Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order

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SCET Workshop

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arXiv:1901.04424 and in preparation

Outline

1. Some challenges in gravitational wave physics.

2. Nontriviality of gravity calculations.

3. Modern idea from particle theory:
    — Effective field theory (EFT).
    — Generalized unitarity.
    — Double copy and duality between color and kinematics.

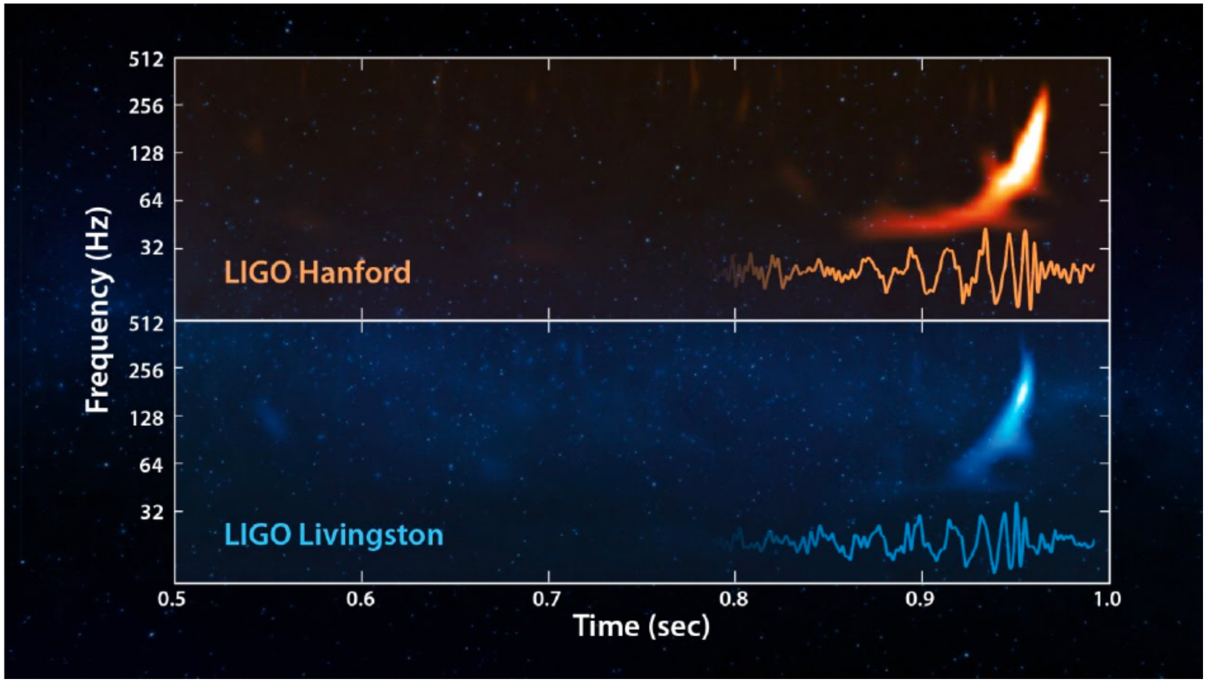
4. Can we calculate something beyond state of the art of direct interest to LIGO/Virgo?

5. 3rd post-Minkowskian order of conservative 2 body potential.

