s^6 \qquad \boxed{\cancel{s^3 \partial^2 s}} \qquad \boxed{\cancel{(\partial^2 s)^2}}$$

If the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu s)^2 - \frac{1}{2}m^2 s^2 - \frac{1}{3!}\mu s^3 - \frac{1}{4!}\lambda s^4 \cdots$$

The equation of motion for s can be written as:

$$-\partial^2 s = m^2 s + \frac{1}{2}\mu s^2 + \frac{1}{3!}\lambda s^3 + \cdots$$

An instance of $\partial^2 s$ can be replaced with other operators, yielding operators already in the basis.

Best to think of this as a field redefinition.

An Example

Real singlet scalar s in 3+1 spacetime dimensions.

The singlet operators at dimension $d = 5$ and 6 are:

$$s^5 \qquad \boxed{s(\partial_\mu s)(\partial^\mu s)} \qquad s^6 \qquad \cancel{s^3 \partial^2 s} \qquad \cancel{(\partial^2 s)^2}$$

This operator can be written as:

$$s(\partial_\mu s)(\partial^\mu s) = \frac{1}{2} \underbrace{\partial_\mu (s^2 \partial^\mu s)}_{\text{total derivative}} - \frac{1}{2} \underbrace{s^2 \partial^2 s}_{\text{EOM}}$$

So this operator is redundant.

Relations like this get more and more complicated as one goes to higher dimensions.

An Example

Real singlet scalar s in 3+1 spacetime dimensions.

The singlet operators at dimension $d = 5$ and 6 are:

$$s^5 \qquad \boxed{\cancel{s(\partial_\mu s)(\partial^\mu s)}} \qquad s^6 \qquad \cancel{s^3 \partial^2 s} \qquad \cancel{(\partial^2 s)^2}$$

This operator can be written as:

$$s(\partial_\mu s)(\partial^\mu s) = \frac{1}{2} \underbrace{\partial_\mu (s^2 \partial^\mu s)}_{\text{total derivative}} - \frac{1}{2} \underbrace{s^2 \partial^2 s}_{\text{EOM}}$$

So this operator is redundant.

Relations like this get more and more complicated as one goes to higher dimensions.

An Example

Up to and including $d = 8$, a non-redundant set of operators is:

$$s^5 \qquad s^6 \qquad s^7 \qquad s^8 \qquad (\partial_\mu s)^4 \qquad \cdots$$

Operator bases are difficult to construct by hand. Try it.

History: The SMEFT

Weinberg figured out the SMEFT operator basis at $d = 5$ in 1980.

[Weinberg, (1980)]

It took until 2010 to figure out the set of non-redundant operators for the SMEFT with one family at $d = 6$.

[Grzadkowski, Iskrzynski, Misiak, and Rosiek, (2010)]

...and until 2013 for with three families at $d = 6$

[Alonso, Jenkins, Manohar, Trott, (2013)]

...and until 2015 until for $d = 7,\ 8,\ 9, \ldots$

[Henning, Lu, Meliac, Murayama, (2015)]

With plenty of missteps along the way...

HQET

Consider a field theory with one heavy fermion, subject to external gauge fields, with rotational symmetry in the rest frame:

$$\mathcal{L} = \psi^\dagger \left[iD_t + c_2 \frac{\mathbf{D}^2}{2M} + c_F \frac{\mathbf{s} \cdot \mathbf{B}}{2M} + \cdots \right] \psi$$

Here, ψ is a 2-component Pauli spinor, $[D^k, D_t] = igE^k$, $\frac{1}{2}[D^j, D^k]\epsilon^{jkm} = igB^m$, and $\mathbf{s} \equiv \boldsymbol{\sigma}/2$.

Additional constraints on the coefficients from reparameterization invariance, but that does not change the number of operators.

HQET

There has been little agreement in the literature regarding the number of operators at each order in $1/M$ in HQET:

Operator dimension	5	6	7	8
Mannel (1994)	2	2	7	-
Manohar (1997)	2	2	11	-
Dassinger, Mannel, Turczyk (2007)	2	2	5	-
Mannel, Turczyk, Uraltsev (2010)	2	2	9	18
Gunawardana, Paz (2017, v1)	2	2	9	18

Goal: resolve this.

