Recent Development of the Quasi-PDF Approach to Calculate Parton Physics in Lattice QCD

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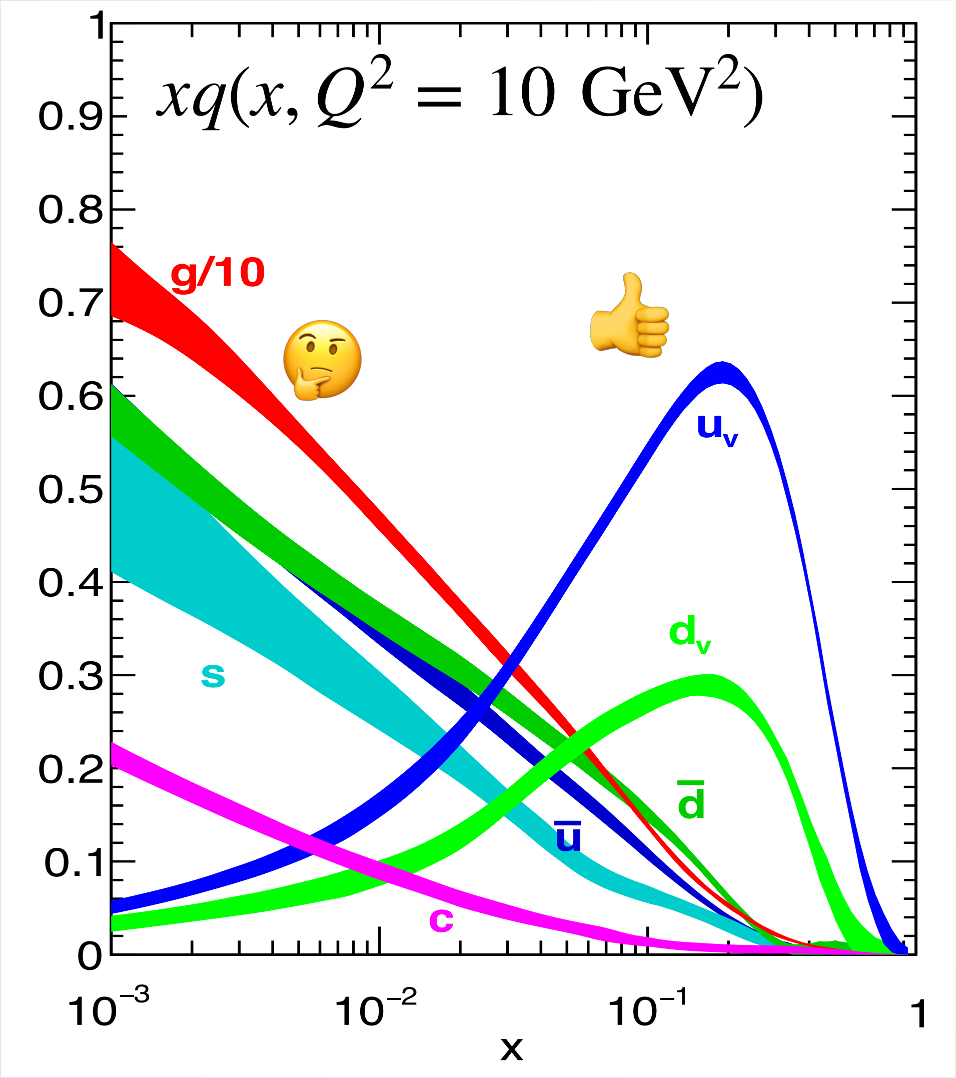
Outline

- Quasi-PDF approach
  - Physical picture and factorization formula
  - Systematic procedure to calculate parton distributions
- Quasi-TMDPDF
  - Relation of the quasi-TMDPDF and physical TMDPDF
  - Collins-Soper Kernel from lattice QCD

So far our knowledge of the PDFs mostly comes from the analysis of high-energy scattering data

Unpolarized PDF



NNPDF 3.1, EPJ C77 (2017)

Gluon PDF is key to the Standard Model predictions at LHC.

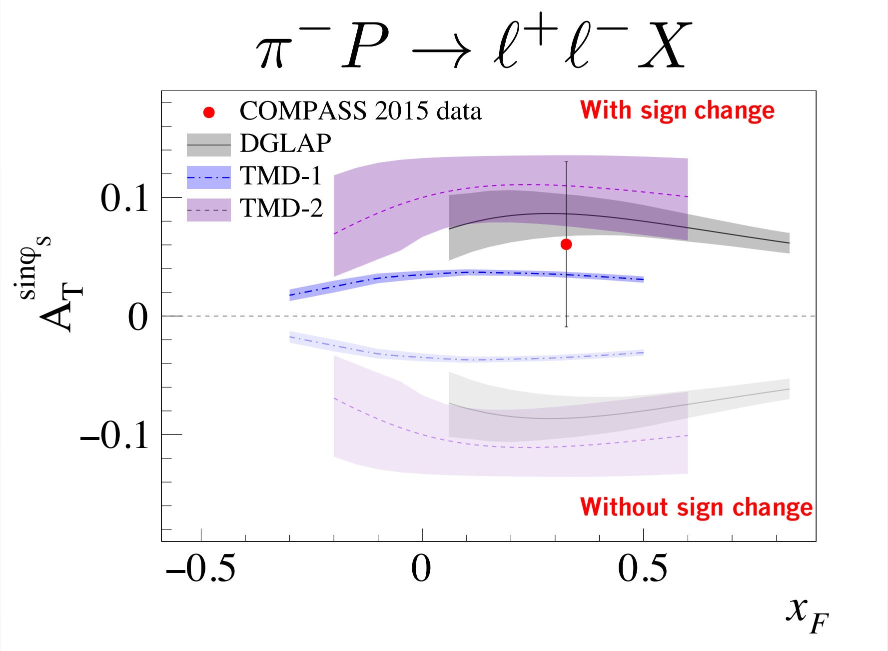
TMD PDF

Existing global analyses of TMDPDFs or TMD fragmentation functions rely on the modeling of their nonperturbative evolution.

- Kang, Prokudin, Sun and Yuan, PRD93 (2016);
- Bacchetta et al., JHEP1706 (2017);
- Bertone, Scimemi and Vladimirov, arXiv:1902.08474.

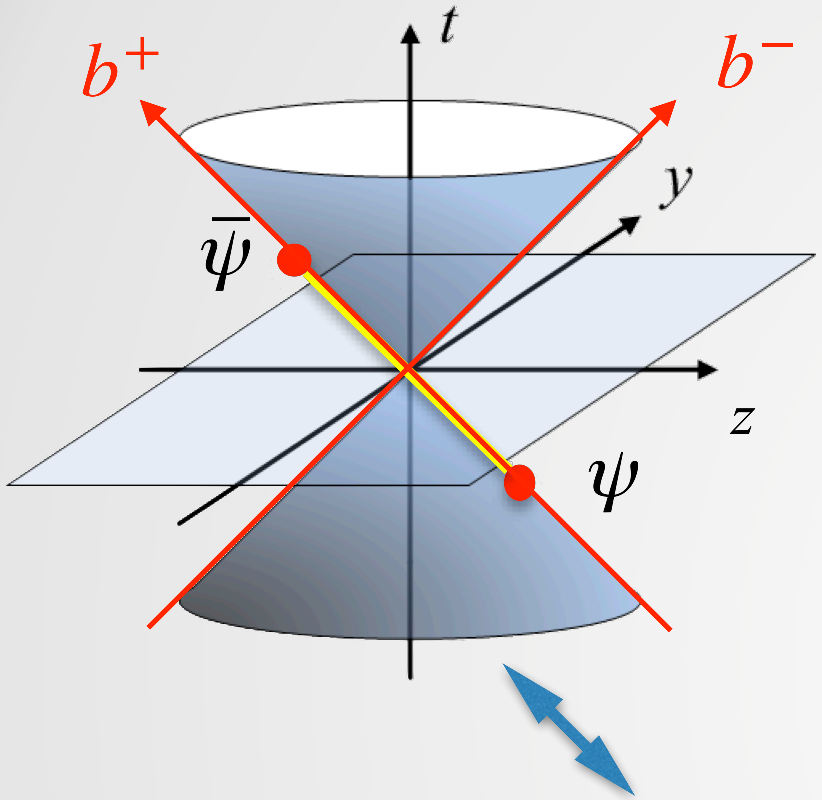
The most definite experimental finding so far is the sign change of the Sivers function in SIDIS and Drell-Yan processes.

COMPASS Phys.Rev.Lett. 119 (2017) 12002



See also STAR Collaboration, PRL116 (2016).

Lattice QCD calculation of PDFs?

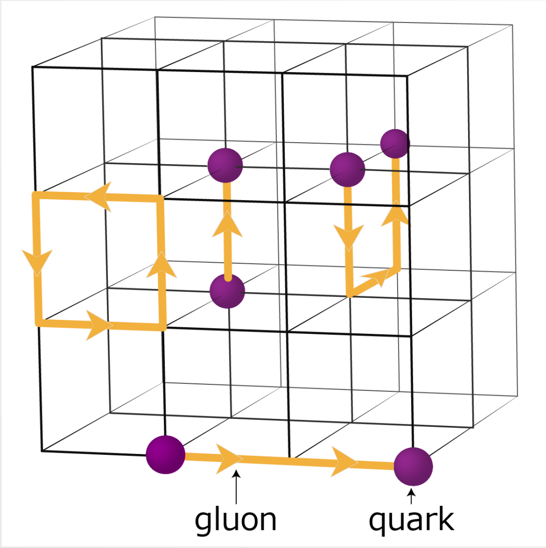


PDF:

$$q(x,\mu) = \int \frac{db^+}{4\pi} e^{-i\frac{1}{2}b^+(xP^-)} \langle P | \bar{\psi}(b^+) \frac{\gamma^-}{2} W[b^+,0] \psi(0) | P \rangle$$

$$b^\pm = t \mp z$$

- Minkowski space, real time;
- Defined on the light-cone which depends on the real time.