6. Prospects for future.

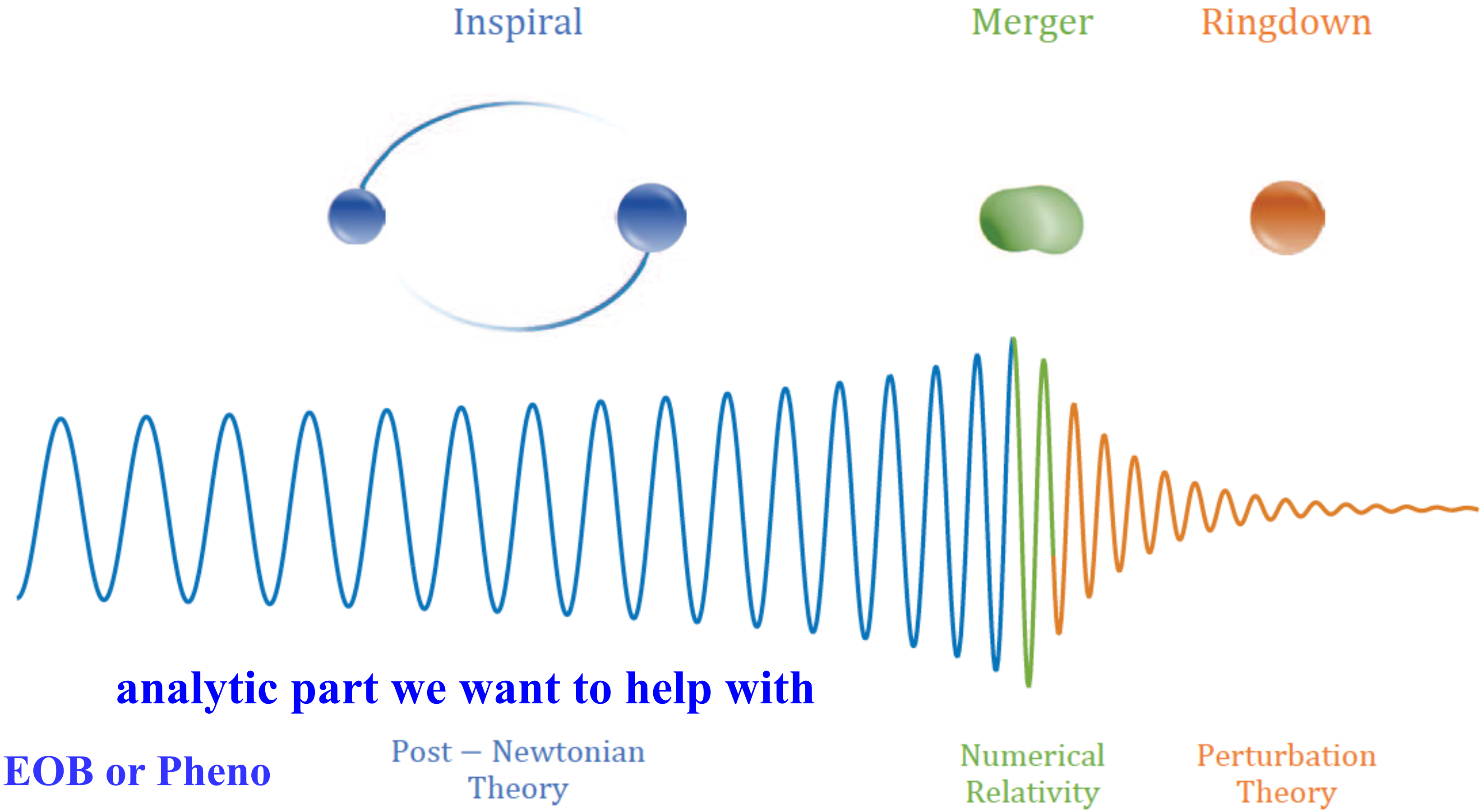
LIGO/Virgo and Templates

Era of gravitational wave astronomy begins!



Signal extracted and compared against ~250K templates computed from theory.

Goal: Improve on post-Newtonian Theory



analytic part we want to help with

PN + EOB or Pheno

Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Post Newtonian Approximation

For orbital mechanics:

Expand in *G* and v^2

$$v^2 \sim \frac{GM}{r} \ll 1$$

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.
4PN: Damour, Jaranowski and Schaefer (2014)

