FROM THE BFKL EQUATION TO TWO-PARTON SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

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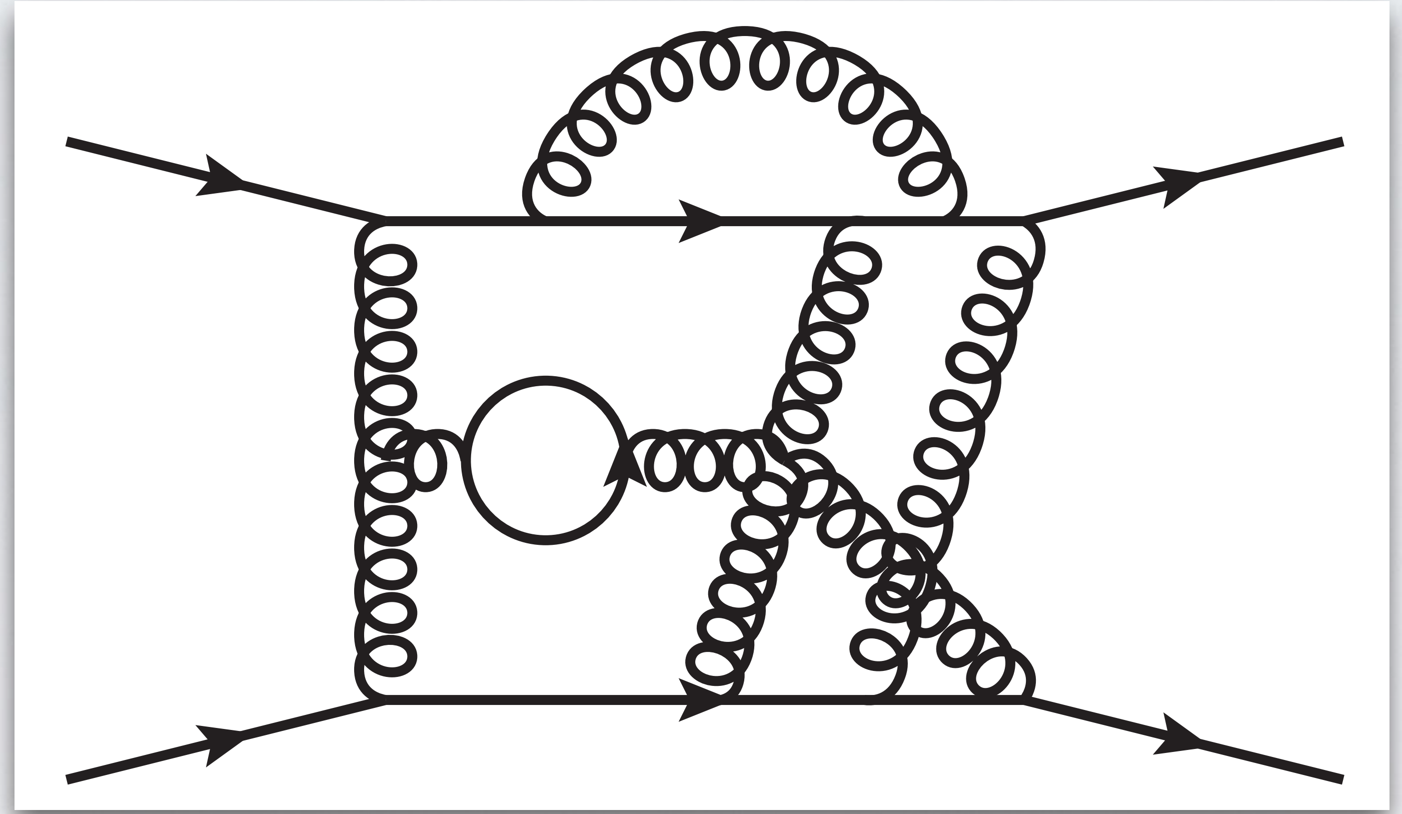
OUTLINE

- **Factorisation of amplitudes in the high-energy limit**
- **The two-Reggeon cut: scattering amplitudes by iterated solution of the BFKL equation**
- **Infrared singularities to all orders**
- **The finite wavefunction and amplitude**

  - ***In collaboration with***
    Simon Caron-Huot, Einan Gardi and Joscha Reichel
  - ***Based on***
    arXiv:1711.04850 (JHEP 1803 (2018) 098),
    and work in progress

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY



HIGH-ENERGY LIMIT

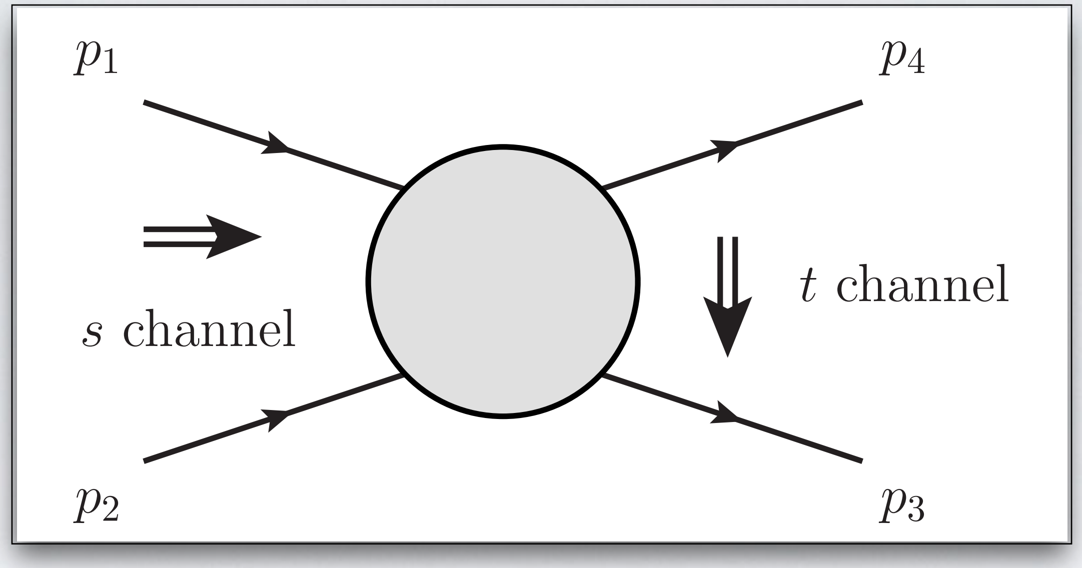
MRK in N=4 SYM: Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2016, …

- Very interesting theoretical problem:
  - retain rich dynamic in the 2D transverse plane,
  - toy model for full amplitude,
  - non-trivial function spaces,
  - predict amplitudes and other observables in overlapping limits:
    - → soft limit, infrared divergences.

- Relevant for phenomenology at the LHC and future colliders:
  - perturbative phenomenology of forward scattering, e.g.
    - → Deep inelastic scattering/saturation (small x = Regge, large Q^2 = perturbative),
    - → Mueller-Navelet: pp → X+2jets, forward and backward.

See e.g. Andersen, Smillie, 2011; Andersen, Medley Smillie, 2016; Andersen, Hapola, Maier, Smillie, 2017; …

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



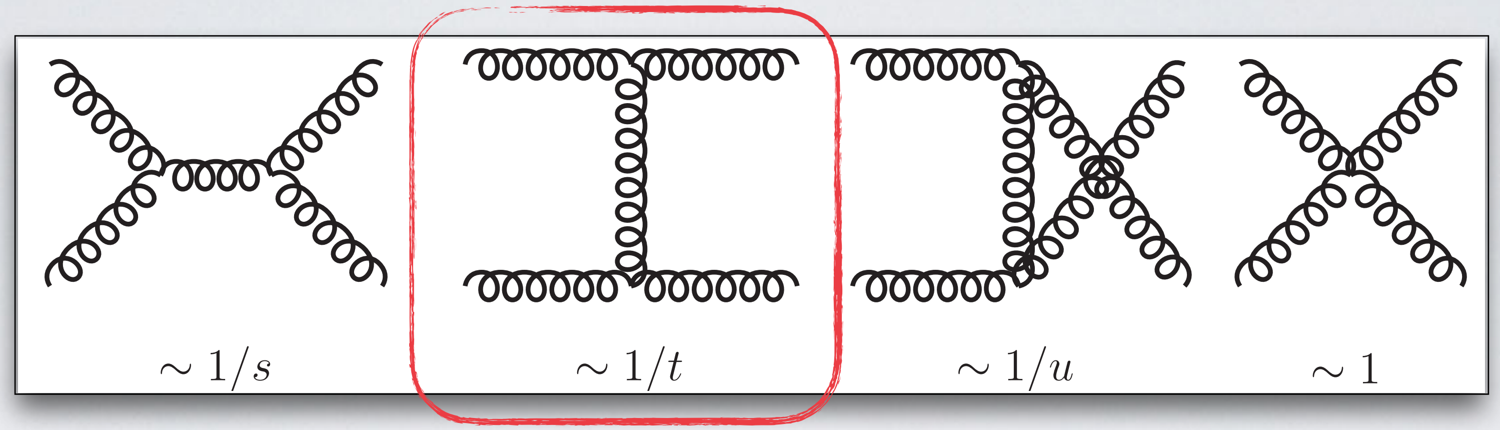
- Consider 2 → 2 scattering amplitudes in the high-energy limit:

$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0.$$

- The amplitude is expanded in the small ratio $|t/s|$; we consider here the leading power term:

$$\mathcal{M}_{ij\to ij}(s,t,\mu^2) = \frac{s}{t}\mathcal{M}^{[-1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \mathcal{M}^{[0]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \frac{t}{s}\mathcal{M}^{[1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \ldots$$