The Hilbert series = tool to aid construction of an operator basis.

[see any paper by Hanany; Manohar, Jenkins (2009); Lehman, Martin (2014, 2015, 2016); Henning, Lu, Melia, Murayama (2015, 2016, 2017)].

Hilbert Series

3 ingredients:

- Ordinary multiplication of characters of a group represent tensor products of group representations.
- Symmetric products of characters are generated by a Taylor expansion of this function:

$$\exp\left[\sum_{n=0}^{\infty}\frac{\chi(z^n,y^n,\cdots)}{n}\right]$$

- Products of derivatives are symmetric.

Method:

- Generate all operators, singlets or not
- Subtract all operators with equation of motion relations
- Subtract total derivatives
- Pick out the singlets

1st Ingredient: Representations

In $SU(2)$, we know $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \otimes \mathbf{1}$:

$$\underbrace{\left(z + \frac{1}{z}\right)}_{\mathbf{2}} \underbrace{\left(z + \frac{1}{z}\right)}_{\mathbf{2}} = \underbrace{\left(z^2 + 1 + \frac{1}{z^2}\right)}_{\mathbf{3}} + \underbrace{1}_{\mathbf{1}}$$

What are these polynomials? They're the characters of these representations of $SU(2)$. For example:

$$\chi_{2j+1}^{SU(2)} = \sum_{|m| \leq j} \langle m| \, e^{i\theta J_3} \, |m\rangle \, , \qquad z \equiv e^{i\theta/2}$$

Or, put another way,

$$\chi_{\mathbf{2}}^{SU(2)} \chi_{\mathbf{2}}^{SU(2)} = \chi_{\mathbf{3}}^{SU(2)} + \chi_{\mathbf{1}}^{SU(2)}$$

2nd Ingredient: Exponentials

For all symmetric products of ϕ, Taylor expand in ϕ:

$$\exp\left[\sum_{n=1}^{\infty}\frac{\phi^n}{n}\chi_{\mathbf{2}}^{SU(2)}(z^n)\right] = 1 + \chi_{\mathbf{2}}^{SU(2)}\phi + \chi_{\mathbf{3}}^{SU(2)}\phi^2 + \cdots$$

And for all antisymmetric products of ϕ:

$$\exp\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\phi^n}{n}\chi_{\mathbf{2}}^{SU(2)}(z^n)\right] = 1 + \chi_{\mathbf{2}}^{SU(2)}\phi + \phi^2$$

ϕ is just a mathematical flag, not a field.

3rd Ingredient: Derivatives

Generate all possible symmetric products of derivatives:

$$\exp\left[\sum_{n=1}^{\infty} P(\alpha^n)\frac{\phi^n}{n}\chi_{\mathbf{2}}^{SU(2)}(z^n)\right]$$
$$= 1 + \chi_{\mathbf{2}}^{SU(2)}\phi + \chi_{\mathbf{3}}^{SO(3)}\chi_{\mathbf{2}}^{SU(2)}\mathcal{D}_\perp\phi + \chi_{\mathbf{2}}^{SU(2)}\mathcal{D}_t\phi + \cdots$$

$$P(\alpha) = \exp\left[\sum_{n=1}^{\infty}\frac{\mathcal{D}_t^n}{n}\right]\cdot\exp\left[\sum_{n=1}^{\infty}\frac{\mathcal{D}_\perp^n}{n}\chi_{\mathbf{3}}^{SO(3)}(\alpha^n)\right]$$

$\mathcal{D}_\perp$ and $\mathcal{D}_t$ are just mathematical flags for spatial and time derivatives, not actual derivatives.

