Infrared Finiteness and Forward Scattering

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arXiv: 1810.10022

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March 27, 2019

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Motivation

- Improved understanding of IR divergences
- Define a finite S-matrix
- Precision collider physics

Introduction

- IR divergences: Soft and collinear divergences in matrix elements and cross sections
- Ideas on how to get IR finite quantities:
    1. Cross section method → IR finite σ
    2. Coherent states → IR finite S-matrix

Theorems on IR divergences

Bloch-Nordsieck: Soft IR divergences cancel in QED when summing over final state radiation with finite energy resolution.

KLN Theorem: Transition amplitude squared is IR finite when summing over final states **and** initial states within some energy window:

$$\sum_{f \in D(E),\, i \in D(E)} |\langle f|\, S\, |i\rangle|^2 < \infty$$

Stronger KLN Theorem: Transition amplitude squared is IR finite when summing over final states **or** initial states:

$$\sum_f |\langle f|\, S\, |i\rangle|^2 < \infty, \qquad \sum_i |\langle f|\, S\, |i\rangle|^2 < \infty$$

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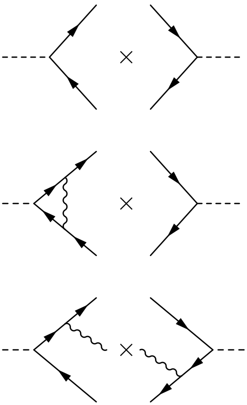
$$\sum_f |\langle f | S | i \rangle|^2 < \infty, \qquad \sum_i |\langle f | S | i \rangle|^2 < \infty$$

Both KLN and the stronger version require a term where $f = i$, which corresponds to forward scattering

$Z \to e^+e^-$ + final state radiation

$\mathcal{M} \quad \times \quad \mathcal{M}^\dagger$



$Z \to e^+e^-$ + final state radiation

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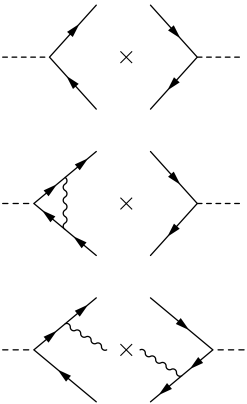


$$\Gamma = \Gamma_0 \ \text{(finite)}$$

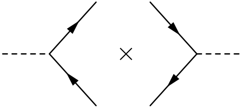
$$\frac{\Gamma}{\Gamma_0} \propto -\frac{1}{4\epsilon^2} - \frac{3}{8\epsilon}$$

$$\frac{\Gamma}{\Gamma_0} \propto \frac{1}{4\epsilon^2} + \frac{3}{8\epsilon}$$

$m_e = 0$
Dim reg
CM frame

$Z \to e^+e^-$ + final state radiation

$\mathcal{M} \times \mathcal{M}^\dagger$



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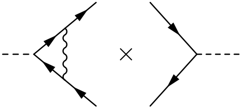

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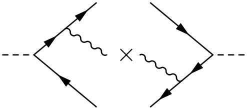

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$m_e = 0$
Dim reg
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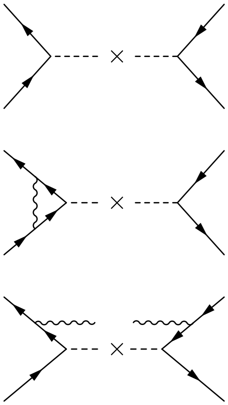
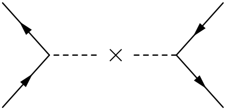
Soft singularities cancel by the Bloch-Nordsieck Theorem
Collinear singularities happen to also cancel

$e^+e^- \to Z$ + final state radiation

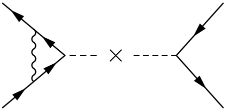


$e^+e^- \to Z$ + final state radiation

$m_e = 0$
$z = m_z^2/E_{CM}^2$
Dim reg
CM frame



$$\sigma = \sigma_0\,\delta(1-z)\ \text{(finite)}$$



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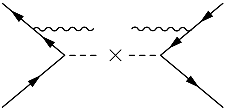

$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

$e^+e^- \to Z$ + final state radiation

$m_e = 0$

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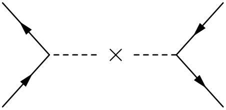