HIGH ENERGY LIMIT AT LL: REGGE POLES



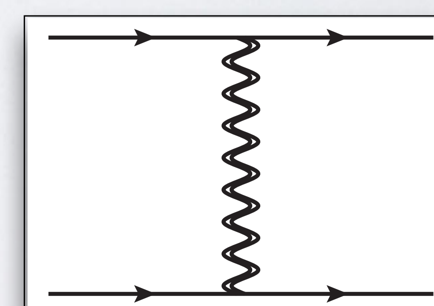
- At leading power, gluon exchanges in the t-channel:

$$\mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t}\,(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}\,.$$

- The amplitude contains logarithms of the ratio $|s/t|$.
  → Characterised in terms of Regge poles at LL:

Regge, Gribov

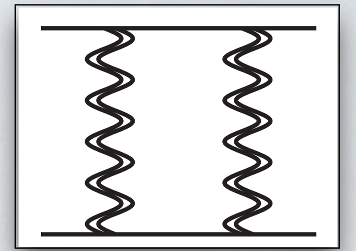
$$\mathcal{M}_{ij\to ij}|_{\rm LL} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}\,C_A\,\alpha_g^{(1)}(t)} 4\pi\alpha_s\,\mathcal{M}^{(0)}_{ij\to ij},$$



- The function $\alpha_g(t)$ is known as the Regge trajectory:

$$\alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \overset{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon}, \qquad r_\Gamma = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1-\frac{1}{2}\zeta_2\,\epsilon^2-\frac{7}{3}\zeta_3\,\epsilon^3+\ldots$$

BEYOND LL: REGGE CUTS



- Crossing symmetry $s \leftrightarrow u$:

  → project onto eigenstates of signature:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

  → Express amplitudes in terms of the signature-even combination of logs:

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

  → $M^{(+)}$ and $M^{(-)}$ are respectively imaginary and real.

- Color: beyond tree level

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\,\mathcal{M}^{[i]}(s,t).$$

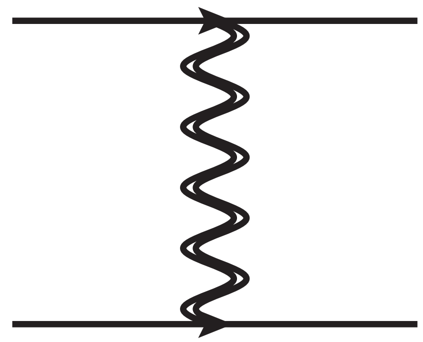
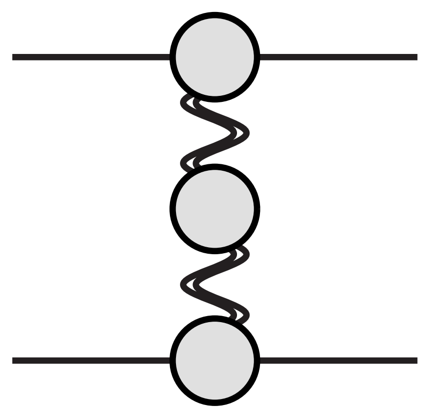
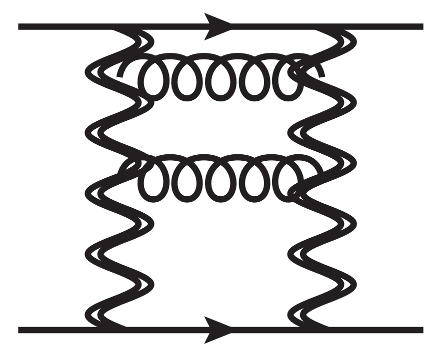
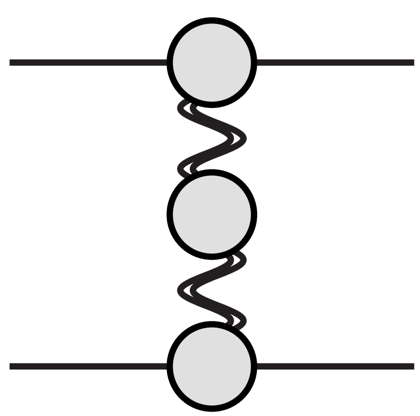
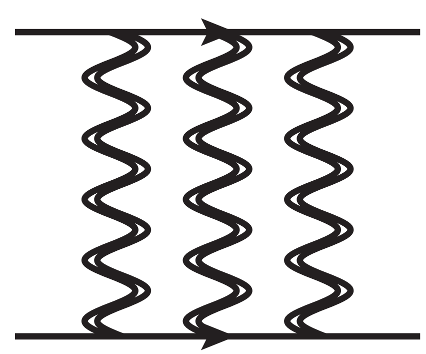
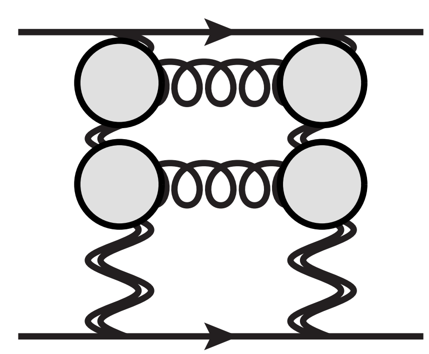
  → Decompose the amplitude in a color orthonormal basis in the t-channel

$$8 \otimes 8 = 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27$$

  → Invoking Bose symmetry we deduce (gg scattering)

$$\text{odd: } \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even: } \mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}.$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT

	Odd : $\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\bar{10}]}$	Even : $\mathcal{M}^{[1]}, \mathcal{M}^{[8_s]}, \mathcal{M}^{[27]}$
LL		
NLL		
NNLL	 + 	

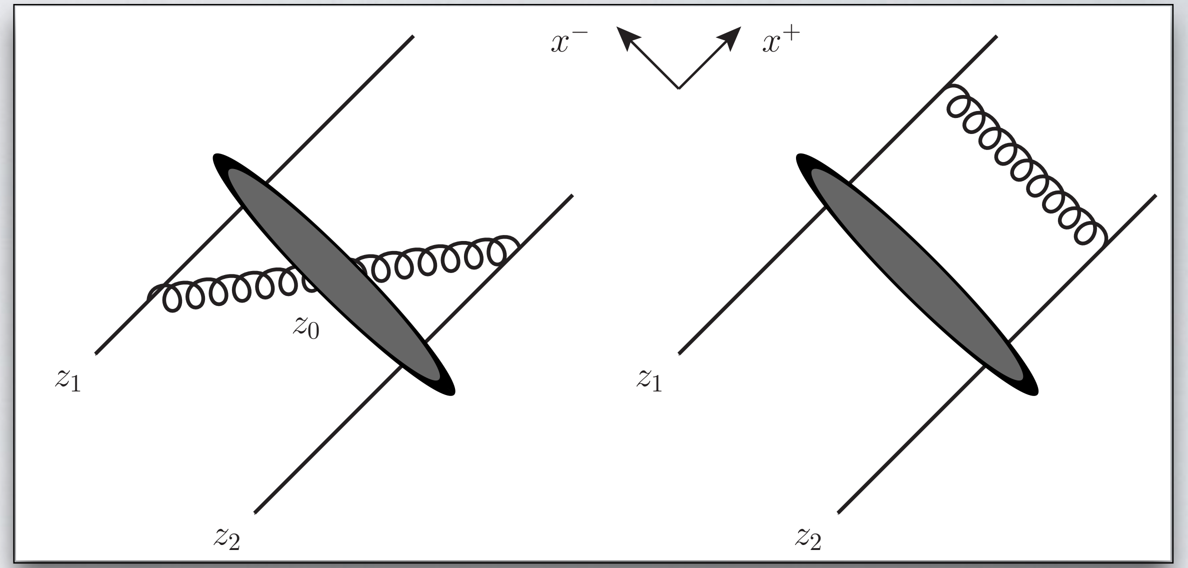
Caron-Huot, 2013; Del Duca, Falcioni, Magnea, LV, 2014, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- High-energy limit = forward scattering: to leading power, the fast projectile and target described in terms of Wilson lines:

$$U(z_\perp) = \mathcal{P}\exp\left[ig_s \int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+ T^a\right].$$

Korchemskaya, Korchemsky, 1994, 1996; Babansky, Balitsky, 2002, Caron-Huot, 2013



- The Wilson line stretches from $-\infty$ to $+\infty$ and thus develops rapidity divergencies. The regularised Wilson lines obeys the (non linear!) Balitsky-JIMWLK evolution equation:

$$-\frac{d}{d\eta}\Big[U(z_1)\ldots U(z_n)\Big] = \sum_{i,j=1}^{n} H_{ij}\cdot\Big[U(z_1)\ldots U(z_n)\Big],$$

with

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int [dz_i][dz_j][dz_0]K_{ij;0}\Big[T_{i,L}^a T_{j,L}^a + T_{i,R}^a T_{j,R}^a - U_{\rm ad}^{ab}(z_0)\left(T_{i,L}^a T_{j,R}^b + T_{j,L}^a T_{i,R}^b\right)\Big] + \mathcal{O}(\alpha_s^2).$$

- Evolution in rapidity resumes the high-energy log:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

FROM BALITSKY-JIMWLK TO AMPLITUDES

- In perturbation theory the unitary matrices $U(z) \sim \mathbb{1}$: parametrize in terms of a field W

$$U(z) = e^{i g_s \, T^a W^a(z)} \, .$$

Kovner Lublinsky, 2005; Caron-Huot, 2013

- The color-adjoint field W sources a BFKL Reggeised gluon: a generic projectile is expanded at weak coupling as

$$|\psi_i\rangle \equiv g_s \, D_{i,1}(t) \, |W\rangle + g_s^2 \, D_{i,2}(t) \, |WW\rangle + g_s^3 \, D_{i,3}(t) \, |WWW\rangle + \ldots$$

- Focus on the Regge-cut contributions: define a "reduced" amplitude by removing the Reggeized gluon and collinear divergences

$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\, \mathbf{T}_t^2 \, \alpha_g(t) \, L} \, \mathcal{M}_{ij\to ij} \, ,$$

- Scattering amplitude: expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s} \hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s} \left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij} \right) \equiv \langle \psi_j^{(+)} | e^{-\hat{H} L} | \psi_i^{(+)} \rangle + \langle \psi_j^{(-)} | e^{-\hat{H} L} | \psi_i^{(-)} \rangle .$$

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

FROM BALITSKY-JIMWLK TO AMPLITUDES

- An $m \to m+k$ transition from the leading-order Balitsky-JIMWLK equation is proportional to g_s^{2l+k}. Thus for $k \geq 0$, all the interactions can be extracted from the leading-order equation.