Lattice QCD:

$$t = i\tau, \quad e^{iS} \to e^{-S}, \quad \langle O \rangle = \int D\psi D\bar{\psi} DA \; O(x) e^{-S}$$

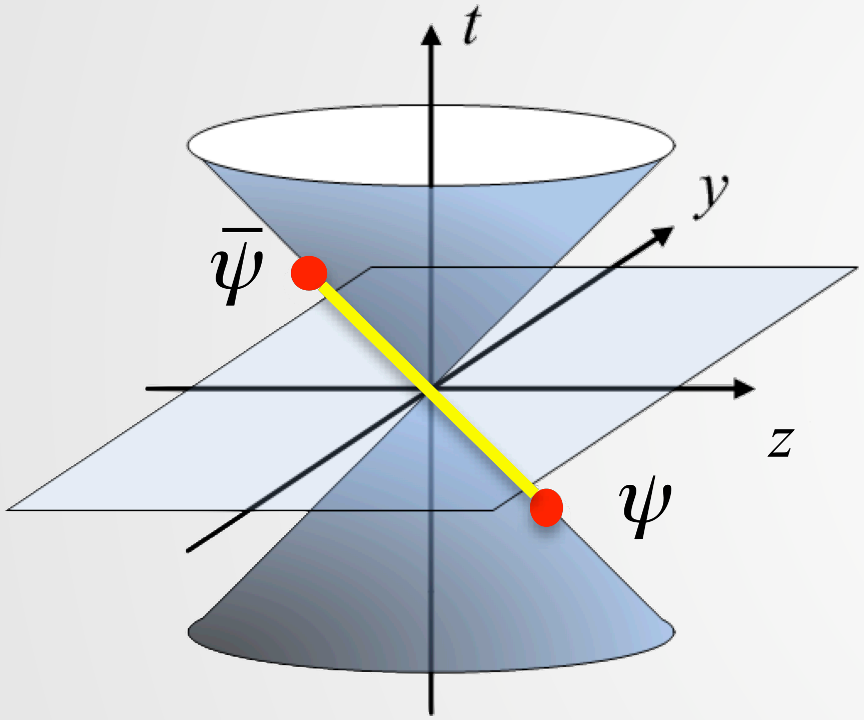
- Euclidean space, imaginary time;
- General difficulty of analytically continuing to real time.

Light-cone PDFs not directly accessible from the lattice!

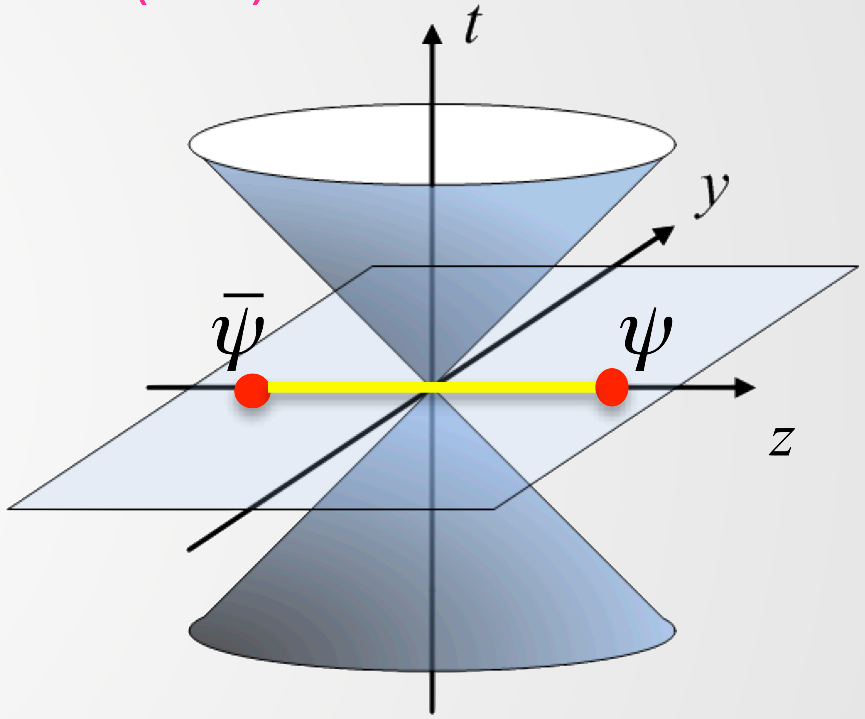
A novel approach to calculate light-cone PDFs

- Large-Momentum Effective Theory:

- Ji, PRL110 (2013);
- X. Ji, J.-H. Zhang, and Y.Z., PRL111 (2013);
- Ji, SCPMA57 (2014).



PDF $q(x)$:
Cannot be calculated on the lattice



Quasi-PDF $\tilde{q}(x, P^z)$:
Directly calculable on the lattice

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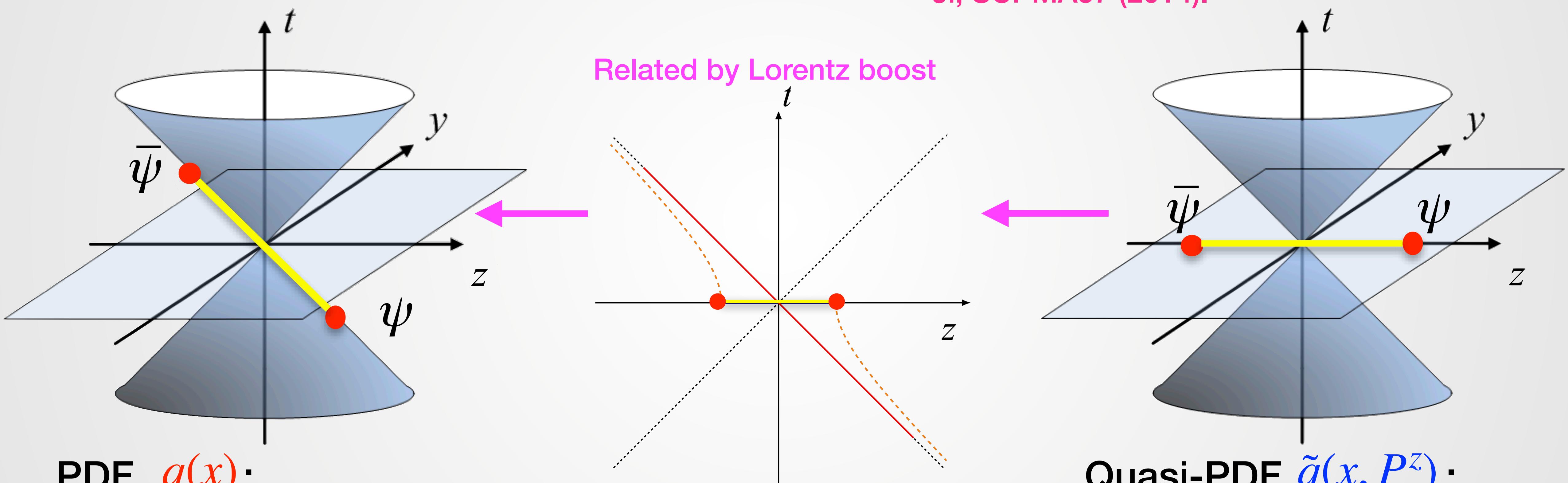

PDF $q(x)$:
Cannot be calculated on the lattice

Quasi-PDF $\tilde{q}(x, P^z)$:
Directly calculable on the lattice

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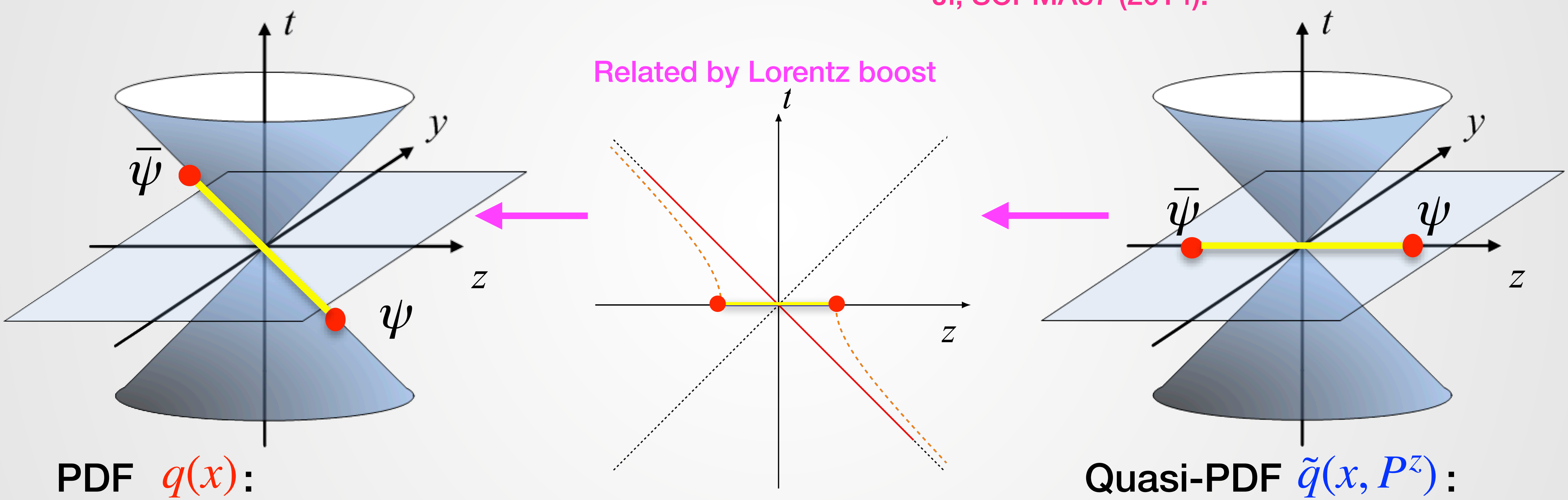