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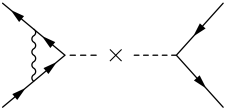

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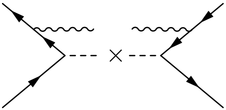


$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

Soft singularities cancel by the Bloch-Nordsieck Theorem

Collinear singularities **do not cancel**

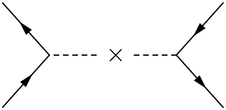
$e^+e^- \to Z$ + initial state absorption

$m_e = 0$

$z = m_z^2/E_{CM}^2$

Dim reg

CM frame



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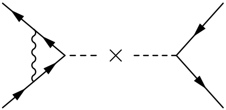

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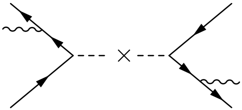

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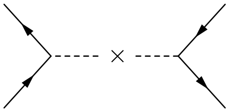
$e^+e^- \to Z$ + initial state absorption



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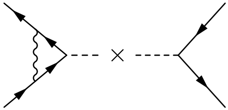

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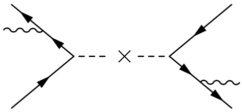


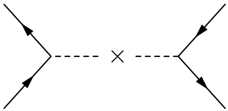
$$\frac{\sigma}{\sigma_0} \propto \delta(1-z) \left(\frac{1}{4\epsilon^2} + \frac{3}{8\epsilon} \right)$$

Including initial state absorption diagram:

Soft and collinear singularities cancel

$e^+e^- \to Z$ + initial state absorption & final state radiation

$m_e = 0$
$z = m_z^2/E_{CM}^2$
Dim reg
CM frame



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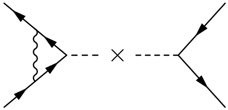

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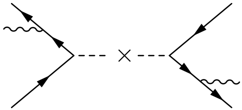

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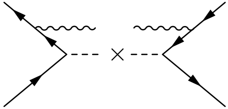

$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

$e^+e^- \to Z$ + initial state absorption & final state radiation

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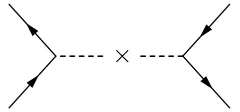
Dim reg

CM frame



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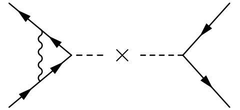

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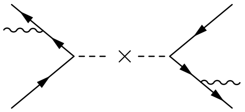

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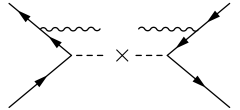


$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

No reason to exclude $\to$ leftover singularity

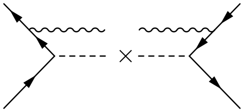
Cancelling IR singularities

- Which diagrams cancel the leftover singularity from the final state radiation diagram?
- **KLN Theorem:** Transition amplitude squared is IR finite when summing over final states **and** initial states within some energy window:

$$\sum_{f \in D(E),\, i \in D(E)} |\langle f|\, S\, |i\rangle|^2 < \infty$$

- Not including all possible diagrams

$e^+e^- \to Z$ + initial state absorption & final state radiation



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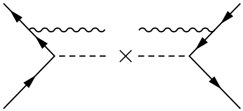

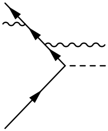

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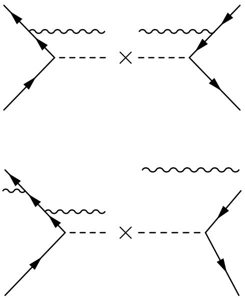

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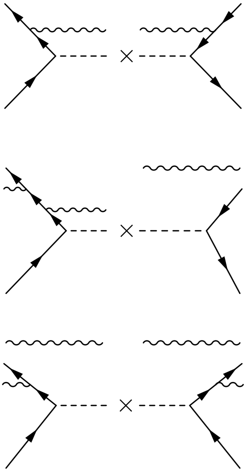


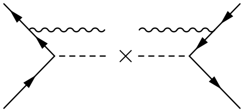
$e^+e^- \to Z$ + initial state absorption & final state radiation

$m_e = 0$

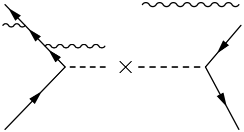
$z = m_z^2/E_{CM}^2$

Dim reg

CM frame



$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$



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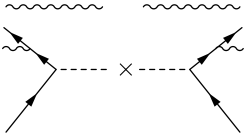