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

$$
H \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}
$$

$$
\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} .
$$

LO BFKL kernel

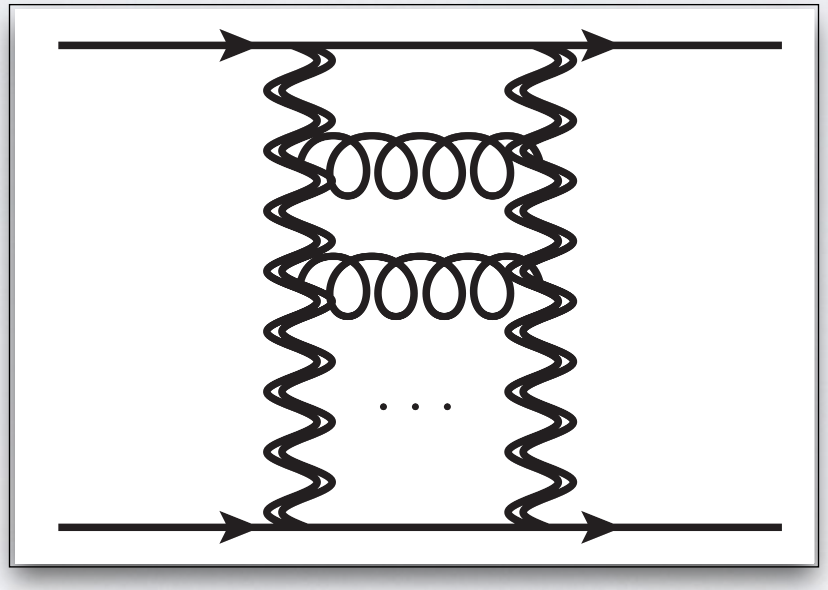
From LO B-JIMWLK

Terms in NNLO B-JIMWLK - predicted by symmetry $H = H^T$

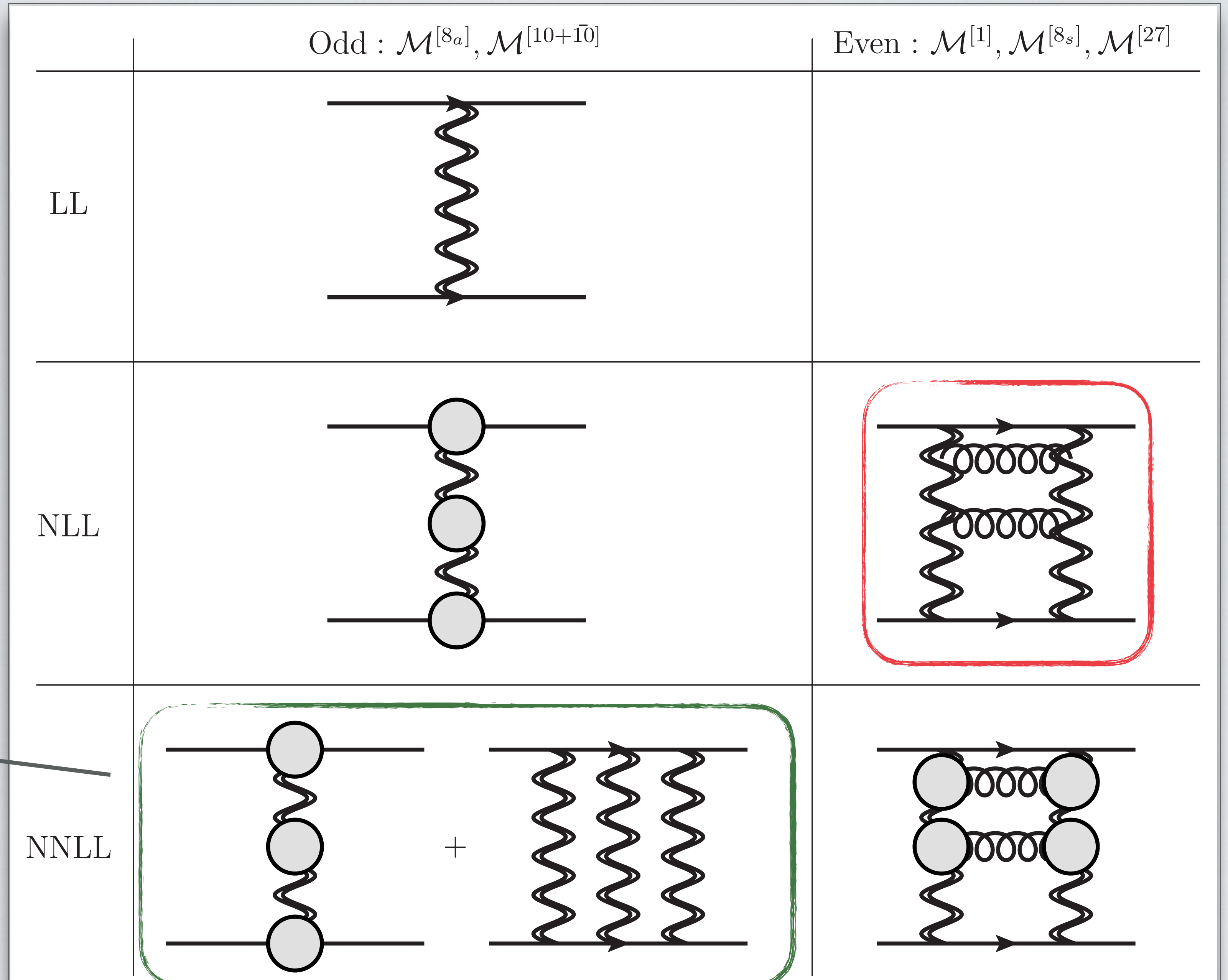
- Interactions with $k < 0$ are suppressed by at least $g_s^{2l+|k|}$, which means that they can first appear in the $(|k|+l)$-loop Balitsky-JIMWLK Hamiltonian.
- At NLL we need $m \to m$ transition only → the LO BFKL kernel.

"Reggeon field theory" still elusive; see Rothstein, Stewart 2016 for a SCET approach

THE TWO REGGEON CUT



THE 2-REGGEON CUT



See **Caron-Huot, Gardi, LV, 2017**

$$\frac{i}{2s}\hat{\mathcal{M}}^{(+),\mathrm{NLL}}_{ij\to ij} \equiv \langle \psi^{(+)}_{j,2} | e^{-\hat{H}L} | \psi^{(+)}_{i,2} \rangle .$$

THE TWO-REGGEON CUT

- The amplitude takes the form of an iterated integral over the BFKL kernel:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k] \, \frac{p^2}{k^2(k-p)^2} \, \Omega^{(\ell-1)}(p,k) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)}, \qquad B_0 = e^{\epsilon\gamma_{\rm E}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- One "rung" = apply once the BFKL kernel on the "target averaged wave function":

$$\Omega^{(\ell-1)}(p,k) = \hat{H} \, \Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2) \, \hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2) \, \hat{H}_{\rm m}$$

- "Integration" part:

$$\hat{H}_{\rm i} \, \Psi(p,k) = \int [{\rm D}k'] \, f(p,k,k') \, \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

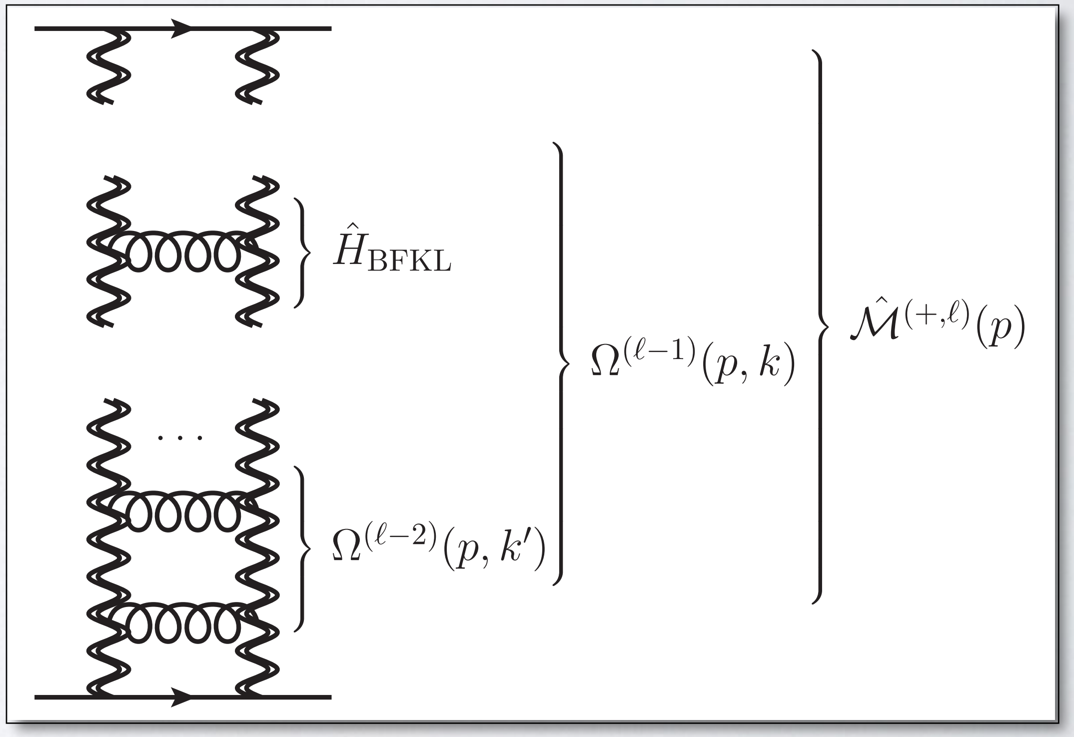
$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2}.$$