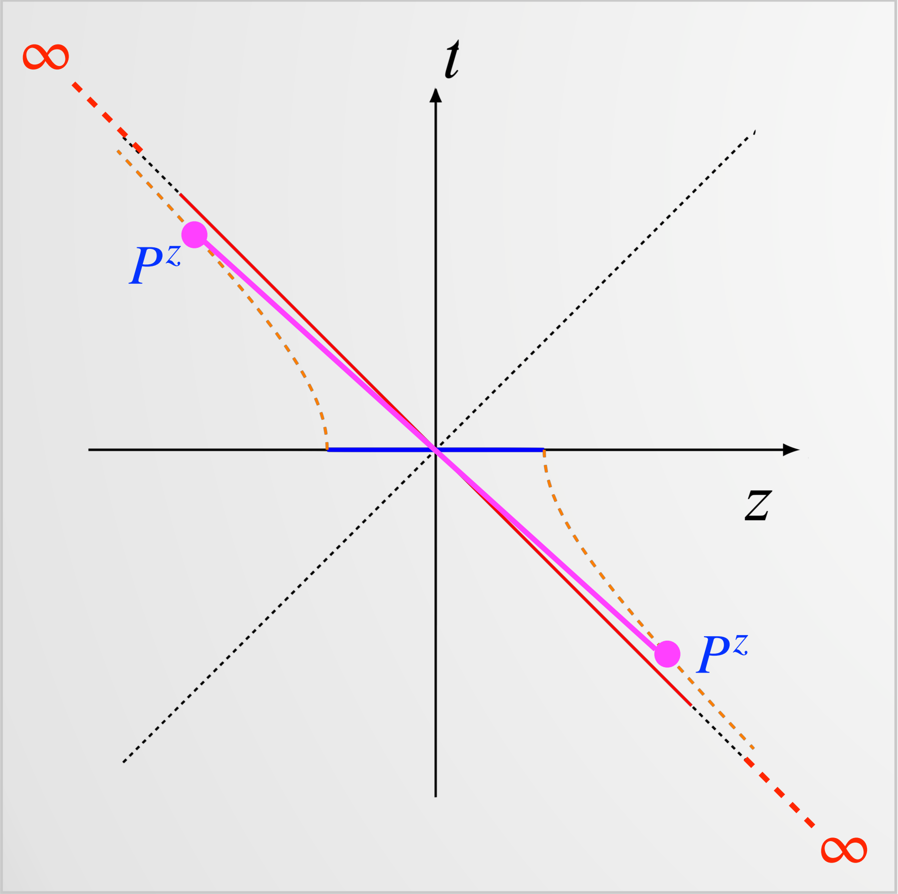
PDF $q(x)$:
Cannot be calculated on the lattice

Calculating the quasi-PDF at hadron momentum P^z is equivalent to boosting it.

Quasi-PDF $\tilde{q}(x, P^z)$:
Directly calculable on the lattice

A novel approach to calculate light-cone PDFs

$$\lim_{P^z \to \infty} \tilde{q}(x, P^z) = ?$$ ✗



Instead of taking $P^z \to \infty$ limit, one can perform an expansion for large but finite P^z:

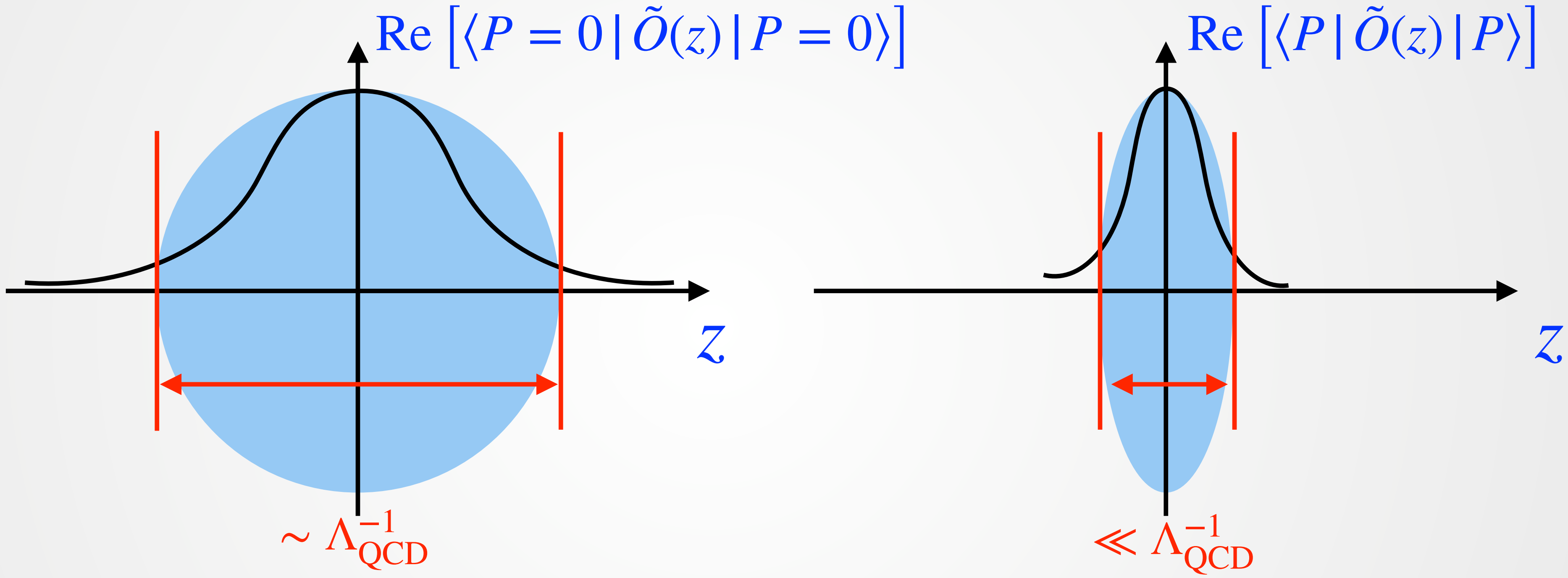
$$\tilde{q}(x, P^z) = C\left(x, P^z\right) \otimes q(x) + O\left(1/(P^z)^2\right)$$

- X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);

- $\tilde{q}(x, P^z)$ and $q(x)$ have the same infrared physics (nonperturbative), but different ultraviolet (UV) physics (perturbative);
- Therefore, the matching coefficient $C(x, P^z)$ is perturbative, which controls the logarithmic dependences on P^z.

The quasi-PDF factorization

- Spatial correlator in a highly boosted hadron:



- Operator product expansion at small distance:

H Dorn, Fortschr. Phys. 34 (1986)

$$\tilde{O}(z,\epsilon) = \bar{\psi}(z)\gamma^z P \exp\left[-ig\int_0^z dz' A^z(z')\right]\psi(0) = \Big\langle \underbrace{\bar{\psi}(z)\gamma^z Q(z)}_{j_1(z)}\ \underbrace{\bar{Q}(0)\psi(0)}_{j_2(0)} \Big\rangle_{\mathscr{L}_Q=\bar{Q}in\cdot DQ}$$

$$\tilde{O}(z,\mu) = Z_{j_1}^{-1} Z_{j_2}^{-1} e^{\delta m|z|}\tilde{O}(z,\epsilon)$$

δm renormalizes linear divergence in Wilson line self energy (under lattice regularization).

- X. Ji, J.-H. Zhang, and Y.Z., PRL120 (2018);
- J. Green et al., PRL121 (2018);
- T. Ishikawa, Y.-Q. Ma, J. Qiu, S. Yoshida, PRD96 (2017).

Factorization formula

- Operator product expansion (for the non-singlet case):

$$\tilde{O}(z,\mu) = \sum_{n=0} C_n(\mu^2 z^2)\frac{(-iz)^n}{n!}\hat{z}_{\mu_1}\cdots\hat{z}_{\mu_n}O^{z\mu_1\cdots\mu_n} + \text{higher twist}$$

$$O^{\mu_0\mu_1\cdots\mu_n}(\mu) = Z^{-1}_{n+1}(\epsilon,\mu)\ \left[\bar{\psi}\gamma^{\{\mu_0}i\overleftrightarrow{D}^{\mu_1}\cdots i\overleftrightarrow{D}^{\mu_n\}}\psi - \text{traces}\right]$$

$$a_{n+1}(\mu) = \int_{-1}^{1} dy\ y^n q(y,\mu)$$

$$\langle P|\tilde{O}(z,\mu)|P\rangle = 2\sum_{n=0} C_n(\mu^2 z^2)\frac{(-iz)^n}{n!}a_{n+1}(\mu)\left[(P^z)^{n+1} - O\Big(\frac{M^2}{P_z^2}\Big)\right] + O(z^2\Lambda^2_{\text{QCD}})$$

$$C\left(\frac{x}{y},\frac{\mu}{yP^z}\right) \equiv \int \frac{d(yzP^z)}{2\pi}e^{i\frac{x}{y}(yzP^z)}\sum_{n=0} C_n\Big(\frac{\mu^2}{y^2P_z^2}(yzP^z)^2\Big)\frac{(-iyzP^z)^n}{n!}$$

$$\tilde{q}(x,P^z,\mu) = \int\frac{dz}{4\pi}e^{ixP^z z}\langle P|\tilde{O}(z,\mu)|P\rangle = \int_{-1}^{1}\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{yP^z}\right)q(y,\mu) + O\left(\frac{M^2}{P_z^2},\frac{\Lambda^2_{\text{QCD}}}{x^2P_z^2}\right)$$

- Y.-Q. Ma and J. Qiu, PRD98 (2018), PRL 120 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018).

Systematic procedure of calculating the PDFs

1. Simulation of the quasi PDF in lattice QCD

$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y, \mu) + O\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

Systematic procedure of calculating the PDFs

$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y, \mu) + O\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

2. Renormalization of the lattice quasi PDF, and then taking the continuum limit

Nonperturbative renormalization on the lattice:

- I. Stewart and Y.Z., PRD97 (2018);
- J.-W. Chen, Y.Z. et al., LP3 Collaboration, PRD97 (2018).
- Constantinou and Panagopoulos, PRD96 (2017); C. Alexandrou et al., ETM Collaboration, NPB923 (2017).

Systematic procedure of calculating the PDFs

- O Nachtmann, NPB63 (1973);
- J.W. Chen et al. (LP3), NPB911 (2016).

3. Subtraction of power corrections

$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y, \mu) + O\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

$$q(x) \cdot O\left(\frac{\Lambda_{\rm QCD}^2}{x^2(1-x)P_z^2}\right)$$

Renormalon contribution to the power correction:
Braun, Vladimirov, and Zhang, PRD99 (2019).