$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{5z^2-4z+1}{4\,\epsilon\,[1-z]_+}$$

$e^+e^- \to Z$ + initial state absorption & final state radiation

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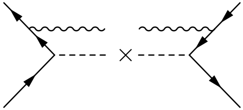

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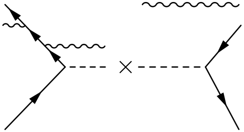

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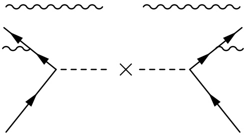

$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{5z^2-4z+1}{4\,\epsilon\,[1-z]_+}$$

Soft and collinear singularities cancel

$e^+e^- \to Z$ + initial state absorption & final state radiation

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Dim reg

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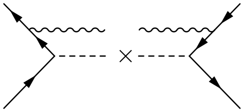

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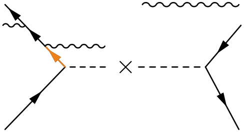

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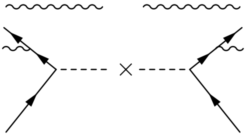

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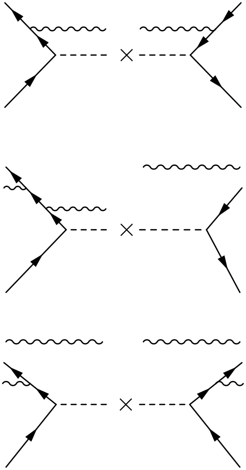
Soft and collinear singularities cancel

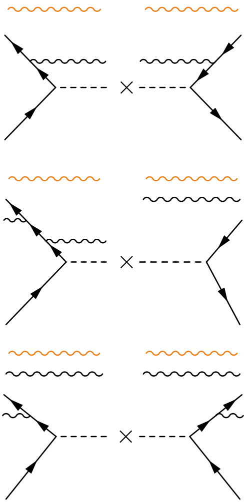
On-shell intermediate propagators: Treat $\frac{\delta(p^2)}{p^2+i\epsilon}$ as a distribution

$e^+e^- \to Z$ + initial state absorption & final state radiation



No reason to stop at 1 disconnected photons

$e^+e^- \to Z$ + initial state absorption & final state radiation



No reason to stop at 1 disconnected photons

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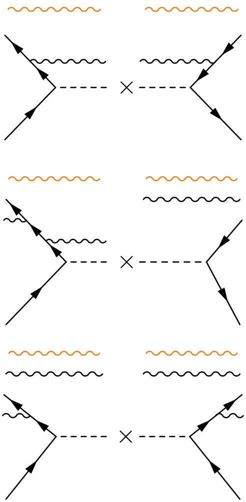
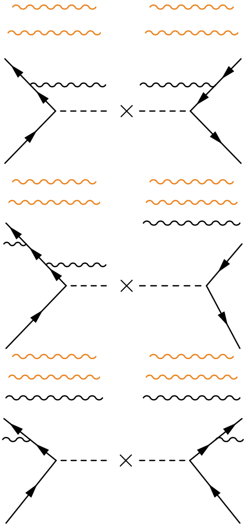


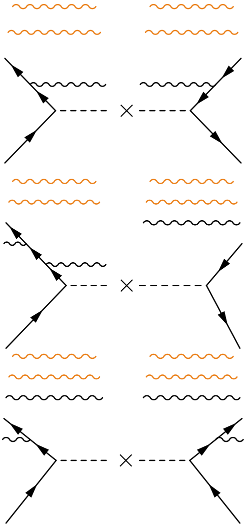
No reason to stop at 2 disconnected photons

$e^+e^- \to Z$ + initial state absorption & final state radiation



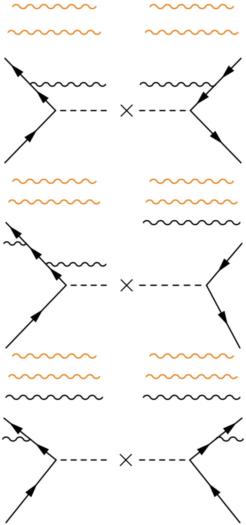
No reason to stop at n disconnected photons