- "Multiplication" part:

$$\hat{H}_{\rm m} \Psi(p,k) = \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^{\epsilon} - \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon}\right] \Psi(p,k).$$

- Initial condition

$$\Omega^{(0)}(p,k) = 1.$$



THE 2-REGGEON CUT

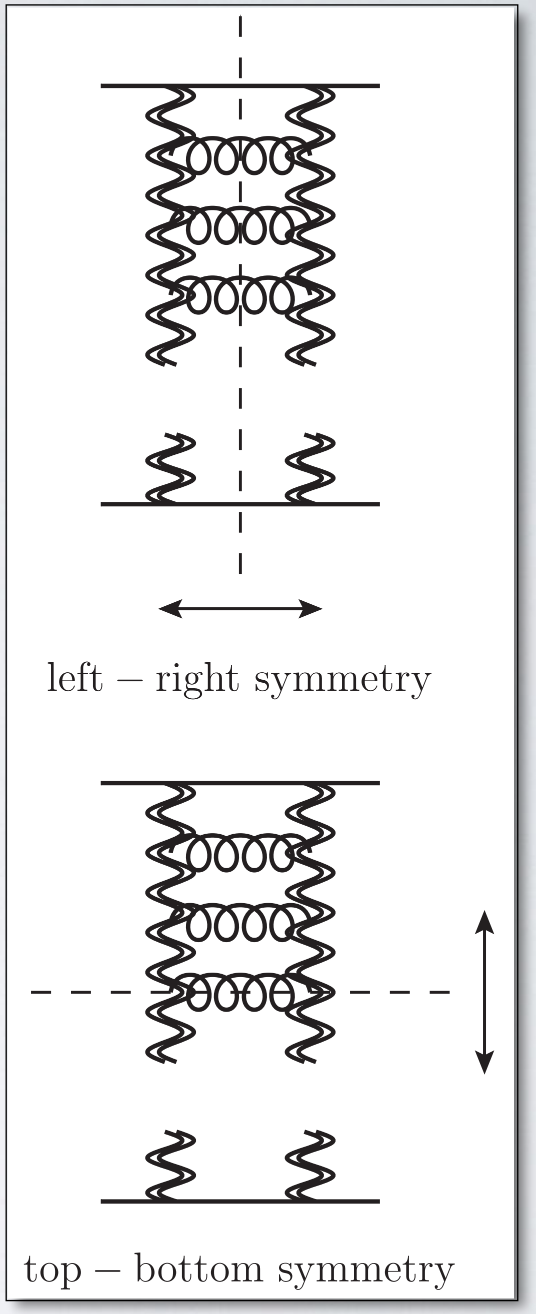
- Exact solution in the adjoint channel: $\Omega = I$.
- Cases where eigenfunctions are known:
  - → Color singlet dipoles,
  - → Color adjoint.

  Lipatov
- For $d \neq 2$/other color representations eigenfunctions are not known:
  - → Iterative solution.
- General features:
  - → "top-bottom" and "left-right" ladder symmetry;
  - → outermost rungs are always easy (multiplication);
  - → first non-trivial integration at 4-loops:

  Caron-Huot, 2013

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,4)} = i\pi \frac{(B_0)^4}{4!}\Big\{(C_A - \mathbf{T}_t^2)^3\left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) + C_A(C_A - \mathbf{T}_t^2)^2\left(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2)\right)\Big\}\,\mathbf{T}_{s-u}^2\, M^{(0)}.$$

- Integration in d=2-2ϵ involves Appell functions starting at 4 loops.
  - → How to predict higher orders?



THE 2-REGGEON CUT

- Observations:

  1) The wavefunction $\Omega(n)(p,k)$ is finite as $\epsilon \to 0$:

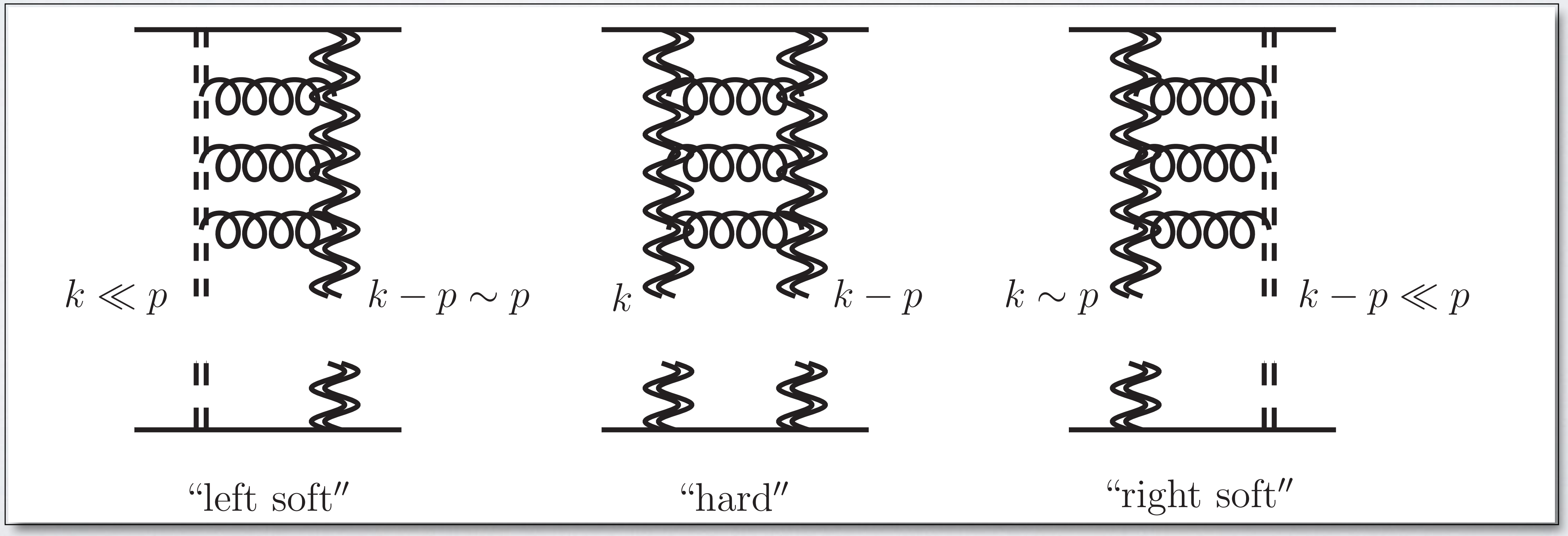
     → poles can only appear from final integration.

  2) Evolution closes in the soft limit:

$$\int_{k\to 0} \Omega^{(\ell)}(p,k).$$

     → IR divergences occur only when a full rail goes soft!

     → compute evolution in the (left) soft region and multiply by two.



2-REGGEON CUT: SOFT APPROXIMATION

- The soft function is polynomial in $(p^2/k^2)^\epsilon$:

Γ functions

$$\hat{H}_{\rm i}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon},$$

$$\hat{H}_{\rm m}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = \frac{1}{2\epsilon}\left[\left(\frac{p^2}{k^2}\right)^{n\epsilon} - \left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}\right].$$

- It is easy to compute to all orders, and integrate to $O(\epsilon^{-1})$: we get the reduced amplitude

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL}|_s = i\pi\,\frac{1}{(2\epsilon)^\ell}\,\frac{B_0^\ell(\epsilon)}{\ell!}\,(1+\hat{B}_{-1})\,(C_A-\mathbf{T}_t^2)^{\ell-1}\sum_{n=1}^{\ell}(-1)^{n+1}\binom{\ell}{n} \times\prod_{m=0}^{n-2}\left[1+\hat{B}_m(\epsilon)\frac{2C_A-\mathbf{T}_t^2}{C_A-\mathbf{T}_t^2}\right]\mathbf{T}_{s-u}^2\,\mathcal{M}^{(0)}+\mathcal{O}(\epsilon^0).$$

- The result is highly constrained, the wavefunction has to be finite!

  → The amplitude reduces to a geometric series, to $O(\epsilon^{-1})$:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL}|_s = i\pi\,\frac{1}{(2\epsilon)^\ell}\,\frac{B_0^\ell(\epsilon)}{\ell!}\,\left(1-R(\epsilon)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}(C_A-\mathbf{T}_t^2)^{\ell-1}\,\mathbf{T}_{s-u}^2\,\mathcal{M}^{(0)}+\mathcal{O}(\epsilon^0),$$

where

Caron-Huot, Gardi, Reichel, LV, 2017

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2+10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO REGGEON CUT: SOFT APPROXIMATION

- A few orders:

Caron-Huot, Gardi, Reichel, LV, 2017

$$\bar{\mathcal{M}}^{(+,1)}_{\rm NLL}|_s = i\pi\left[\frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,2)}_{\rm NLL}|_s = i\pi\frac{C_A - \mathbf{T}_t^2}{2!}\left[\frac{1}{(2\epsilon)^2} + \mathcal{O}(\epsilon^0)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,3)}_{\rm NLL}|_s = i\pi\frac{(C_A - \mathbf{T}_t^2)^2}{3!}\left[\frac{1}{(2\epsilon)^3} + \mathcal{O}(\epsilon^0)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