Systematic procedure of calculating the PDFs

$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y, \mu) + O\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

- Matching for the quasi-PDF:
  - X. Xiong, X. Ji, J.-H. Zhang and Y.Z., PRD90 (2014);
  - I. Stewart and Y.Z., PRD97 (2018);
  - Y.-S. Liu, Y.Z. et al. (LP3), arXiv:1807.06566;
  - T. Izubuchi, X. Ji, L. Jin, I. Stewart, and Y.Z., PRD98 (2018);
  - Y.-S. Liu, Y.Z. et al., arXiv:1810.10879;
  - Y.Z., Int.J.Mod.Phys. A33 (2019);

4. Matching to the PDF.

Systematic procedure of calculating the PDFs

5. Extract $q(y)$

$$\tilde{q}(x, P^z, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) q(y,\mu) + O\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

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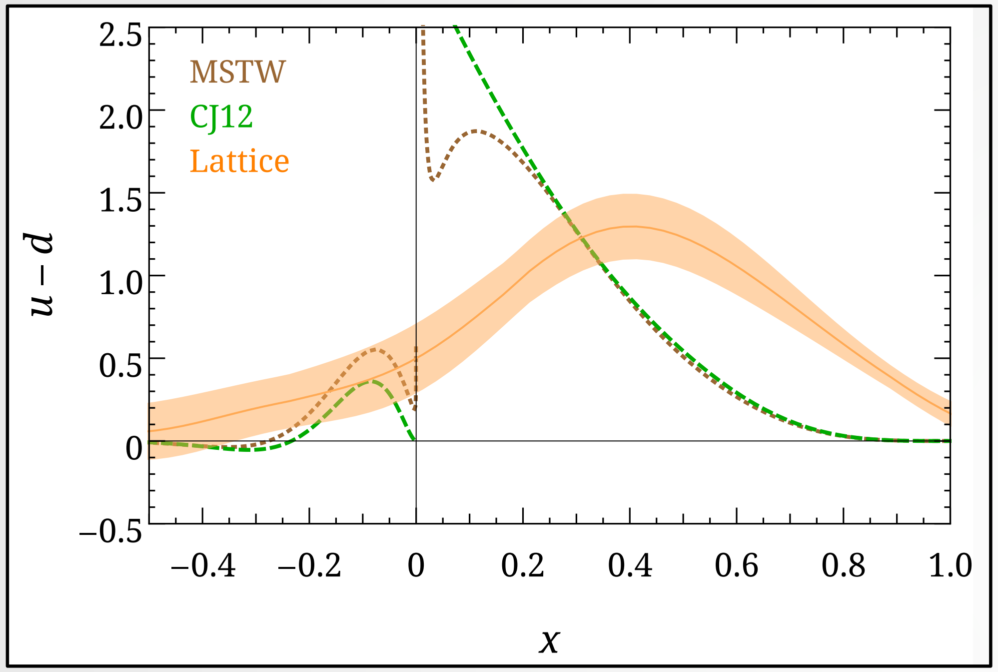
Lattice calculation of the iso-vector PDF $u(x) - d(x)$

2014



Lin et al., PRD91 (2015)

1.
- **Nucleon momentum**, from 1.4 GeV to 3.0 GeV;
- **Pion mass**, from 310 MeV to 135 MeV (physical point).

2. +

Lattice renormalization:
- X. Ji, J.-H. Zhang, and Y.Z., PRL120 (2018);
- J.-W. Chen, Y.Z. et al. (LP3), PRD97 (2018).

Perturbative matching:
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Improved Fourier transform:
- H.-W. Lin et al. (LP3), PRD98 (2018).

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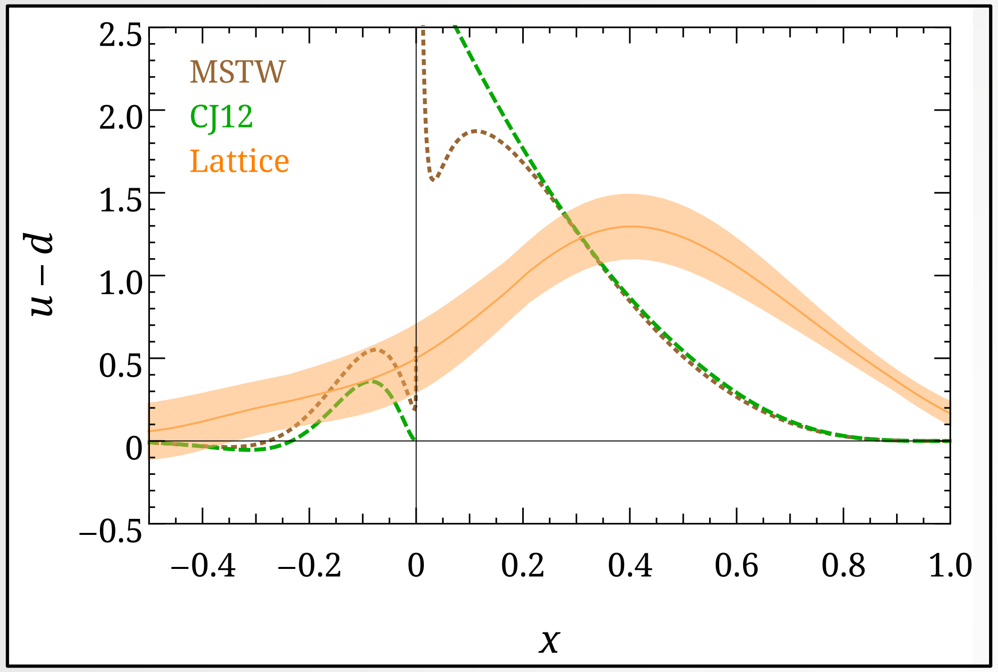

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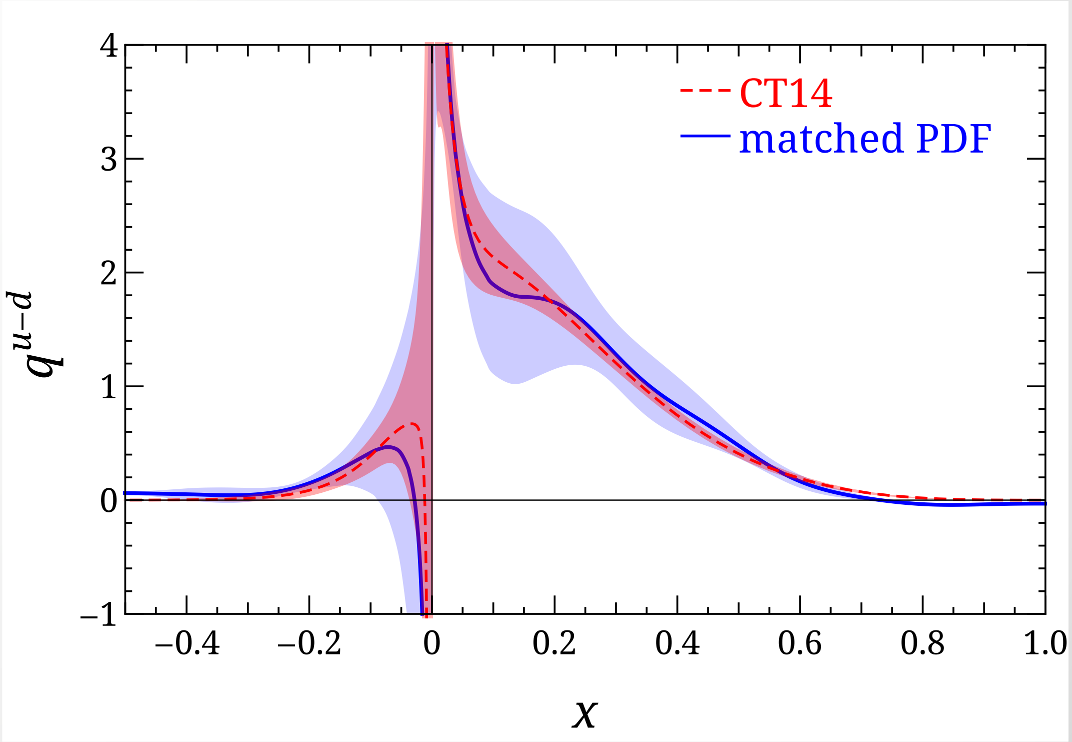

J.W. Chen, Y.Z. et al. (LP3), arXiv:1803.04393.

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Improved Fourier transform:

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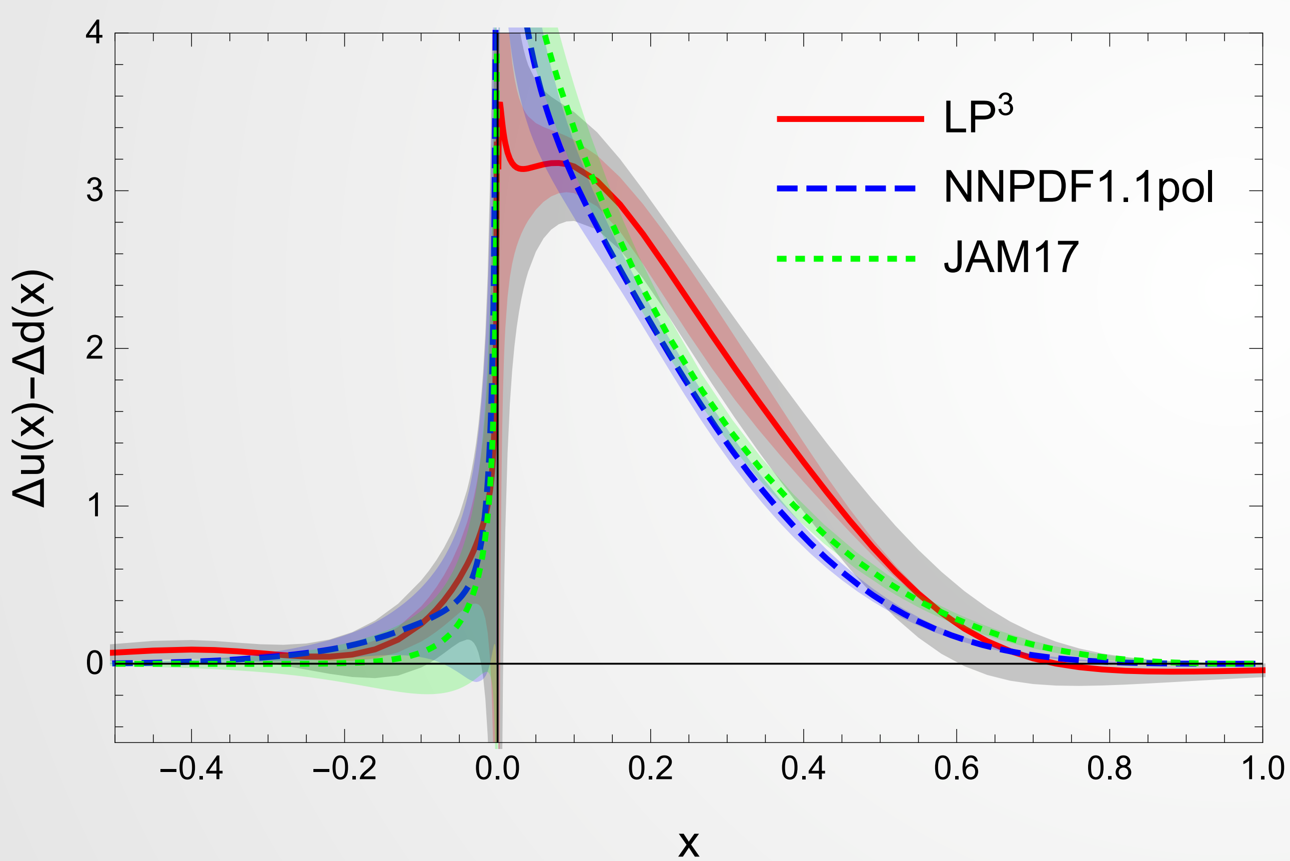
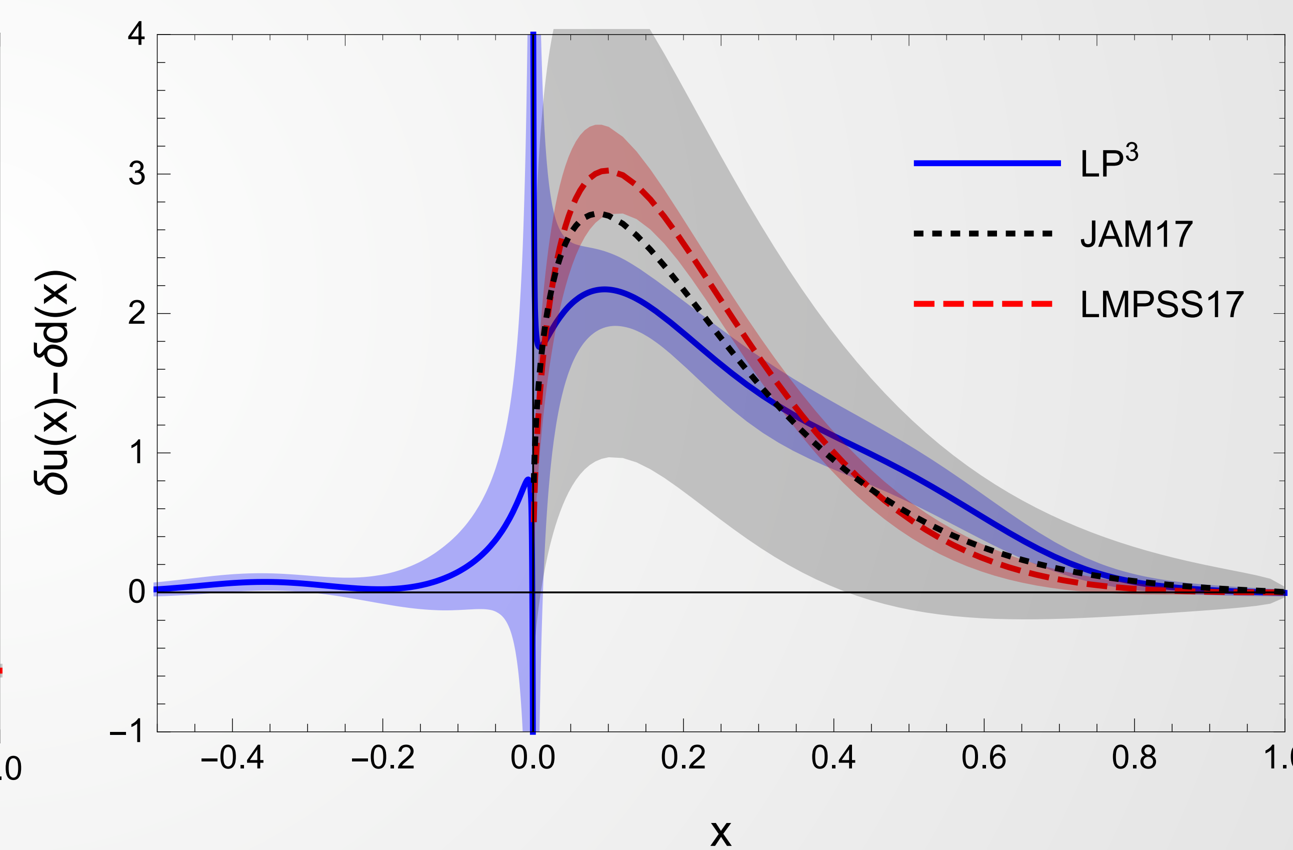
Lattice calculation of the iso-vector PDFs

Helicity PDF



H.W. Lin, Y.Z., et al. (LP3), Phys.Rev.Lett. 121 (2018).

Transversity PDF



Y.-S. Liu, Y.Z., et al. (LP3), arXiv:1810.05043.

Similar improvements also achieved by
ETMC collaboration, PRL121 (2018), PRD98 (2018).

The first case of PDFs where lattice calculation outperforms experiments!

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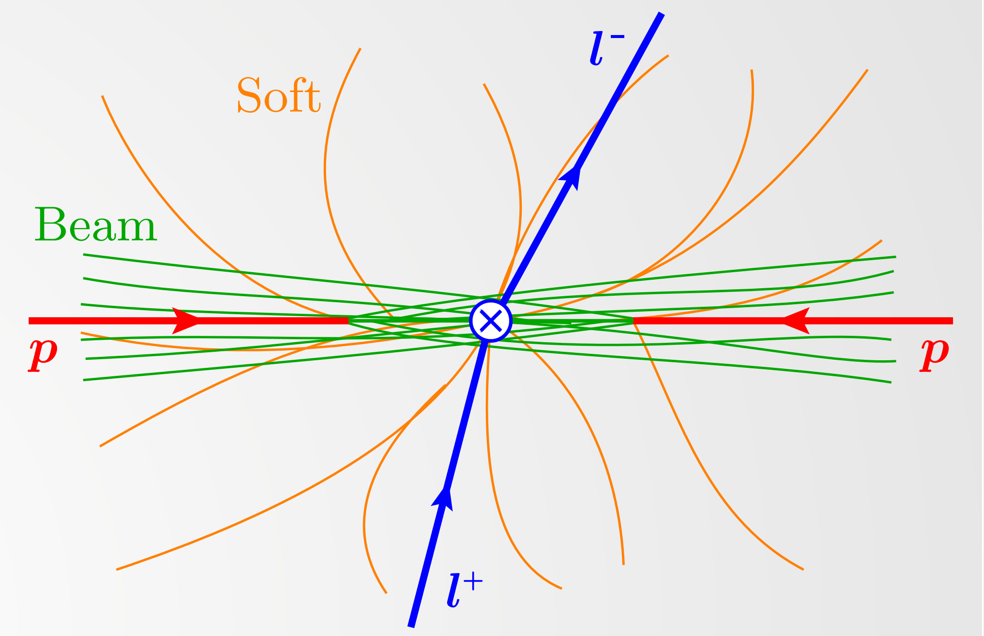
- Quasi-PDF approach
    - Physical picture and factorization formula
    - Systematic procedure to calculate parton distributions
- Quasi-TMDPDF
    - Relation of the quasi-TMDPDF and physical TMDPDF
    - Collins-Soper Kernel from lattice QCD

TMDPDFs

- **TMDPDF factorization for Drell-Yan:**
  - Collins, Soper and Sterman, NPB250 (1985); Collins, 2011;
  - Becher and Neubert, EPJC71 (2011);
  - Echevarria, Idilbi and Scimemi, JHEP07 (2012), PLB26 (2013);
  - Chiu, Jain, Neil and Rothstein, JHEP05 (2012), PRL108 (2012);
  - Li, Neil and Zhu, arXiv: 1604.00392.