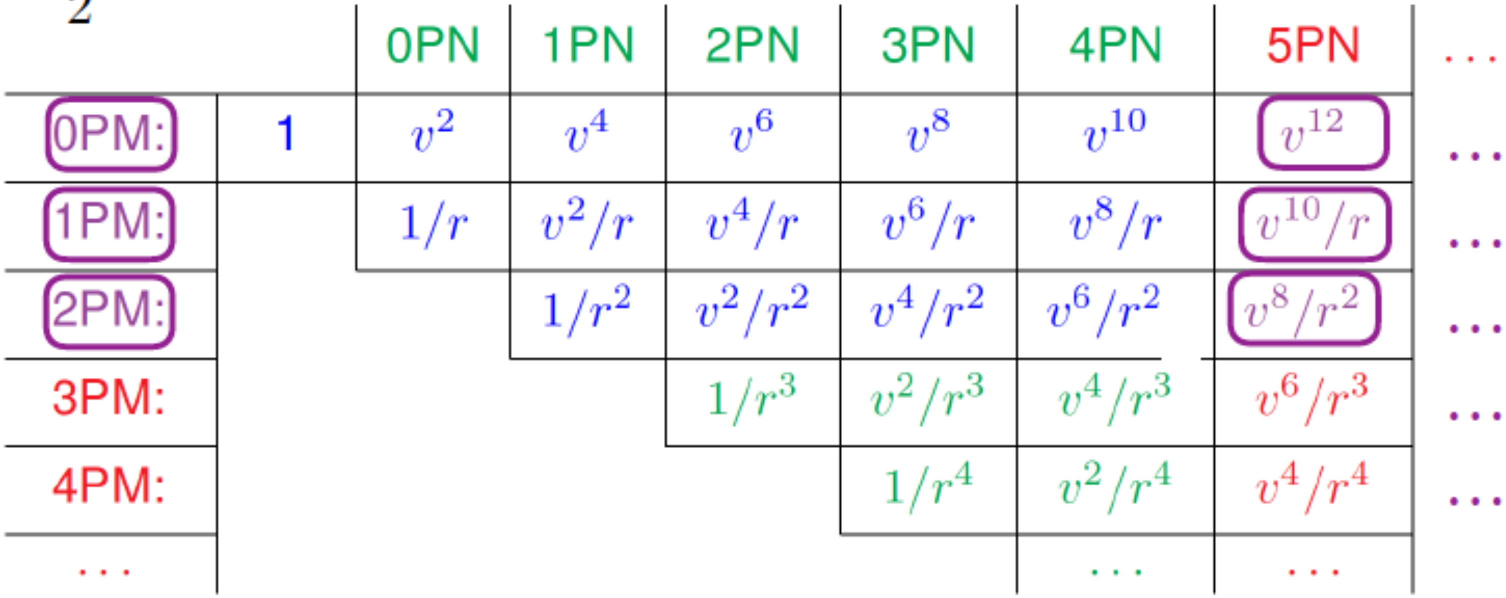
PN versus PM expansion for conservative two-body dynamics

From Buonanno Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\Big[\ldots\Big] + \frac{1}{c^4}\Big[\ldots\Big] + \ldots$$

$$E(v) = -\frac{\mu}{2}\,v^2 + \cdots$$

non-spinning compact objects



		0PN	1PN	2PN	3PN	4PN	5PN	…
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	…
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	…
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	…
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	…
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	…
…						…	…	

(credit: Justin Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results

current known PM results

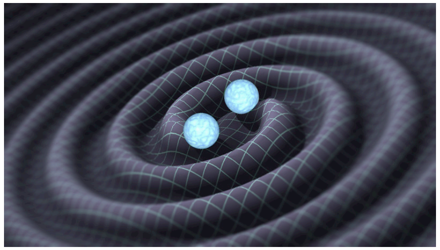
overlap between PN & PM results

unknown

- **PM results** (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

Which problem to solve?

Some problems for (analytic) theorists:



1. Spin.
2. Finite size effects.
3. Radiation.
4. High orders in perturbation theory.

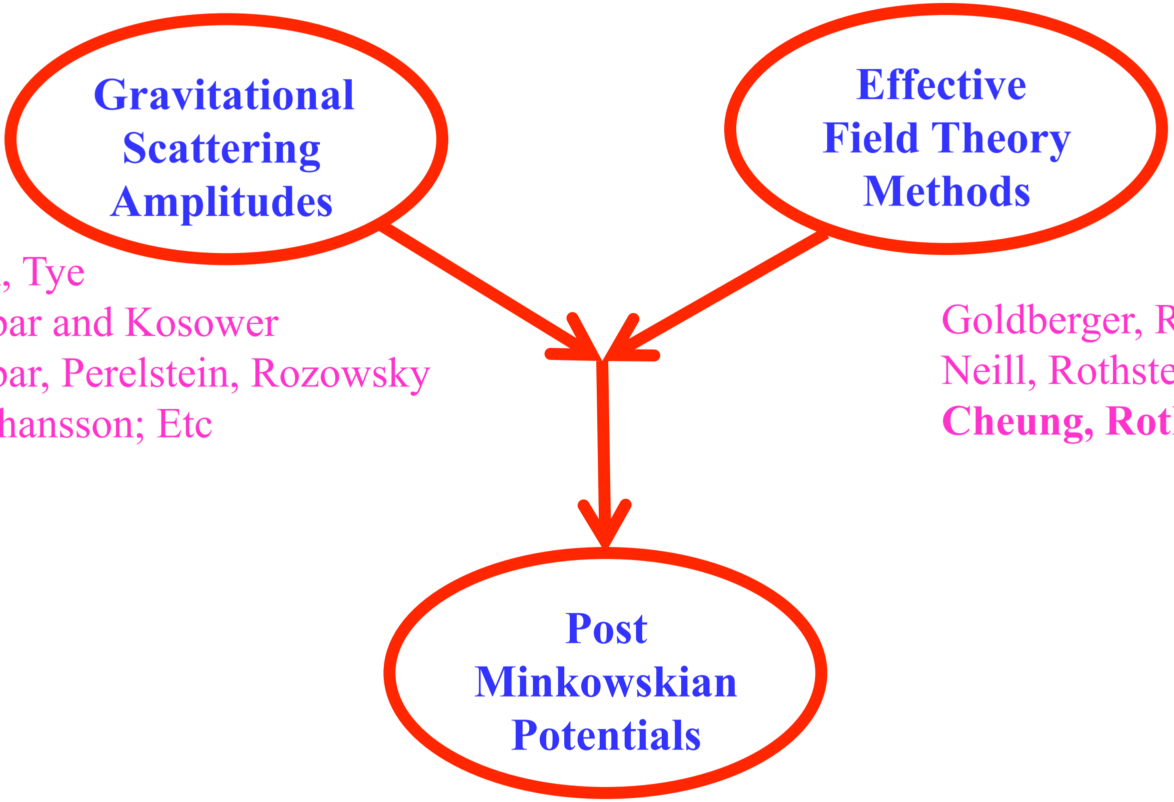
Which one to solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of direct interest to LIGO.
- Needs to be in a form that LIGO can use.

3rd post-Minkowskian order 2-body Hamiltonian

Our Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng



Amplitudes community

Gravitational Scattering Amplitudes

Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

EFT community

Effective Field Theory Methods

Goldberger, Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

Post Minkowskian Potentials

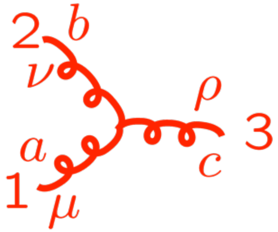
Inefficient: **Start with quantum theory and take $\hbar \to 0$**

Efficient: **Almost magical simplifications for gravity amplitudes.**
EFT methods efficiently target pieces we want.

Will show efficiency wins

Three Vertices

Standard Feynman diagram approach.



Three-gluon vertex:

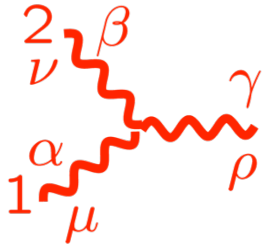
$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}
&\text{sym}[\,-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\
&\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\
&\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\
&\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

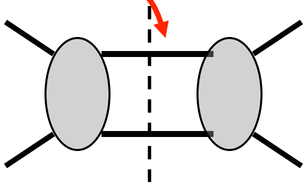
Generalized Unitarity Method

No Feynman rules; no need for virtual particles.