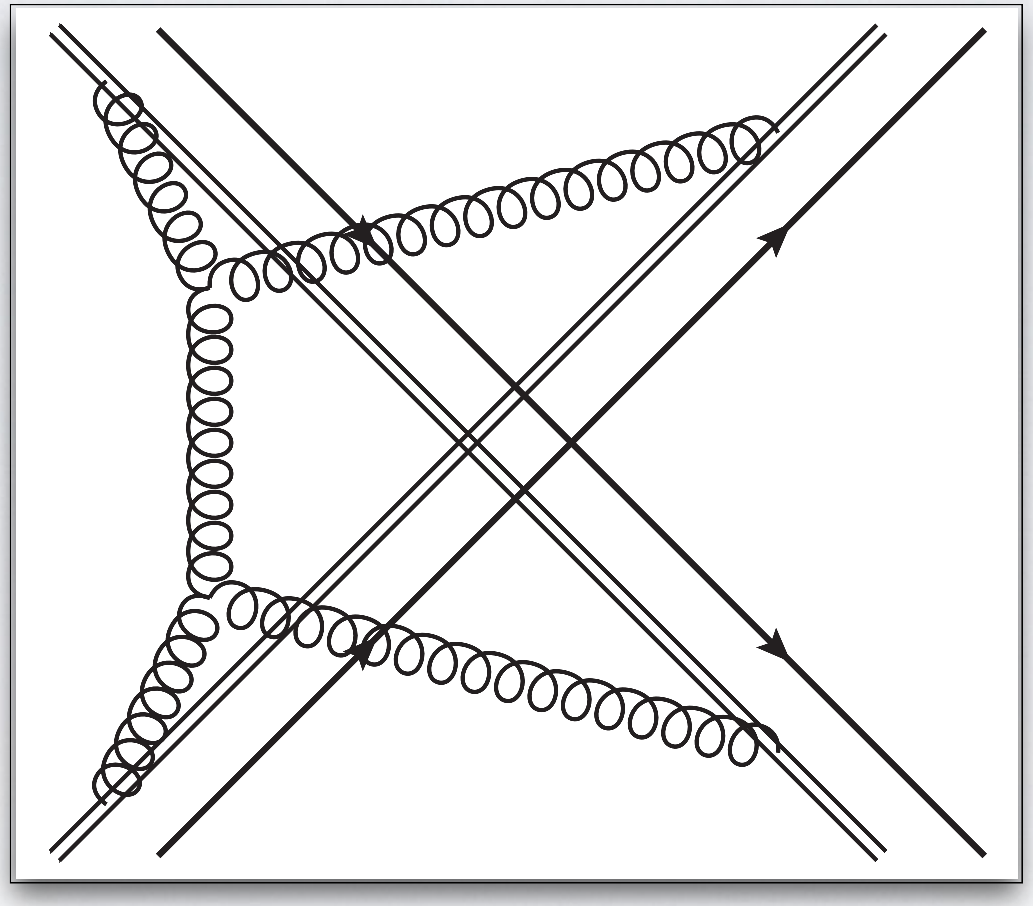
$$\bar{\mathcal{M}}^{(+,4)}_{\rm NLL}|_s = i\pi\frac{(C_A - \mathbf{T}_t^2)^3}{4!}\left[\frac{1}{(2\epsilon)^4} - \frac{1}{2\epsilon}\frac{\zeta_3 C_A}{4(C_A - \mathbf{T}_t^2)} + \mathcal{O}(\epsilon^0)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\bar{\mathcal{M}}^{(+,5)}_{\rm NLL}|_s = i\pi\frac{(C_A - \mathbf{T}_t^2)^4}{5!}\left[\frac{1}{(2\epsilon)^5} - \frac{1}{(2\epsilon)^2} - \frac{1}{2\epsilon}\frac{3\zeta_4 C_A}{16(C_A - \mathbf{T}_t^2)} + \mathcal{O}(\epsilon^0)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)}.$$

iteration of lower loops

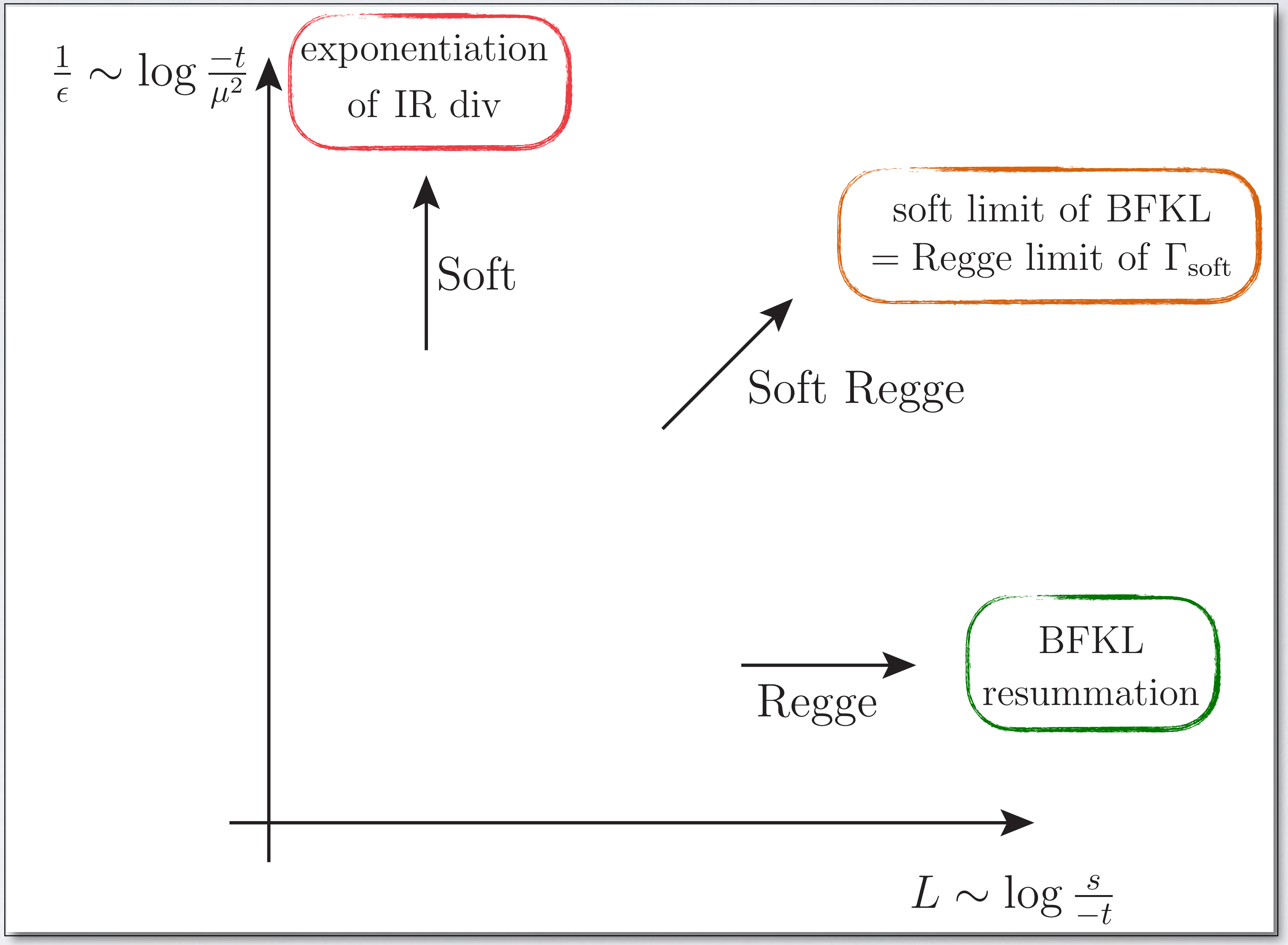
determines the soft anomalous dimension

2-REGGEON CUT: INFRARED SINGULARITIES



REGGE VS INFRARED FACTORISATION

- $2 \to 2$ kinematic limits:



- Application: test (and predict) the analytic structure of infrared divergences.

REGGE VS INFRARED FACTORISATION

- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\mathbf{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

Becher, Neubert, 2009; Gardi, Magnea, 2009

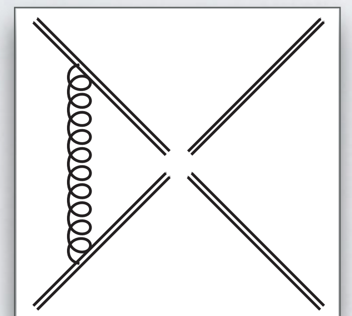
$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

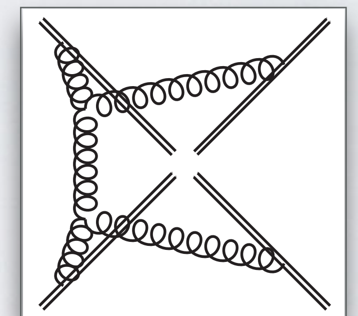
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

REGGE VS INFRARED FACTORISATION

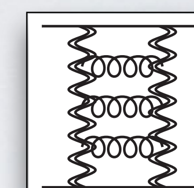
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$



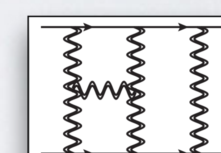
"dipole formula"



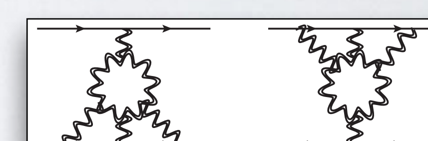
"quadrupole correction"



- Early studies of constraints from soft-collinear factorisation, collinear limits, and the high-energy limit in Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012;
- First evidence of "beyond dipole" contribution at four loops in Caron-Huot, 2013;



- Calculated a three loops in Almelid, Duhr, Gardi, 2015, 2016;
- Confirmed, in 2 ➔ 2 scattering in N=4 SYM in Henn, Mistlberger, 2016;



- Confirmed, in the high energy limit, in Caron-Huot, Gardi, LV, 2017;
- Re-derived based on a bootstrap approach in Almelid, Duhr, Gardi, McLeod, White, 2017.