  For a review of different schemes, see:
  - Ebert, Stewart and Y.Z., arXiv:1901.03685 (Appendix B).



$$\frac{d\sigma}{dQdYd^2q_T} = \sum_{ij} H_{ij}(Q,\mu) \int d^2b_T\, e^{i\vec{b}_T\cdot\vec{q}_T} B_i(x_a, \vec{b}_T, \mu, \frac{\zeta_a}{\nu^2})\, B_j(x_b, \vec{b}_T, \mu, \frac{\zeta_b}{\nu^2}) S_{ij}(b_T, \mu, \nu)$$

$$= \sum_{ij} H_{ij}(Q,\mu) \int d^2b_T\, e^{i\vec{b}_T\cdot\vec{q}_T}\, f_i^{\rm TMD}(x_a, \vec{b}_T, \mu, \zeta_a) f_j^{\rm TMD}(x_b, \vec{b}_T, \mu, \zeta_b)$$

$ij = q\bar{q}$ for DY

$ij = gg$ for H

$$f_i^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) = B_i(x, \vec{b}_T, \mu, \frac{\zeta}{\nu^2}) \sqrt{S^i(b_T, \mu, \nu)}$$

ζ: Collins-Soper Scale $\quad \zeta_a\zeta_b = Q^4$

Evolution of TMDPDFs

- **Renormalization scale evolution:**

$$\mu\frac{d}{d\mu}f_i^{\rm TMD}(x,b_T,\mu,\zeta)=\gamma_\mu^i(\mu,\zeta)f_i^{\rm TMD}(x,b_T,\mu,\zeta)$$

- **Collins-Soper evolution:**

$$\zeta\frac{d}{d\zeta}f_i^{\rm TMD}(x,b_T,\mu,\zeta)=\frac{1}{2}\gamma_\zeta^i(\mu,b_T)f_i^{\rm TMD}(x,b_T,\mu,\zeta)$$

$\gamma_\zeta(\mu,b_T)$: **Collins-Soper kernel, nonperturbative when b_T~$\Lambda_{\rm QCD}^{-1}$.**

- **Solution to evolution equations:**

$$f_i^{\rm TMD}(x,\vec{b}_T,\mu,\zeta)=f_i^{\rm TMD}(x,\vec{b}_T,\mu_0,\zeta_0)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_\mu^i(\mu',\zeta_0)\right]\exp\left[\frac{1}{2}\gamma_\zeta^i(\mu,b_T)\ln\frac{\zeta}{\zeta_0}\right]$$

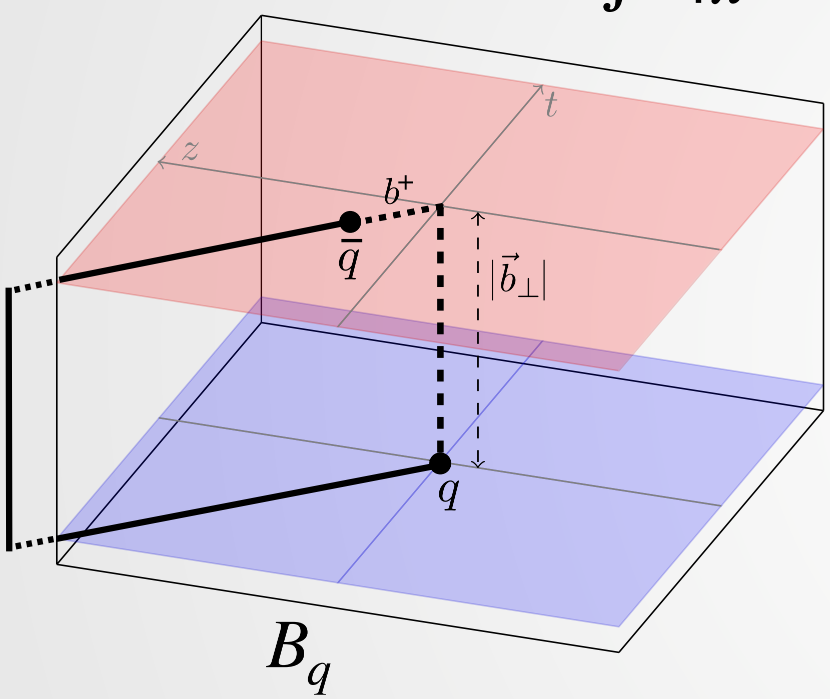
  - μ, ζ: **factorization scales,** $\mu>>\Lambda_{\rm QCD}$, ζ~Q^2;
  - μ_0, ζ_0: **initial or reference scales, measured in experiments or determined from lattice (~2 GeV).**

Quasi-beam function

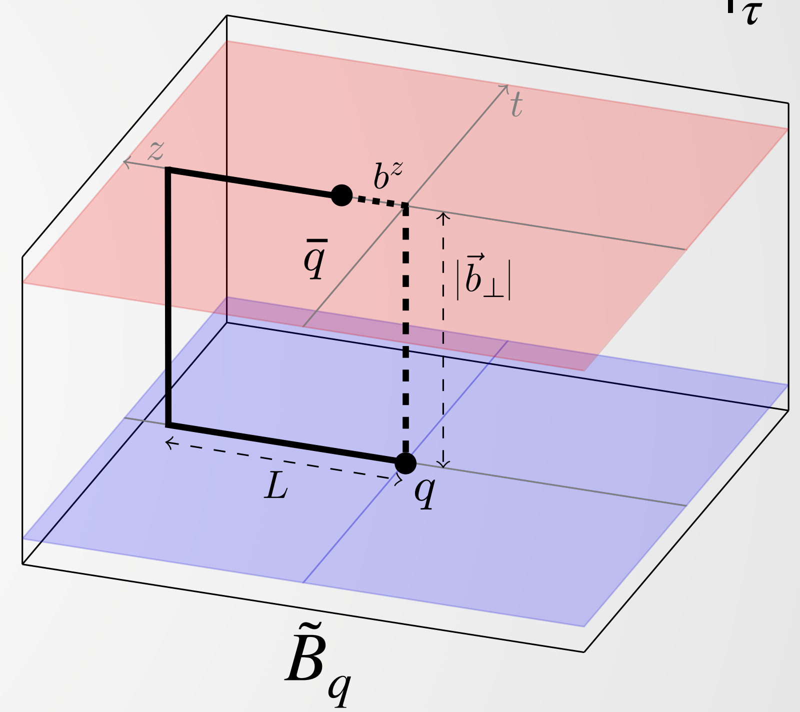
- Beam function:

ϵ : UV regulator τ : rapidity regulator

$$B_q(x, \vec{b}_T, \epsilon, \tau, xP^-) = \int \frac{db^+}{4\pi} e^{-i\frac{1}{2}b^+(xP^-)} \langle P | \bar{q}(b^\mu) W(b^\mu) \frac{\gamma^-}{2} W_T(-\infty \bar{n}; \vec{b}_T, \vec{0}_T) W^\dagger(0) q(0) \Big|_\tau | P \rangle$$



Lorentz boost and $L \to \infty$



- Quasi-beam function on lattice:

$$\tilde{B}_q(x, \vec{b}_T, a, L, P^z) = \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \tilde{B}_q(b^z, \vec{b}_T, a, L, P^z)$$

$$= \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{q}(b^\mu) W_{\hat{z}}(b^\mu; L - b^z) \frac{\Gamma}{2} W_T(L\hat{z}; \vec{b}_T, \vec{0}_T) W^\dagger_{\hat{z}}(0) q(0) | P \rangle$$

Finite Wilson line length L due to the finite lattice volume.

Quasi-soft function

- **Soft function:**

$$S_q(b_T,\epsilon,\tau) = \frac{1}{N_c}\langle 0|\,\mathrm{Tr}\left[S_n^\dagger(\vec{b}_T)S_{\bar n}(\vec{b}_T)S_T(-\infty\bar n;\vec{b}_T,\vec{0}_T)S_{\bar n}^\dagger(\vec{0}_T)S_n(\vec{0}_T)S_T^\dagger(-\infty n;\vec{b}_T,\vec{0}_T)\right]\Big|_\tau\, |0\rangle$$

- **Quasi-soft function on lattice (naive definition):**

$$\tilde S_q(b_T,a,L) = \frac{1}{N_c}\langle 0|\,\mathrm{Tr}\left[S_{\hat z}^\dagger(\vec{b}_T;L)S_{-\hat z}(\vec{b}_T;L)S_T(L\hat z;\vec{b}_T,\vec{0}_T)S_{-\hat z}^\dagger(\vec{0}_T;L)S_n(\vec{0}_T;L)S_T^\dagger(-L\hat z;\vec{b}_T,\vec{0}_T)\right]|0\rangle$$

Impact of finite-length Wilson lines

- Linear power divergence under lattice regularization $\sim L/a$
- Finite *L* regulates rapidity divergences:
  - Light-like Wilson lines

$$g_s t^a n^\mu \frac{1-e^{ik^+L}}{k^+} \xrightarrow{L\to\infty} g_s t^a n^\mu \frac{1}{k^+}$$

$$I_{\rm div} = \int dk^+ dk^- \frac{1}{(k^+k^-)^{1+\epsilon}} \longrightarrow \int dk^+ dk^- \frac{1}{(k^+k^-)^{\epsilon}} \frac{1-e^{ik^+L}}{k^+} \frac{1-e^{ik^-L}}{k^-}$$

  - Space-like Wilson lines

$$\tilde{I}_{\rm div} = \int dk_0 dk_z \frac{1}{(k_0^2-k_z^2)^\epsilon} \frac{1}{k_z^2} \longrightarrow \int dk_0 dk_z \frac{1}{(k_0^2-k_z^2)^\epsilon} \frac{1-e^{ik^zL}}{k^z} \frac{1-e^{-ik^zL}}{k^z}$$

- By construction the *L* dependence has to be canceled out between the quasi-beam and soft functions.

Quasi-TMDPDF

- Quasi-TMDPDF in the MSbar scheme:

$$\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z) = \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \tilde{Z}'(b^z, \mu, \tilde{\mu}) \tilde{Z}_{\mathrm{UV}}(b^z, \mu, a) \frac{\tilde{B}_q(b^z, \vec{b}_T, a, L, P^z)}{\sqrt{\tilde{S}_q(b_T, a, L)}}$$

- Schematic factorization formula:

$$\tilde{f}_i^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z) \sim \sum_j C_{ij}^{\mathrm{TMD}}\left(x, \mu, P^z\right) \exp\left[\frac{1}{2}\gamma_\zeta^j(\mu, b_T) \ln \frac{(2xP^z)^2}{\zeta}\right]$$
$$\times f_j^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}\left(\frac{b_T}{L}, \frac{1}{b_T P^z}, \frac{1}{P^z L}\right)$$

Hierarchy of scales: $b^z \sim \frac{1}{P^z} \ll b_T \ll L, \quad b_T \sim \Lambda_{\mathrm{QCD}}^{-1}$

One-loop test

- **Physical TMDPDF:**

$$f_q^{\mathrm{TMD}(1)}(x, \vec{b}_T, \epsilon, \zeta) = \frac{\alpha_s C_F}{2\pi}\left[-\left(\frac{1}{\epsilon_{\mathrm{IR}}} + \mathbf{L}_b\right)\frac{1+x^2}{1-x} + (1-x)\right]_+ \theta(x)\theta(1-x)$$

$$b_0 = 2e^{-\gamma_E}$$

$$\mathbf{L}_b = \ln\frac{b_T^2\mu^2}{b_0^2}$$

$$\mathbf{L}_\zeta = \ln\frac{\mu^2}{\zeta}$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[\frac{1}{\epsilon_{\mathrm{UV}}^2} + \frac{1}{\epsilon_{\mathrm{UV}}}\left(\frac{3}{2} + \mathbf{L}_\zeta\right) + \frac{1}{2} - \frac{\pi^2}{12}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[-\frac{1}{2}\mathbf{L}_b^2 + \frac{3}{2}\mathbf{L}_b + \mathbf{L}_b\mathbf{L}_\zeta\right]$$

- Ji, Jin, Yuan, Zhang and Y.Z., arXiv:1801.05930;
- Ebert, Stewart and Y.Z., arXiv:1901.03685.