Subtracting Equations of Motion

Don't want operators with $D_t\phi$? Just subtract it.

$$\exp\left[\sum_{n=1}^{\infty} P(\alpha^n)\frac{\phi^n}{n}\chi_{\mathbf{2}}^{SU(2)}(z^n)(1-\mathcal{D}_t)\right]$$

$$= 1 + \chi_{\mathbf{2}}^{SU(2)}\phi + \chi_{\mathbf{3}}^{SO(3)}\chi_{\mathbf{2}}^{SU(2)}\mathcal{D}_\perp\phi + \cdots$$

Subtracting Total Derivatives

Let S be the set of all operators.

$$\begin{aligned} S &= \mathcal{O}_1 + \mathcal{O}_2 + \cdots + \underbrace{\mathcal{D}(\mathcal{O}_1 + \mathcal{O}_2 + \cdots) + \mathcal{D}^2(\mathcal{O}_1 + \mathcal{O}_2 + \cdots) + \cdots}_{\text{total derivatives}} \\ &= \underbrace{(1 + \mathcal{D} + \mathcal{D}^2 + \cdots)}_{P} \underbrace{(\mathcal{O}_1 + \mathcal{O}_2 + \cdots)}_{\text{no total derivatives}} \end{aligned}$$

P generates all terms with derivatives (and the number 1)

$$S_{\text{no total derivatives}} = \frac{1}{P} S$$

Picking Out The Singlets

If you want to pick out the singlets, use character orthogonality:

$$\oint [d\mu]_G \; \chi_R(z) \; \chi_{R'}(z) = \delta_{RR'}$$

since $\chi_{\text{singlet}} = 1$,

$$\oint [d\mu]_G \; \chi_R = 1 \quad \text{iff } \chi_R = \chi_{\text{singlet}} = 1$$

$[d\mu]_G$ is the Haar measure for the group G.

Building Blocks for HQET

In the rest frame:

	$SO(3)_{\text{rot}}$	$SU(3)_{\text{color}}$	$SU(2)_{\text{spin}}$
ψ	1	$\mathbf{3}$	$\mathbf{2}$
$\psi^\dagger$	1	$\overline{\mathbf{3}}$	$\mathbf{2}$
E	$\mathbf{3}$	$\mathbf{8}$	1
B	$\mathbf{3}$	$\mathbf{8}$	1
s	$\mathbf{3}$	1	$\mathbf{3}$
$\mathcal{D}_t$	1	1	1
$\mathcal{D}_\perp$	$\mathbf{3}$	1	1

Input into Hilbert series...

The Result

The output from the Hilbert series is, after imposing invariance under parity...

$$
\begin{aligned}
HS_{d=5} &= \mathcal{D}_\perp^2 + sB, \\
HS_{d=6} &= 2E\mathcal{D}_\perp + sE\mathcal{D}_\perp, \\
HS_{d=7} &= \mathcal{D}_\perp^4 + 2E^2 + 2B^2 + sE^2 + sB^2 + B\mathcal{D}_\perp^2 + 5sB\mathcal{D}_\perp^2, \\
HS_{d=8} &= B^2\mathcal{D}_t + E^2\mathcal{D}_t + 2sB^2\mathcal{D}_t + 2sE^2\mathcal{D}_t + 6EB\mathcal{D}_\perp + 3sE\mathcal{D}_\perp^3 \\
&\quad + 5E\mathcal{D}_\perp^3 + 21sEB\mathcal{D}_\perp
\end{aligned}
$$

These are the operators $\mathcal{O}$ that are sandwiched between $\psi^\dagger \mathcal{O} \psi$.

The numerical coefficients count the number of singlets with those degrees of freedom.

One can automatically impose invariance under parity if the fermion is charged under $SU(3)$, but not time reversal, since f_{abc} is T-odd.