$e^+e^- \to Z$ + initial state absorption & final state radiation



Soft and collinear singularities cancel in each triplet of diagrams

$e^+e^- \to Z$ + initial state absorption & final state radiation



Soft and collinear singularities cancel in each triplet of diagrams

$$\sum_m \sigma_{m,n} \propto -\frac{(1-z)^3}{z^2 n^4} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

m : No. of initial state photons

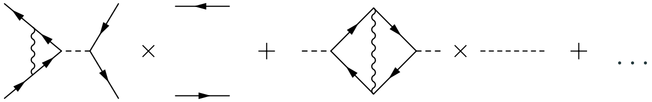
n : No. of final state photons

$z = m_z^2/E_{CM}^2$

- Why did it work to sum over disconnected photons?
- **KLN Theorem:** Transition amplitude squared is IR finite when summing over final states **and** initial states within some energy window:

$$\sum_{f \in D(E),\, i \in D(E)} |\langle f|\, S\, |i\rangle|^2 < \infty$$

KLN requires a term where $f = i \to$ forward scattering



The IR singularity cancellation only worked since the forward scattering diagrams $Z + n\,\gamma \to Z + n\,\gamma$ are finite for any n

Cancelling IR singularities - KLN interpretation

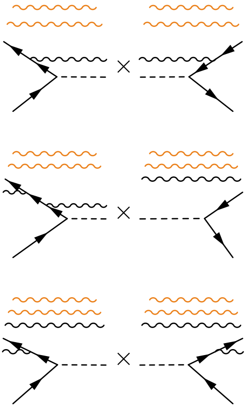
- **Stronger KLN Theorem:** Transition amplitude squared is IR finite when summing over final states **or** initial states:

$$\sum_f |\langle f|\, S \,|i\rangle|^2 < \infty, \qquad \sum_i |\langle f|\, S \,|i\rangle|^2 < \infty$$

  - Trivial consequence of unitarity:
    - Probability of $i \to$ anything is $1 < \infty$
    - Probability of anything $\to f$ is $1 < \infty$
  - Proof in old-fashioned perturbation theory
  - Fix state and cut up squared diagrams in all possible ways
- 3 ways of making $e^+e^- \to Z$ finite:
  1. With infinite number disconnected photons, but no forward scattering
  2. Final state sum, including forward scattering
  3. Initial state sum, including forward scattering

3 ways of making $e^+e^- \to Z$ finite:
1. Disconnected photons

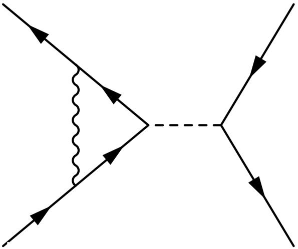
Sum of triplets of diagrams are IR finite



Forward scattering diagrams $Z + n\,\gamma \to Z + n\,\gamma$ are IR finite

3 ways of making $e^+e^- \to Z$ finite:
2. Final state sum: Virtual corrections

Procedure: Take any squared diagram, fix initial state and make all possible final state cuts



The sum of the corresponding cross sections should be zero

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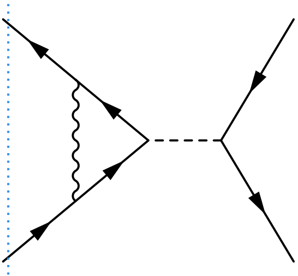

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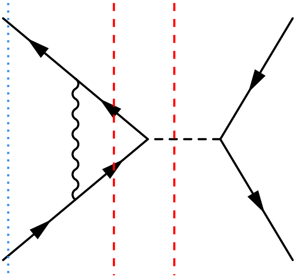

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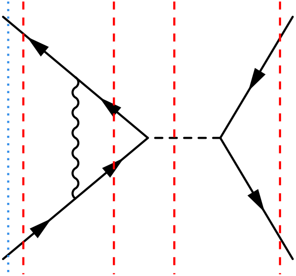

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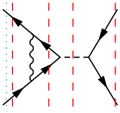

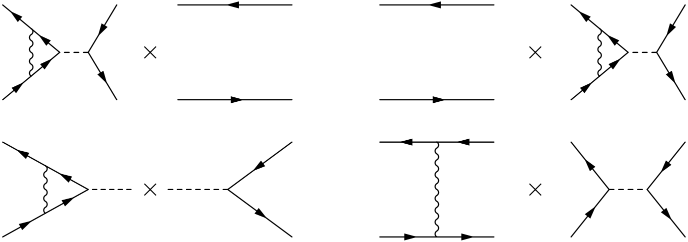

The inclusive cross sections of these contributions add up to 0

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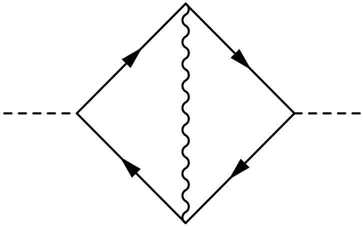