- **Naive quasi-TMDPDF:**

$$\tilde{f}_q^{\mathrm{TMD}(1)}(x, \vec{b}_T, \epsilon, P^z) = \frac{\alpha_s C_F}{2\pi}\left[-\left(\frac{1}{\epsilon_{\mathrm{IR}}} + \mathbf{L}_b\right)\frac{1+x^2}{1-x} + (1-x)\right]_+ \theta(x)\theta(1-x)$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[\frac{3}{2}\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{2}\ln^2\frac{\mu^2}{(2xP^z)^2} - \ln\frac{\mu^2}{(2xP^z)^2} - \frac{3}{2}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[-\frac{1}{2}\mathbf{L}_b^2 + \frac{5}{2}\mathbf{L}_b + \mathbf{L}_b\ln\frac{\mu^2}{(2xP^z)^2}\right]$$

- Same collinear divergence;
- b_T dependences do not match even if one sets $\zeta = 2xP^z$
- No perturbative matching when $b_T \sim \Lambda_{\mathrm{QCD}}^{-1}$

One-loop test

- **Physical TMDPDF:**

$$f_q^{\mathrm{TMD}(1)}(x, \vec{b}_T, \epsilon, \zeta) = \frac{\alpha_s C_F}{2\pi}\left[-\left(\frac{1}{\epsilon_{\mathrm{IR}}} + \mathbf{L}_b\right)\frac{1+x^2}{1-x} + (1-x)\right]_+ \theta(x)\theta(1-x)$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[\frac{1}{\epsilon_{\mathrm{UV}}^2} + \frac{1}{\epsilon_{\mathrm{UV}}}\left(\frac{3}{2} + \mathbf{L}_\zeta\right) + \frac{1}{2} - \frac{\pi^2}{12}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[-\frac{1}{2}\mathbf{L}_b^2 + \frac{3}{2}\mathbf{L}_b + \mathbf{L}_b\mathbf{L}_\zeta\right]$$

$$b_0 = 2e^{-\gamma_E}$$

$$\mathbf{L}_b = \ln\frac{b_T^2\mu^2}{b_0^2}$$

$$\mathbf{L}_\zeta = \ln\frac{\mu^2}{\zeta}$$

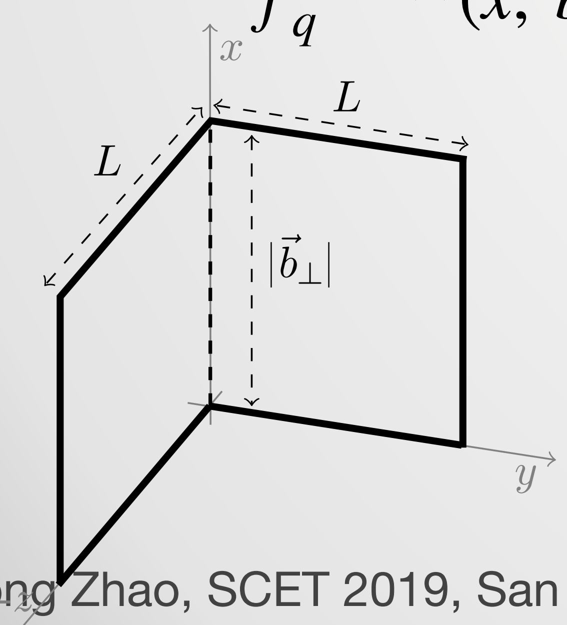
- Ebert, Stewart and Y.Z., arXiv:1901.03685.

- **Bent quasi-soft function:**

$$\tilde{f}_q^{\mathrm{TMD}(1)}(x, \vec{b}_T, \epsilon, P^z) = \frac{\alpha_s C_F}{2\pi}\left[-\left(\frac{1}{\epsilon_{\mathrm{IR}}} + \mathbf{L}_b\right)\frac{1+x^2}{1-x} + (1-x)\right]_+ \theta(x)\theta(1-x)$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[\frac{3}{2}\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{2}\ln^2\frac{\mu^2}{(2xP^z)^2} - \ln\frac{\mu^2}{(2xP^z)^2} - \frac{3}{2}\right]$$

$$+\frac{\alpha_s C_F}{2\pi}\delta(1-x)\left[-\frac{1}{2}\mathbf{L}_b^2 + \frac{3}{2}\mathbf{L}_b + \mathbf{L}_b\ln\frac{\mu^2}{(2xP^z)^2}\right]$$



Correct relation between quasi-TMDPDF and TMDPDF

Ebert, Stewart and Y.Z., PRD99 (2019), arXiv:1901.03685.

- Factorization formula (for the non-singlet case):

$$\tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P^z) = C_{\rm ns}^{\rm TMD}\left(\mu, xP^z\right) \, g_q^S(b_T, \mu) \; \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T)\ln\frac{(2xP^z)^2}{\zeta}\right] f_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, \zeta)$$

$$C_{\rm ns}^{\rm TMD}\left(\mu, xP^z\right) = 1 + \frac{\alpha_s C_F}{2\pi}\left[-\frac{1}{2}\ln^2\frac{(2xP^z)^2}{\mu^2} + \ln\frac{(2xP^z)^2}{\mu^2} - 2 + \frac{\pi^2}{12}\right]$$

$$g_q^{S_{\rm naive}}(b_T, \mu) = 1 + \frac{\alpha_s C_F}{2\pi}\mathbf{L}_b + O(\alpha_s^2)$$

$$g_q^{S_{\rm bent}}(b_T, \mu) = 1 + O(\alpha_s^2)$$

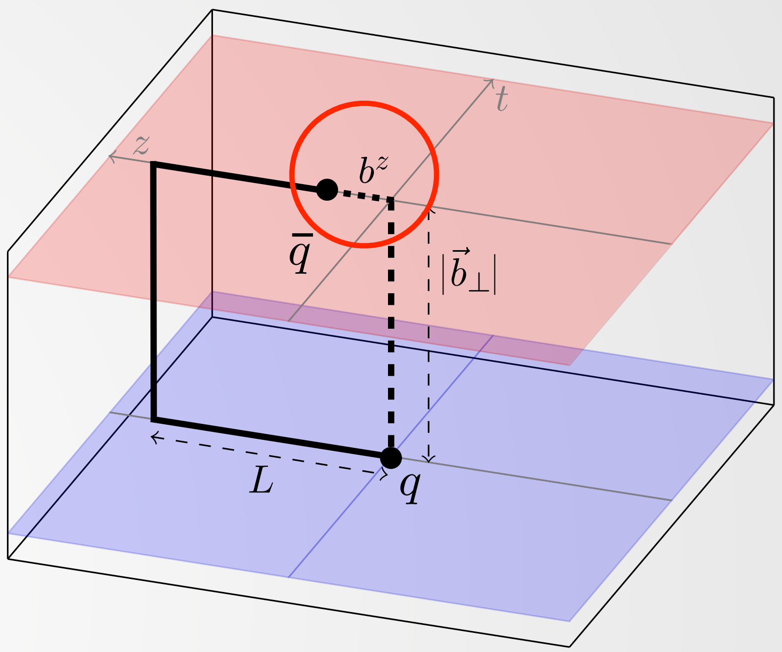
  - $g^S{}_i$ does not depend on the external state or quark flavor, but can be different between quark and gluon;
  - For the bent quasi-soft function, $g^S{}_i$ needs to be checked at higher loop orders;
  - One can form ratios of TMDPDFs to cancel out $g^S{}_i$.

Lattice calculation of the ratios of the TMDPDF moments:
Hagler, Musch, Engelhardt, Yoon, et al., EPL88 (2009), PRD83 (2011), PRD85 (2012), PRD93 (2016), arXiv:1601.05717, PRD96 (2017)

Collins-Soper kernel from lattice

- **Proposal to calculate** $\gamma_\zeta(\mu, b_T)$ **with Lattice QCD**

Ebert, Stewart and Y.Z., PRD99 (2019).



$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

$$= \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\int db^z\, e^{ib^z xP_1^z}\tilde{Z}'(b^z, \mu, \tilde{\mu})\tilde{Z}_{\rm UV}(b^z, \tilde{\mu}, a)\tilde{B}_{\rm ns}(b^z, \vec{b}_T, a, L, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\int db^z\, e^{ib^z xP_2^z}\tilde{Z}'(b^z, \mu, \tilde{\mu})\tilde{Z}_{\rm UV}(b^z, \tilde{\mu}, a)\tilde{B}_{\rm ns}(b^z, \vec{b}_T, a, L, P_2^z)}$$

- g^S as well as the quasi-soft function gets canceled in the ratio;
- The quasi-beam function includes a linear power divergence that depends on b^z/a, which needs to be nonperturbatively renormalized by $\tilde{Z}_{\rm UV}(b^z, \tilde{\mu}, a)$ before the Fourier transform.

Summary and Outlook

In 5~10 years, expect:

- Lattice calculation of quark PDFs to be within 10% accuracy or even better;
- Determination of sea quark distributions to be better than experiments;
- Reaching smaller *x* region with larger nucleon momentum;
- Lattice calculation of gluon PDFs;

  Renormalization: Zhang et al., arXiv:1808.10824; Li et al., PRL122 (2019);
  Perturbative matching: Wang et al., EPJC78 (2018), JHEP1805 (2018);
  First lattice attempt: Fan et al., PRL121 (2018).
- Lattice calculation of gluon spin and parton orbital angular momentum (OAM);
  - First lattice calculation of gluon spin: Y.-B. Yang, R. S. Suffian, Y.Z., et al. (χQCD), PRL118 (2017).
  - Method to calculate canonical OAM: Y.Z., Liu and Yang, PRD93 (2016)
- Lattice calculation of GPDs;

  Perturbative matching: Liu, Y.Z. et al., arXiv:1902.00307.
- Lattice calculation of transverse structures such the TMDPDFs.