Order	HS	T even	T odd
$\frac{1}{M}$	$\mathcal{D}_\perp^2$	$(iD_\perp)^2$	
	sB	$s^i B_a^j \delta_{ij} T^a$	
$\frac{1}{M^2}$	$2E\mathcal{D}_\perp$	$[D_\perp^i E^j]_a \delta_{ij} T^a$	$\{E_a^i, iD_\perp^j\} \delta_{ij} T^a$
	$sE\mathcal{D}_\perp$	$s^i \{E_a^j, iD_\perp^k\} \epsilon_{ijk} T^a$	
$\frac{1}{M^3}$	$\mathcal{D}_\perp^4$	$(iD_\perp)^4$	
	$2E^2$	$E_a^i E_b^j \delta_{ij} d^{abc} T_c$ $E_a^i E_b^j \delta_{ij} \delta^{ab}$	
	$2B^2$	$B_a^i B_b^j \delta_{ij} d^{abc} T_c$ $B_a^i B_b^j \delta_{ij} \delta^{ab}$	
	sE^2	$s^i E_a^j E_b^k \epsilon_{ijk} f^{abc} T_c$	
	sB^2	$s^i B_a^j B_b^k \epsilon_{ijk} f^{abc} T_c$	
	$B\mathcal{D}_\perp^2$	$\{[D_\perp^i B^j]_a, iD_\perp^k\} \epsilon_{ijk} T^a$	
	$5sB\mathcal{D}_\perp^2$	$\{s^i B_a^j, (iD_\perp^k)^2\} \delta_{ij} T^a$ $\{s^i B_a^j, iD_\perp^k iD_\perp^l\} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) T^a$ $s^i [D_\perp^2 B^j]_a \delta_{ij} T^a$	$\{s^i [D_\perp^j B^k]_a, iD_\perp^l\} \delta_{ij}\delta_{kl} T^a$ $\{s^i [D_\perp^j B^k]_a, iD_\perp^l\} \delta_{ik}\delta_{jl} T^a$
$\frac{1}{M^4}$	$B^2\mathcal{D}_t$	$B_a^i [D_t B^j]_b \delta_{ij} f^{abc} T_c$	
	$E^2\mathcal{D}_t$	$E_a^i [D_t E^j]_b \delta_{ij} f^{abc} T_c$	
	$2sB^2\mathcal{D}_t$	$s^i B_a^j [D_t B^k]_b \epsilon_{ijk} \delta^{ab}$ $s^i B_a^j [D_t B^k]_b \epsilon_{ijk} d^{abc} T_c$	
	$2sE^2\mathcal{D}_t$	$s^i E_a^j [D_t E^k]_b \epsilon_{ijk} \delta^{ab}$ $s^i E_a^j [D_t E^k]_b \epsilon_{ijk} d^{abc} T_c$	
	$6EB\mathcal{D}_\perp$	$\{E_a^i B_b^j, iD_\perp^k\} \epsilon_{ijk} \delta^{ab}$ $\{E_a^i B_b^j, iD_\perp^k\} \epsilon_{ijk} d^{abc} T_c$ $E_a^i [D_\perp^j B^k]_b \epsilon_{ijk} f^{abc} T_c$	$\{E_a^i B_b^j, iD_\perp^k\} \epsilon_{ijk} f^{abc} T_c$ $E_a^i [D_\perp^j B_b^k]_b \epsilon_{ijk} \delta^{ab}$ $E_a^i [D_\perp^j B^k]_b \epsilon_{ijk} d^{abc} T_c$
	$3sE\mathcal{D}_\perp^3$	$\{s^i E_a^j, iD_\perp^k (iD_\perp)^2\} \epsilon_{ijk} T^a$ $\{s^i [D_\perp^j D_\perp^k E^l]_a, iD_\perp^m\} T^a (\epsilon_{ijm}\delta_{kl} + \epsilon_{ikm}\delta_{jl} + \epsilon_{ilm}\delta_{jk})$	$\{s^i [D_\perp^j E^k]_a, iD_\perp^l iD_\perp^m\} T^a (\epsilon_{ijl}\delta_{km} + \epsilon_{ijm}\delta_{kl} + \epsilon_{ikl}\delta_{jm} + \epsilon_{ikm}\delta_{jl})$
	$5E\mathcal{D}_\perp^3$	$\{[D_\perp^i E^j]_a, (iD_\perp)^2\} \delta_{ij} T^a$ $\{[D_\perp^i E^j]_a, iD_\perp^k iD_\perp^l\} T^a (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $[D_\perp^i D_\perp^j D_\perp^k E^l]_a T^a (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$	$\{E_a^i, iD_\perp^j iD_\perp^k iD_\perp^l\} T^a (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $\{[D_\perp^i D_\perp^j E^k]_a, iD_\perp^l\} T^a (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
	$21sEB\mathcal{D}_\perp$	$\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ij}\delta_{kl} f^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ik}\delta_{jl} f^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{il}\delta_{jk} f^{abc} T_c$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{ik}\delta_{jl} d^{abc} T_c$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{il}\delta_{jk} d^{abc} T_c$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{ik}\delta_{jl} \delta^{ab}$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{il}\delta_{jk} \delta^{ab}$ $s^i B_a^j [D_\perp^k E^l]_b d^{abc} T_c (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [D_\perp^k E^l]_b \delta_{ij}\delta_{kl} d^{abc} T_c$ $s^i B_a^j [D_\perp^k E^l]_b \delta^{ab} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [D_\perp^k E^l]_b \delta^{ab} \delta_{ij}\delta_{kl}$	$\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ij}\delta_{kl} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ik}\delta_{jl} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{il}\delta_{jk} d^{abc} T_c$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ij}\delta_{kl} \delta^{ab}$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{ik}\delta_{jl} \delta^{ab}$ $\{s^i E_a^j B_b^k, iD_\perp^l\} \delta_{il}\delta_{jk} \delta^{ab}$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{ik}\delta_{jl} f^{abc} T_c$ $s^i E_a^j [D_\perp^k B^l]_b \delta_{il}\delta_{jk} f^{abc} T_c$ $s^i B_a^j [D_\perp^k E^l]_b f^{abc} T_c (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ $s^i B_a^j [D_\perp^k E^l]_b \delta_{ij}\delta_{kl} f^{abc} T_c$