2-REGGEON CUT: INFRARED SINGULARITIES

- Expand the soft anomalous dimension in the high-energy logarithm:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s(\lambda), L\right) + \ldots .$$

- At LL gluon Reggeization fixes $\mathbf{\Gamma}_{\rm LL}$ from gluon trajectory:

$$\mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda)\right) = \frac{\alpha_s(\lambda)}{\pi}\frac{\gamma_K^{(1)}}{2} L\, \mathbf{T}_t^2 = \frac{\alpha_s(\lambda)}{\pi} L\, \mathbf{T}_t^2 .$$

- At NLL

$$\mathbf{\Gamma}_{\rm NLL} = \mathbf{\Gamma}_{\rm NLL}^{(+)} + \mathbf{\Gamma}_{\rm NLL}^{(-)},$$

- with

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \frac{\alpha_s(\lambda)}{\pi}\sum_{i=1}^{2}\left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right) + \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\, \mathbf{T}_t^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi\frac{\alpha_s(\lambda)}{\pi}\mathbf{T}_{s-u}^2 + O(\alpha_s^4 L^3) .$$

Del Duca, Duhr, Gardi, Magnea, White, 2011

2-REGGEON CUT: INFRARED SINGULARITIES

- Derive an Infrared-factorised representation of the reduced amplitude: start from

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \exp\left\{ - \frac{\alpha_s(\mu)}{\pi} \frac{B_0(\epsilon)}{2\epsilon} L \mathbf{T}_t^2 \right\} \left[\mathbf{Z}^{(-)}_{\rm NLL}\left(\frac{s}{t}, \mu, \alpha_s(\mu)\right) \mathcal{H}^{(-)}_{\rm LL}\left(\{p_i\}, \mu, \alpha_s(\mu)\right) + \mathbf{Z}^{(+)}_{\rm LL}\left(\frac{s}{t}, \mu, \alpha_s(\mu)\right) \mathcal{H}^{(+)}_{\rm NLL}\left(\{p_i\}, \mu, \alpha_s(\mu)\right) \right],$$

No poles

- we obtain

$$\exp\left\{ \frac{1 - B_0(\epsilon)}{2\epsilon} \frac{\alpha_s}{\pi} L (C_A - \mathbf{T}_t^2) \right\} \hat{\mathcal{M}}_{\rm NLL} = - \int_0^p \frac{d\lambda}{\lambda} \exp\left\{ \frac{1}{2\epsilon} \frac{\alpha_s(p)}{\pi} L (C_A - \mathbf{T}_t^2) \left[1 - \left(\frac{p}{\lambda}\right)^{\epsilon} \right] \right\} \mathbf{\Gamma}^{(-)}_{\rm NLL}\left(\alpha_s(\lambda)\right) \mathcal{M}^{(\rm tree)} + \mathcal{O}(\epsilon^0).$$

- By matching we get the soft anomalous dimension to all orders:

$$\mathbf{\Gamma}^{(-,\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!} \left(1 - R\left(\frac{x}{2}(C_A - \mathbf{T}_t^2)\right) \frac{C_A}{C_A - \mathbf{T}_t^2} \right)^{-1} \Bigg|_{x^{\ell-1}} \mathbf{T}^2_{s-u},$$

with

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \ldots.$$

Caron-Huot, Gardi, Reichel, LV, 2017

2-REGGEON CUT: INFRARED SINGULARITIES

- Explicitly, for the first few orders we have:

$$\mathbf{\Gamma}_{\rm NLL}^{(-,1)} = i\pi\, \mathbf{T}_{s-u}^2, \qquad \mathbf{\Gamma}_{\rm NLL}^{(-,2)} = 0, \qquad \mathbf{\Gamma}_{\rm NLL}^{(-,3)} = 0,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,4)} = -i\pi\, \frac{\zeta_3}{24}\, C_A(C_A - \mathbf{T}_t^2)^2\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,5)} = -i\pi\, \frac{\zeta_4}{128}\, C_A(C_A - \mathbf{T}_t^2)^3\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,6)} = -i\pi\, \frac{\zeta_5}{640}\, C_A(C_A - \mathbf{T}_t^2)^4\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,7)} = i\pi \frac{1}{720}\left[\frac{\zeta_3^2}{16}\, C_A^2(C_A - \mathbf{T}_t^2)^4 + \frac{1}{32}\left(\zeta_3^2 - 5\zeta_6\right)\, C_A(C_A - \mathbf{T}_t^2)^5\right]\mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,8)} = i\pi \frac{1}{5040}\left[\frac{3\zeta_3\zeta_4}{32}\, C_A^2(C_A - \mathbf{T}_t^2)^5 + \frac{3}{64}\left(\zeta_3\zeta_4 - 3\zeta_7\right)\, C_A(C_A - \mathbf{T}_t^2)^6\right]\mathbf{T}_{s-u}^2.$$

Caron-Huot, Gardi, Reichel, LV, 2017

- The result can be used as constraint in a bootstrap approach to the soft anomalous dimension.

See e.g. Almelid, Duhr, Gardi, McLeod, White, 2017

2-REGGEON CUT: INFRARED SINGULARITIES

- Write the soft anomalous dimension as a function $x = L\,\alpha_s/\pi$:

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi \frac{\alpha_s}{\pi} G\left(\frac{\alpha_s}{\pi} L\right) \mathbf{T}_{s-u}^2\,, \qquad G(x) = \sum_{\ell=1}^{\infty} x^{\ell-1} G^{(\ell)}\,.$$

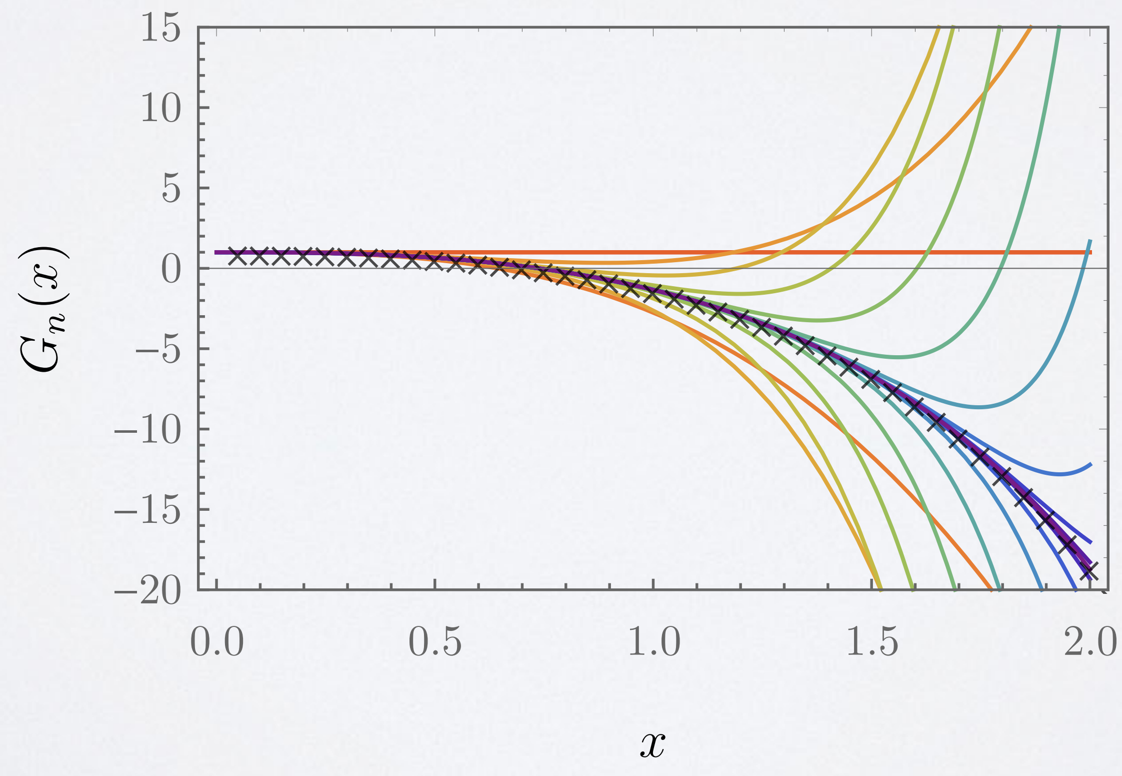
- Write $G(x)$ as the Borel transform of some function $g(1/\eta)$:

$$G(x) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} d\eta\, g\left(\frac{1}{\eta}\right) \mathrm{e}^{\eta x}\,,$$

$g(1/\eta)$ has isolated singularities away from the origin:

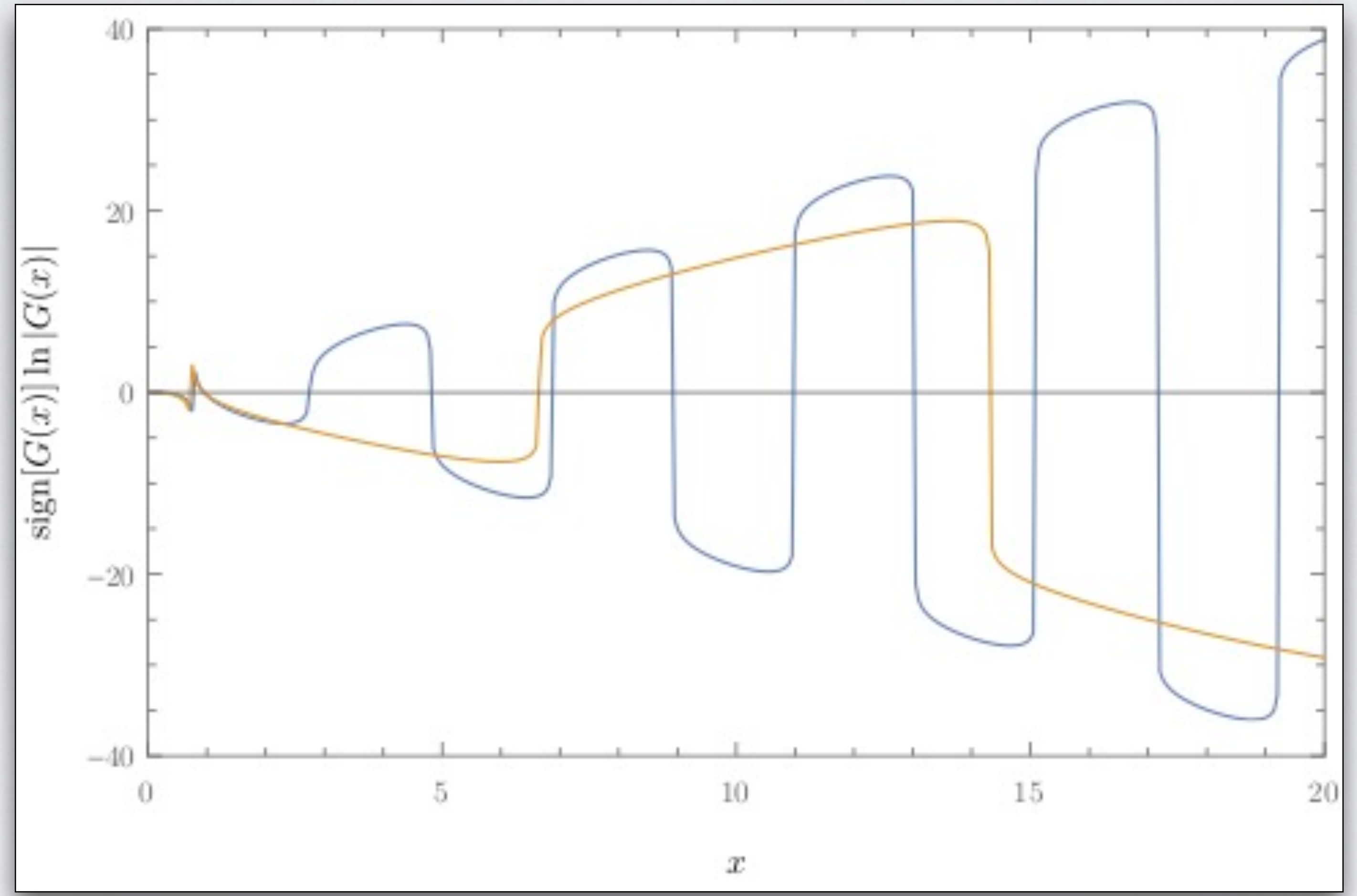
→ $G(x)$ has an infinite radius of convergence.

→ it is an entire function, free of any singularities for any finite x.



2-REGGEON CUT: INFRARED SINGULARITIES

- Plotting $G(x)$ for larger values of x reveals oscillations with a constant period and an exponentially growing amplitude.
- Here we plot the logarithm of $|G(x)|$ weighted by the sign of $G(x)$:



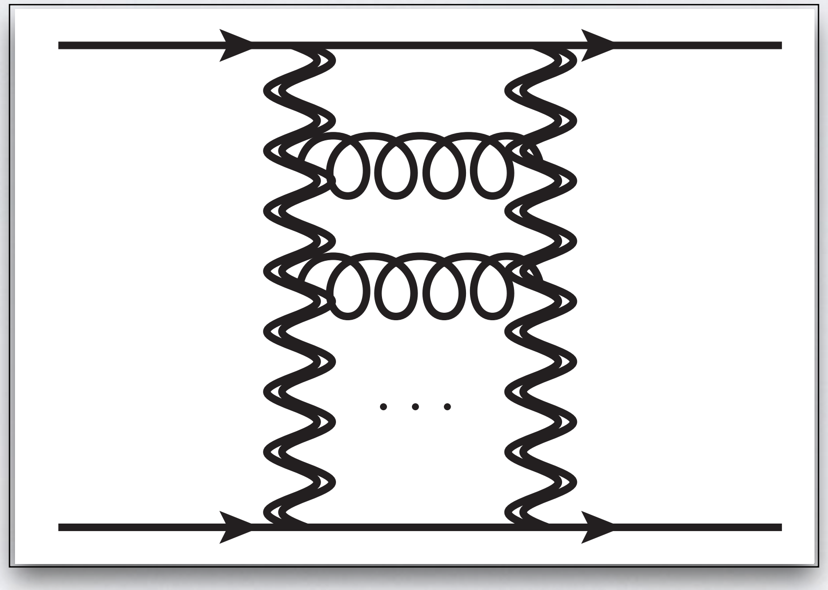
Caron-Huot, Gardi, Reichel, LV, 2017

- The function is well approximated by

$$G(x) \to c\, e^{ax} \cos\left(bx + d\right) \;,$$

	a	b	c	d
1	1.97	1.52	0.25	0.48
27	1.46	0.41	0.58	2.01

FINITE WAVEFUNCTION AND AMPLITUDE

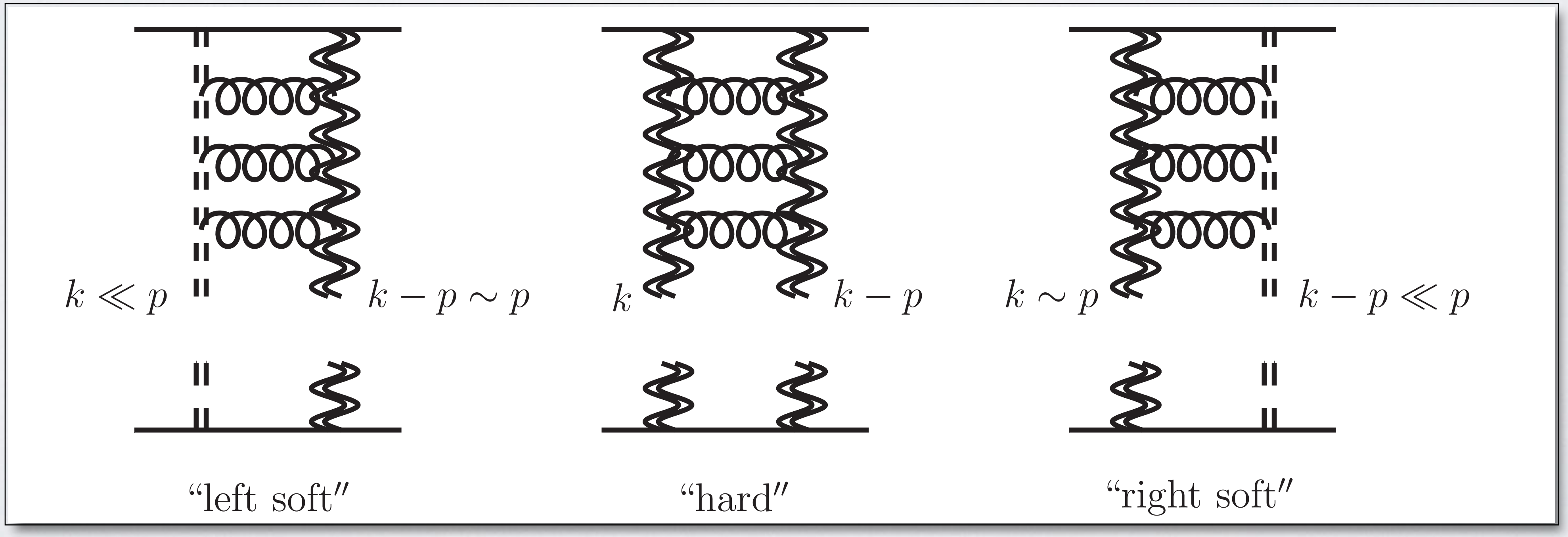


FINITE WAVEFUNCTION AND AMPLITUDE

- What about the finite part?
  - → Claim: $\epsilon \to 0$ limit determined from evolution with $\epsilon = 0$.

$$\mathcal{H}^{(+)}_{\mathrm{NLL}} = \underbrace{\int_{k\ \mathrm{soft}} d^{2-2\epsilon}k\,\Omega(p,k) - (\mathrm{subtractions})}_{\substack{\text{computable using soft limit} \\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\int_{k\ \mathrm{hard}} d^2k\,\Omega(p,k)\Big|_{\epsilon=0}}_{\text{wavefunction in } D=2}.$$

Recall: the wavefunction is finite, singularities are generated upon the last integration for $k \to 0$.



WAVEFUNCTION IN D=2

- Introduce complex variables

$$\frac{k}{p} = \frac{z}{z-1}, \qquad \frac{k'}{p} = \frac{w}{w-1}.$$

- BFKL kernel in D=2:

Caron-Huot, Gardi, Reichel, LV, in progress

$$\hat{H}_{\rm 2d} = (2C_A - \mathbf{T}_t^2)\hat{H}_{\rm 2d,i} + (C_A - \mathbf{T}_t^2)\hat{H}_{\rm 2d,m}$$

- "Integration" part:

$$\hat{H}_{\rm 2d,i} = \frac{1}{4\pi}\int d^2w\, K(w,\bar{w},z,\bar{z})\Big[\Psi(w,\bar{w}) - \Psi(z,\bar{z})\Big],$$

$$K(w,\bar{w},z,\bar{z}) = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}.$$

- "Multiplication" part:

$$\hat{H}_{\rm 2d,m} = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right]\Psi(z,\bar{z}).$$

WAVEFUNCTION IN D=2

- Translate the action of the BFKL kernel into a set of differential equations, thanks to

$$z\frac{d}{dz}\left[\hat{H}_{2\mathrm{d,i}}\Psi(z,\bar{z})\right] = \hat{H}_{2\mathrm{d,i}}\left[z\frac{d}{dz}\Psi(z,\bar{z})\right].$$

- The full algorithms requires taking care of contact terms,

$$\partial_z\partial_{\bar{z}}\,\log(z\bar{z}) = \pi\,\delta^2(z),$$

- and to considering the action of (1-z)d/dz as well.
- The 2D wavefunction is expressed as function of SVHPLs, e.g.