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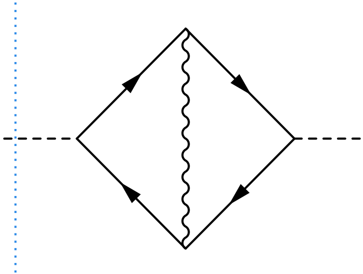

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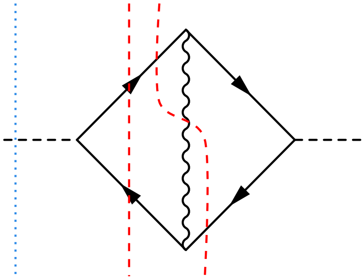

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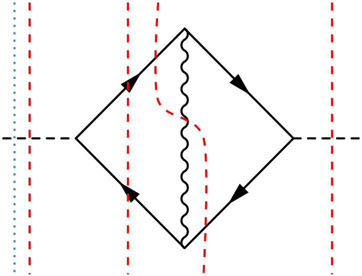

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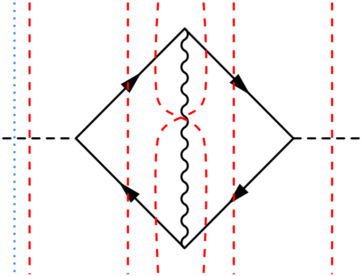

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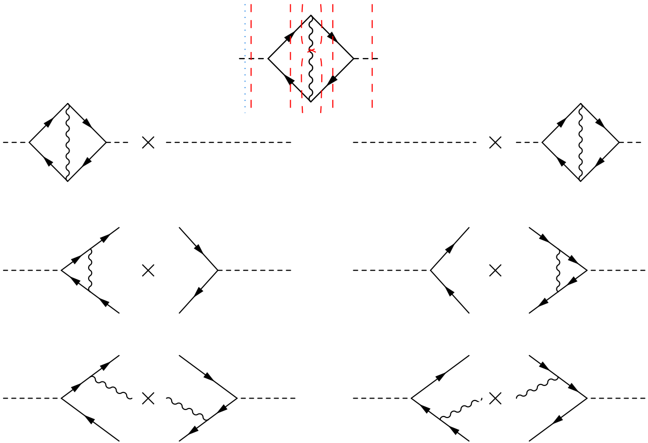

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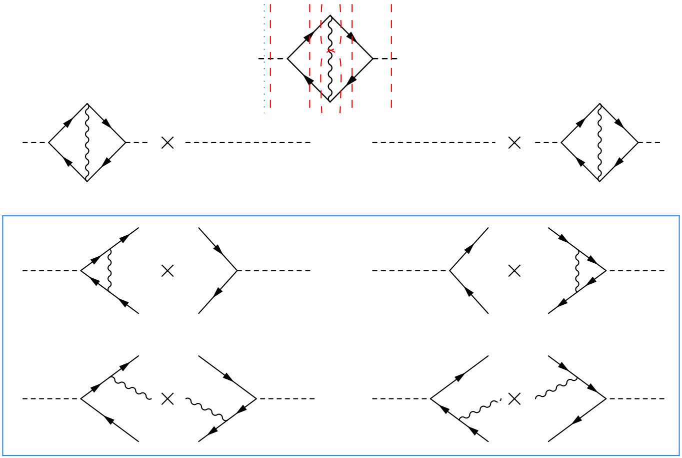