Number of operators in HQET

Operator dimension	5	6	7	8
Mannel (1994)	2	2	7	-
Manohar (1997)	2	2	11	-
Dassinger, Mannel, Turczyk (2007)	2	2	5	-
Mannel, Turczyk, Uraltsev (2010)	2	2	9	18
Gunawardana, Paz (2017, v1)	2	2	9	18
AK, Pal (2017)	2	2	11	25
Gunawardana, Paz (2017, v2)	2	2	11	25

[AK, Pal; Phys. Lett. B772 (2017)]

Aside: Something fun

We removed all operators in our basis with $D_t\psi$, due to the equations of motion. This is functionally equivalent to setting $D_t\psi = 0$.

This looks similar to $\partial_t\psi = 0$, which is the equation of motion for a free fermion. Connection between operator basis and conformal symmetry?

Yes. The HQET operator basis is indeed spanned by a special kind of primary operator of tensor products of the Schrödinger group.

Direct analogy in relativistic theories, where the operator basis is spanned by scalar primaries of tensor products of short representations of the conformal group.

[Henning, Lu, Melia, Murayama (2015, 2016, 2017)]

[AK, Pal; Phys. Lett. B783 (2018)]

Aside: Reparameterization Invariance

The number of operators does does not equal the number of free parameters in HQET, due to reparameterization invariance.

Aided by another Hilbert series we developed, we constructed an operator basis for a theory with a single fermion with degrees of freedom that are manifestly invariant under reparameterization, and showed that this is closely related to Lorentz invariance.

Possible, but cumbersome, to go from this to the heavy in HQET. But that's life.

Maybe possible to employ the Hilbert series to encode RPI from the bottom up?

[AK, Pal; hep-ph:/1810.02356]

Summary

- Writing down operator bases can be tricky. Hilbert-series methods are valuable. Much more than just operator counting.
- We wrote down the HQET Lagrangian up to and including $1/M^4$ terms. Result replicated by other groups.
- Showed that the HQET operator basis is closely related to representations of the Schrödinger group.
- How can reparameterization invariance be encoded in the Hilbert series?
- Similar methods should be applicable for SCET. No one has yet tried.