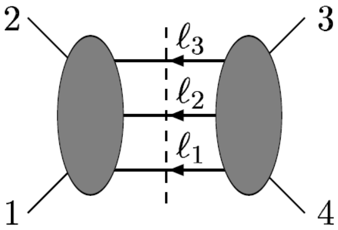
$$E^2 = \vec{p}^{\,2} + m^2 \longleftarrow \text{on-shell}$$

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

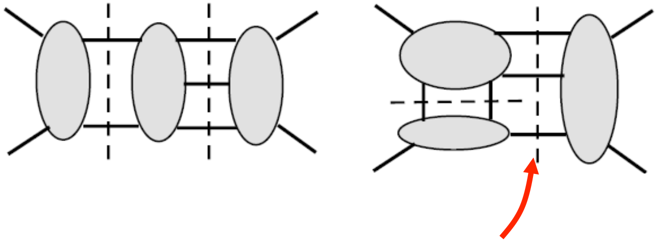


Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool.



on-shell

ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger
and many others

Reproduces Feynman diagrams, except intermediate steps of calculation based on gauge-invariant quantities.

Gravity Amplitudes

KLT (1985)

Kawai-Lewellen-Tye string relations in low energy limit:

gravity **gauge theory color ordered**

$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\,A_4^{\rm tree}(1,2,4,3)\,,$$

$$\begin{aligned} M_5^{\rm tree}(1,2,3,4,5) = {}& is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)\,A_5^{\rm tree}(2,1,4,3,5) \\ & + is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\,A_5^{\rm tree}(3,1,4,2,5) \end{aligned}$$

Pattern gives explicit all-leg form.

1. **Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.**
2. **It is very generally applicable.**

Duality Between Color and Kinematics

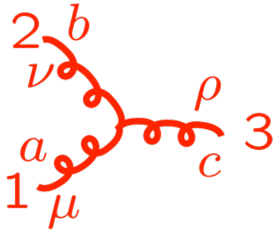
ZB, Carrasco, Johansson (2007)

coupling constant

color factor

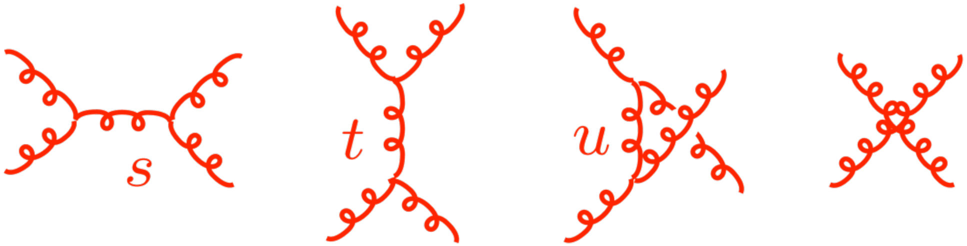
momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1+k_2)^2 \qquad t = (k_1+k_4)^2$

$u = (k_1+k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

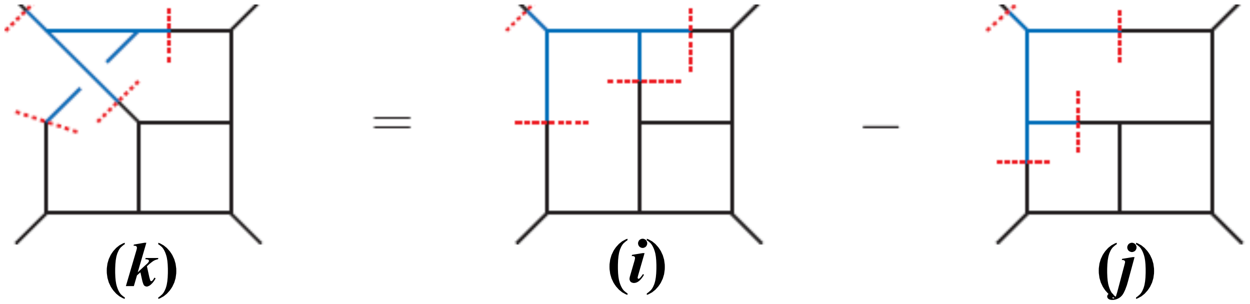
Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