Previously: Sum of virtual loop and absorption diagrams are IR finite

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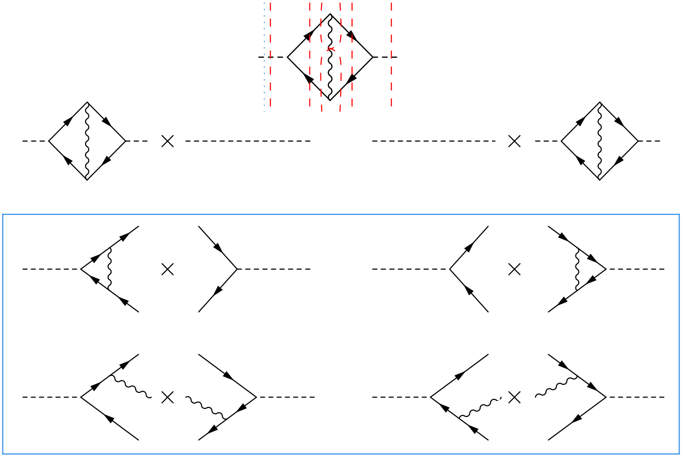
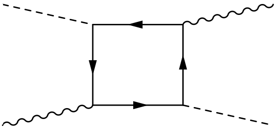


Previously: Sum of virtual loop and absorption diagrams are IR finite

Forward scattering diagrams $Z \to Z$ are IR finite

3 ways of making $e^+e^- \to Z$ finite:
3. Initial state sum: Real emission

Procedure: Fix final state and make all possible initial state cuts through a squared diagram



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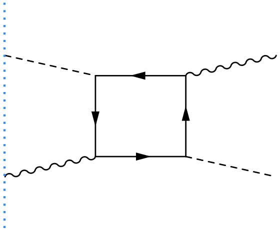

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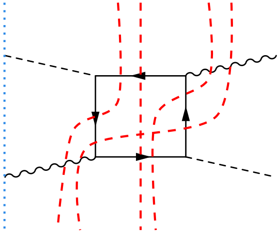

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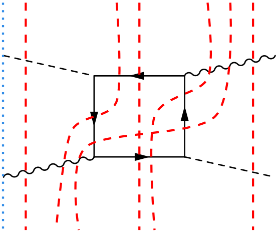

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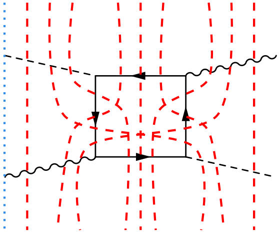

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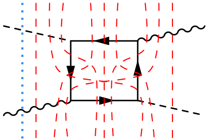
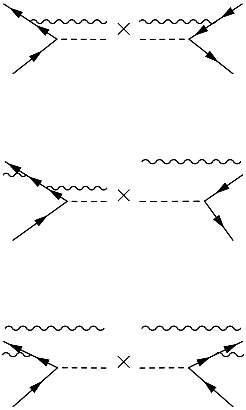
The cuts correspond to IR finite sum of disconnected diagrams





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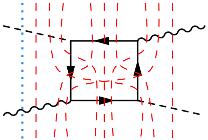

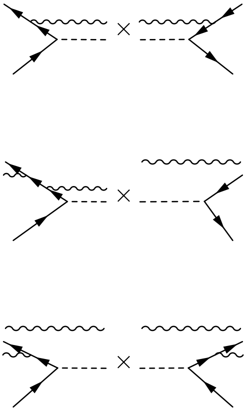

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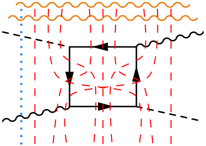
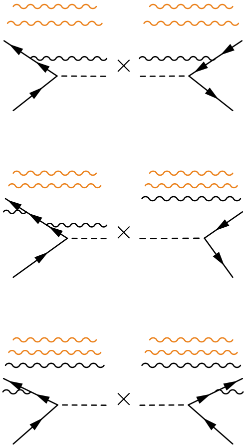
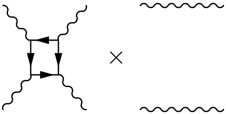




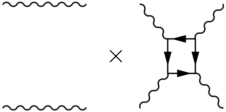
Forward scattering diagrams $Z+{\color{blue}n}\,\gamma \to Z+{\color{blue}n}\,\gamma$ are IR finite

$\gamma\gamma \to \gamma\gamma$ scattering



$$\sigma \propto \frac{1}{2\,\epsilon}$$

$m_e = 0$
Dim reg
CM frame



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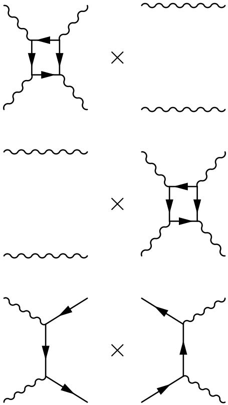