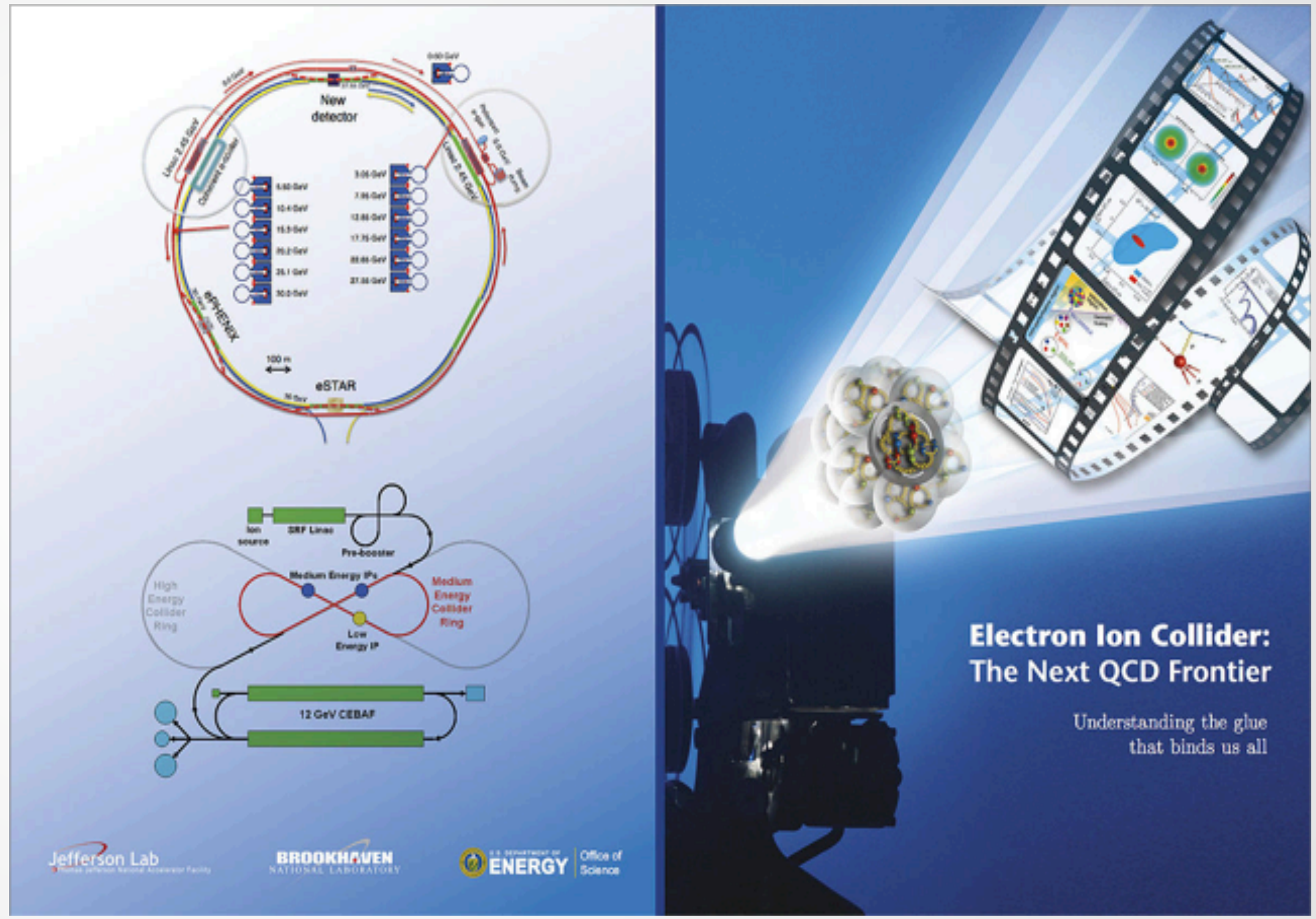
The next frontier of QCD

Electron-Ion Collider!

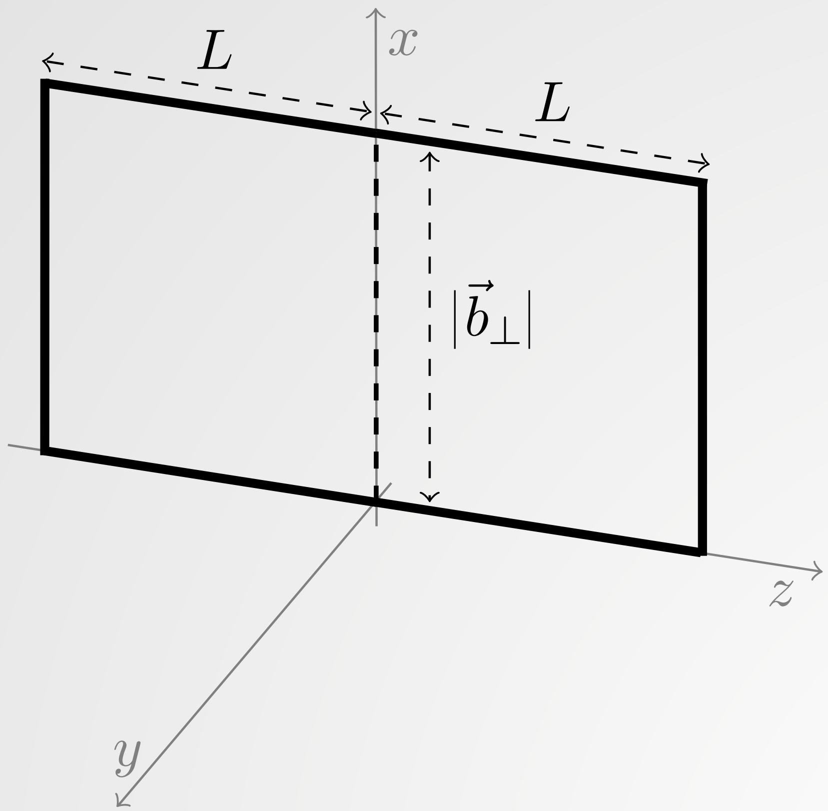
- Highly Polarized Beams
- Large Kinematic Range
- High Intensity Beams

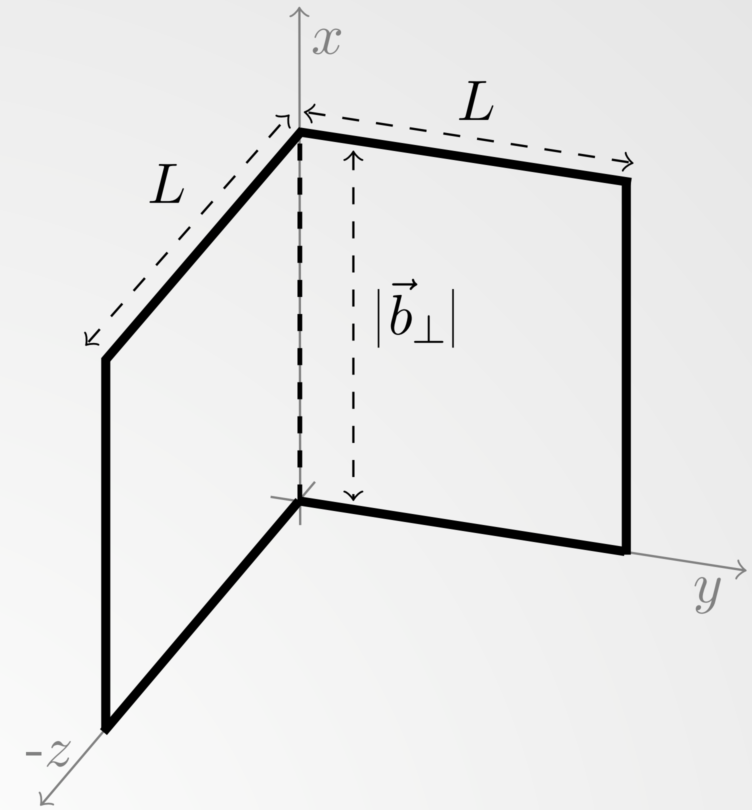
3-D tomography of the nucleon:

- More precise PDFs
- Sea quark distributions
- Gluonic structure of nucleon and nuclei
- Small-*x* physics
- Gluon polarization and parton orbital angular momentum
- TMDPDFs, GPDs
- ……



A. Accardi et al., Eur.Phys.J. A52 (2016) no.9, 268.





$$B_i(x, \overrightarrow{b}_T, \mu, \frac{\zeta}{\nu^2}) = \lim_{\epsilon\to 0, \tau\to 0} Z_B^i(b_T, \mu, \nu, \epsilon, \tau, xP^-)\frac{B_i^{\rm unsub}(x, \overrightarrow{b}_T, \epsilon, \tau, xP^-)}{S_i^0(b_T, \epsilon, \tau)}$$

$$S^i(b_T, \mu, \nu) = \lim_{\epsilon\to 0, \tau\to 0} Z_S^i(b_T, \mu, \nu, \epsilon, \tau) S^i(b_T, \epsilon, \tau)$$

$$\tilde{f}_i^{\rm TMD}(x, \overrightarrow{b}_T, \mu, \tilde{P}^z) \sim \sum_j \int_{-1}^{1} \frac{dy}{|y|} C_{ij}^{\rm TMD}\left(x, y, \mu, \tilde{P}^z, \tilde{\zeta}(x, \tilde{P}^z)\right)$$

$$\times \exp\left[\frac{1}{2}\gamma_\zeta^j(\mu, b_T) \ln\frac{\tilde{\zeta}(x, \tilde{P}^z)}{\zeta}\right] f_j^{\rm TMD}(y, \overrightarrow{b}_T, \mu, \zeta)$$