Brown, 2004, 2013, Schnetz, 2013

Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2016, …

$$\begin{aligned}
\Omega^{(1)}_{2\mathrm{d}} &= \frac{1}{2}C_2\left(\mathcal{L}_0+2\mathcal{L}_1\right)\\
\Omega^{(2)}_{2\mathrm{d}} &= \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0}+2\mathcal{L}_{0,1}+2\mathcal{L}_{1,0}+4\mathcal{L}_{1,1}\right)+\frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1}-\mathcal{L}_{1,0}-2\mathcal{L}_{1,1}\right)\\
\Omega^{(3)}_{2\mathrm{d}} &= \frac{1}{4}C_1C_2^2\left(-2\mathcal{L}_{0,0,1}-3\mathcal{L}_{0,1,0}-7\mathcal{L}_{0,1,1}-2\mathcal{L}_{1,0,0}-7\mathcal{L}_{1,0,1}-7\mathcal{L}_{1,1,0}\right.\\
&\quad\left.-14\mathcal{L}_{1,1,1}+2\zeta_3\right)+\frac{3}{4}C_2^3\left(\mathcal{L}_{0,0,0}+2\mathcal{L}_{0,0,1}+2\mathcal{L}_{0,1,0}+4\mathcal{L}_{0,1,1}+2\mathcal{L}_{1,0,0}\right.\\
&\quad\left.+4\mathcal{L}_{1,0,1}+4\mathcal{L}_{1,1,0}+8\mathcal{L}_{1,1,1}\right)+\frac{1}{16}C_1^2C_2\left(\mathcal{L}_{0,0,1}+2\mathcal{L}_{0,1,0}+4\mathcal{L}_{0,1,1}\right.\\
&\quad\left.+\mathcal{L}_{1,0,0}+4\mathcal{L}_{1,0,1}+4\mathcal{L}_{1,1,0}+8\mathcal{L}_{1,1,1}\right),
\end{aligned}$$

where ($C_1=2C_A-T_t^2$, $C_2 = C_A-T_t^2$) and, e.g.,

$$\mathcal{L}_{0,0,1,1}(z,\bar{z}) = H_{0,0,1,1}(z)+H_{1,1,0,0}(\bar{z})+H_{0,0,1}(z)H_1(\bar{z})+H_0(z)H_{1,1,0}(\bar{z})+H_{0,0}(z)H_{1,1}(\bar{z})-2\zeta_3H_1(\bar{z}).$$

FINITE AMPLITUDE

$$\mathcal{H}^{(+)}_{\rm NLL} = \underbrace{\int_{k\ \rm soft} d^{2-2\epsilon}k\,\Omega(p,k) - (\text{subtractions})}_{\substack{\text{computable using soft limit}\\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\int_{k\ \rm hard} d^2k\,\Omega(p,k)\Big|_{\epsilon=0}}_{\text{wavefunction in } D=2}.$$

- Two methods to perform the last integration, and sum consistently soft and hard region.

$$\hat{\mathcal{M}}^{(1)}|_{\epsilon^0} = 0, \qquad \hat{\mathcal{M}}^{(2)}|_{\epsilon^0} = 0,$$

$$\hat{\mathcal{M}}^{(3)}|_{\epsilon^0} = -i\pi\frac{(B_0)^3}{2!}\left[C_2^2\left(-\frac{11}{4}\zeta_3\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(4)}|_{\epsilon^0} = -i\pi\frac{(B_0)^4}{3!}\left[C_1C_2^2\left(-\frac{3}{16}\zeta_4\right)+C_2^3\left(\frac{3}{16}\zeta_4\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(5)}|_{\epsilon^0} = -i\pi\frac{(B_0)^5}{4!}\left[C_2^4\left(-\frac{717}{16}\zeta_5\right)+C_1C_2^3\left(\frac{333}{16}\zeta_5\right)+C_1^2C_2^2\left(-\frac{5}{2}\zeta_5\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(6)}|_{\epsilon^0} = -i\pi\frac{(B_0)^6}{5!}\left[C_2^5\left(-\frac{2879}{32}\zeta_3^2+\frac{5}{32}\zeta_6\right)+C_1C_2^4\left(\frac{2637}{32}\zeta_3^2-\frac{5}{32}\zeta_6\right)\right.$$
$$\left.+C_1^2C_2^3\left(-\frac{399}{16}\zeta_3^2\right)+C_1^3C_2^2\left(\frac{39}{16}\zeta_3^2\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\ldots$$

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FINITE AMPLITUDE

$$\hat{\mathcal{M}}^{(11)}|_{\epsilon^0} = -i\pi\frac{(B_0)^{11}}{10!}\Bigg[C_2^{10}\bigg(-\frac{958003197}{512}\zeta_3^2\zeta_5 + \frac{15}{512}\zeta_5\zeta_6 + \frac{27}{1024}\zeta_4\zeta_7 + \frac{147}{4096}\zeta_3\zeta_8 - \frac{7424524707}{1024}\zeta_{11}\bigg)$$
$$+ C_1C_2^9\bigg(\frac{2676706539}{512}\zeta_3^2\zeta_5 - \frac{45}{512}\zeta_5\zeta_6 - \frac{81}{1024}\zeta_4\zeta_7 - \frac{525}{4096}\zeta_3\zeta_8 + \frac{19105018467}{1024}\zeta_{11}\bigg)$$
$$+ C_1^2C_2^8\bigg(-\frac{808582491}{128}\zeta_3^2\zeta_5 + \frac{2744295}{256}\zeta_5\zeta_6 - \frac{6586245}{512}\zeta_4\zeta_7 + \frac{63}{512}\zeta_3\zeta_8$$
$$- \frac{3087315}{16}\zeta_2\zeta_9 - \frac{10685853783}{512}\zeta_{11} - \frac{68607}{16}\zeta_{5,3,3}\bigg) + \ldots\Bigg]\mathbf{T}_{s-u}^2\mathcal{M}^{(0)}.$$

- Hard regions contributes only with odd ζ_n, consistent with 2D wavefunction made of SVHPLs.
- Finite (hard) amplitude contains ζ_{533} at 11 loops: no exponentiation in terms of Γ functions.

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FINITE AMPLITUDE

PRELIMINARY

- Series seems alternating;
- Coefficient grows exponentially: finite radius of convergence in $(\alpha_s/\pi\, L)$: define

$$\mathcal{M}^{(n)}|_{\epsilon^0} = -i\pi\, \mathcal{G}^{(n)}_{\text{color}}\, \mathbf{T}^2_{s-u} \mathcal{M}^{(0)}, \qquad \Delta^{(n)}_{\text{color}}\left(\frac{\alpha_s}{\pi}L\right) = \sum_{i=1}^{n} \left(\frac{\alpha_s}{\pi}L\right)^i \mathcal{G}^{(i)}_{\text{color}}.$$

then

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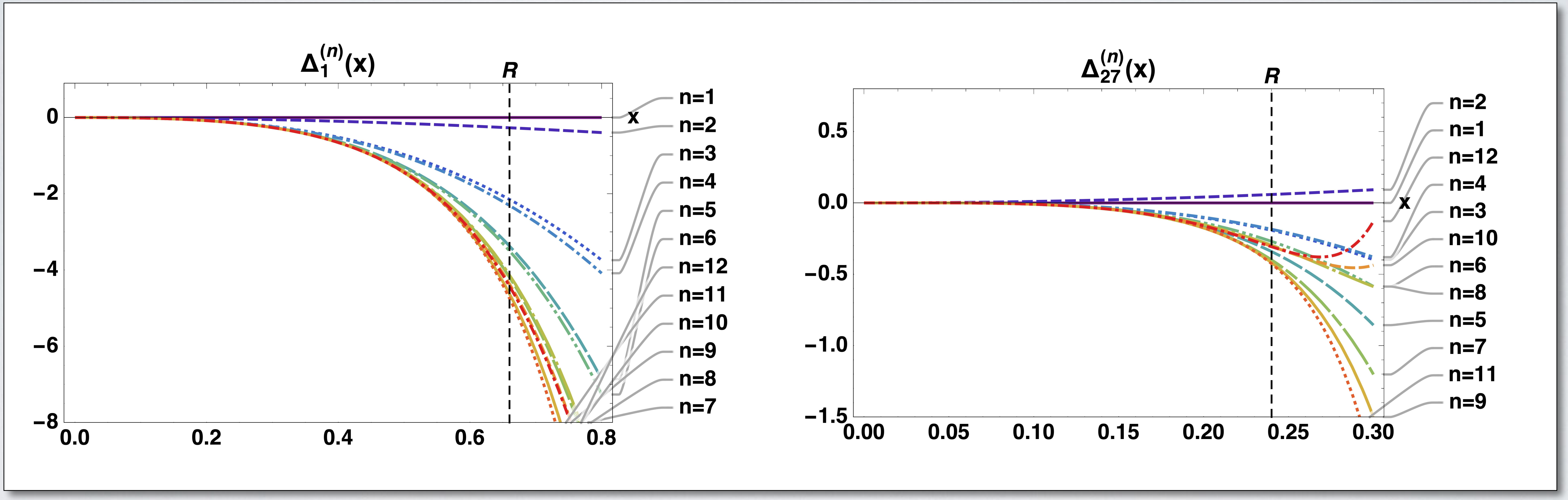

- Radius of convergence stabilises to $|R| \simeq 0.66$ for singlet, $|R| \simeq 0.24$ for 27 representation after a few orders, by means of a Padé approximant (pole at $-|R|$).

See also Larkoski, Moult, Neill, 2016 for a similar analysis in the context of NGL.

CONCLUSION

- Modern approach to high-energy scattering via Wilson lines:
  → new theoretical control up to NNLL.
- 2 → 2 amplitude at NLL obtained by iteration of the BFKL kernel.
- Solved to all orders in the soft limit: gives the soft anomalous dimension in the high-energy limit to all orders.
- Ongoing studies of the finite amplitude:
  → wavefunction expressed in terms of SVHPLs, analytic expression to 12 loops.
  → amplitude expressed in terms of ζ_n values, up to 12 loops.
  → no exponentiation in terms of Γ functions only.
  → finite radius of convergence.