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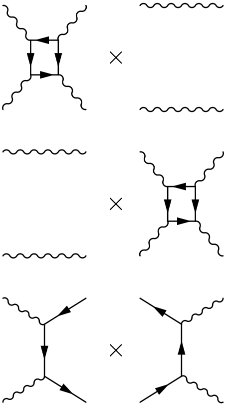


$$\sigma \propto \frac{1}{2\,\epsilon}$$

$$\sigma \propto \frac{1}{2\,\epsilon}$$

$$\sigma \propto -\frac{1}{\epsilon}$$

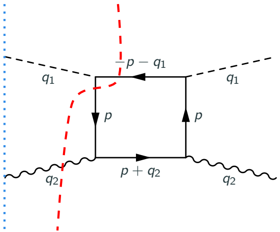
Rate to produce no charged particles
in photon collisions is not IR safe

Conclusions

- Stronger KLN theorem: Transition amplitude squared is IR finite when summing over final states **or** initial states.
- Need forward scattering and disconnected diagrams for finiteness.
- 3 ways of making $e^+e^- \to Z$ finite:
  1. With infinite number disconnected photons, but no forward scattering
  2. Final state sum, including forward scattering
  3. Initial state sum, including forward scattering
- IR divergence in $\gamma\gamma \to \gamma\gamma$ scattering is cancelled by $\gamma\gamma \to e^+e^-$
- Need to revise our understanding of what is observable even in QED in the high-energy ($m_e \to 0$) limit.

Backup slides

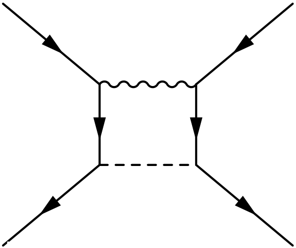
Singular propagator



1. Put cut propagators on-shell: Instead of $\int \frac{d^3p}{(2\pi)^3 2\omega_p}\frac{1}{p^2+i\epsilon}$ with $p^2=0$, write
$$\int \frac{d^4p}{(2\pi)^4}\frac{(2\pi)\delta(p^2)\theta(p^0)}{p^2+i\epsilon}$$
2. Modify cutting rules:
$$\frac{1}{p^2+i\epsilon} \to -2\pi i\delta(p^2-m^2)\theta(p^0)$$
$$\frac{1}{(p^2-m^2+i\varepsilon)^2} f\left(p^0\right) + \text{cc.} \to -2i\pi\delta\left(p^2-m^2\right)\theta\left(p^0\right) 2\omega_p \frac{d}{dp^0}\left[\frac{1}{(p^0+\omega_p)^2} f\left(p^0\right)\right]\Bigg|_{p^0=\omega_p}.$$

3 ways of making $e^+e^- \to Z$ finite:
2. Final state sum: Real emission

Procedure: Take any squared diagram, fix initial state and make all possible final state



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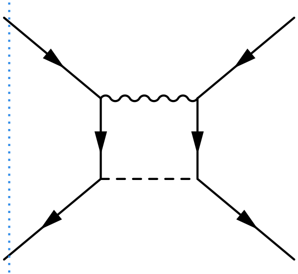

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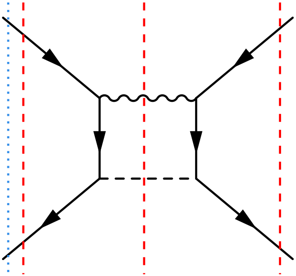

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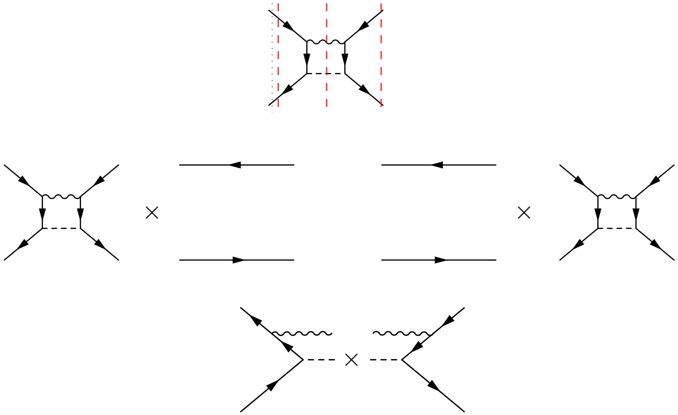

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