BCJ

Gravity Loop Integrands from Gauge Theory

Ideas conjectured to generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

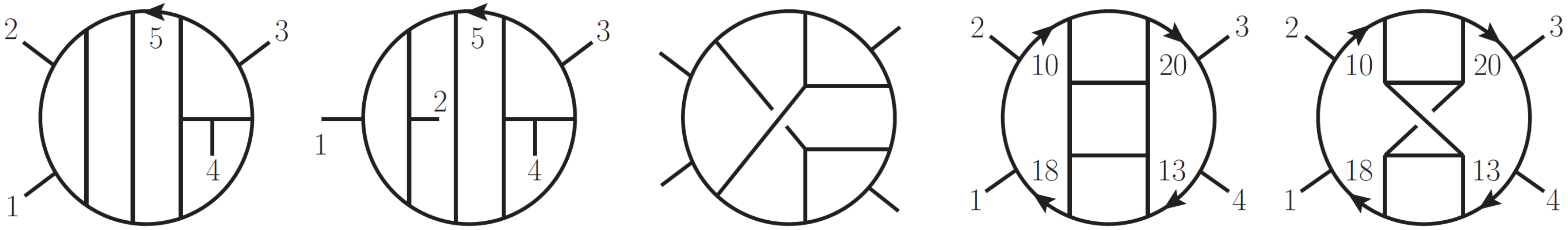
$$c_k \longrightarrow n_k$$

Gravity loop integrands follow from gauge theory!

$\mathcal{N} = 8$ Supergravity at Five loops

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

$\mathcal{N} = 8$ supergravity has best UV properties of supergravity theories.



Evaluated leading divergence behavior to help answer fundamental questions on nonrenormalizability of gravity theories.

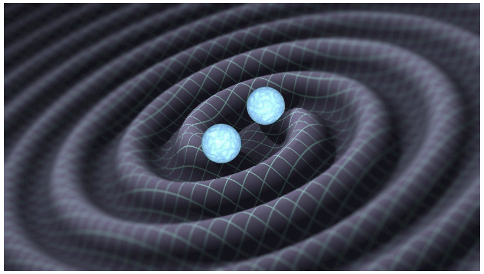
To make a long story short: Even 5 loop calculations are possible.

Key point: "Impossible" gravity calculations are pretty standard by now.

Want to harness these advances for LIGO

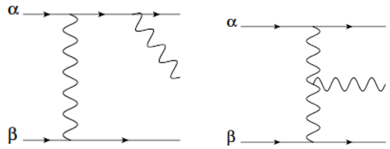
Double Copy and Gravitational Radiation

Can we simplify the types of calculations needed for LIGO?



A small industry has developed to study this.

- **Connection to scattering amplitudes.**
  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Plefka, Steinhoff, Wormsbecher



- **Worldline approach for radiation.**
  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen
- **Removing the dilaton contamination.**
  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

Key Question: Can we calculate something of direct interest to LIGO/Virgo, *beyond* state of the art?

PN versus PM expansion for conservative two-body dynamics

From Buonanno
Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\Big[\dots\Big] + \frac{1}{c^4}\Big[\dots\Big] + \dots$$

$$E(v) = -\frac{\mu}{2}\,v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

$$1 \rightarrow Mc^2, \qquad v^2 \rightarrow \frac{v^2}{c^2}, \qquad \frac{1}{r} \rightarrow \frac{GM}{rc^2}.$$

current known PN results

current known PM results

overlap between PN & PM results

unknown

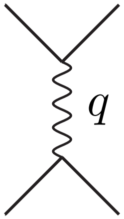
- **PM results** (*Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep*)

Potentials and Amplitudes

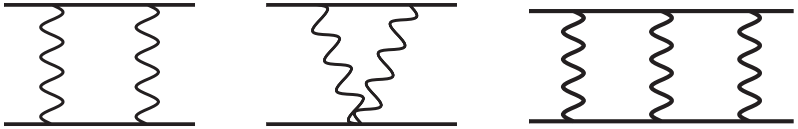
Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein
Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove

Tree-level: Fourier transform gives classical potential.

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



At higher orders things quickly become less obvious:



- **What you learned in grad school on $\hbar$ counting is wrong. Loops can have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar$ scaling of loop amplitudes.**
- **Non-uniqueness of potential.**
- **Cross terms between $1/\hbar$ and $\hbar$**

$$e^{iS_{\text{classical}}/\hbar}$$

$1/\hbar^L$ at L loops

Piece of loops are classical: Our task is to extract these pieces.

We harness EFT to clean up confusion

EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

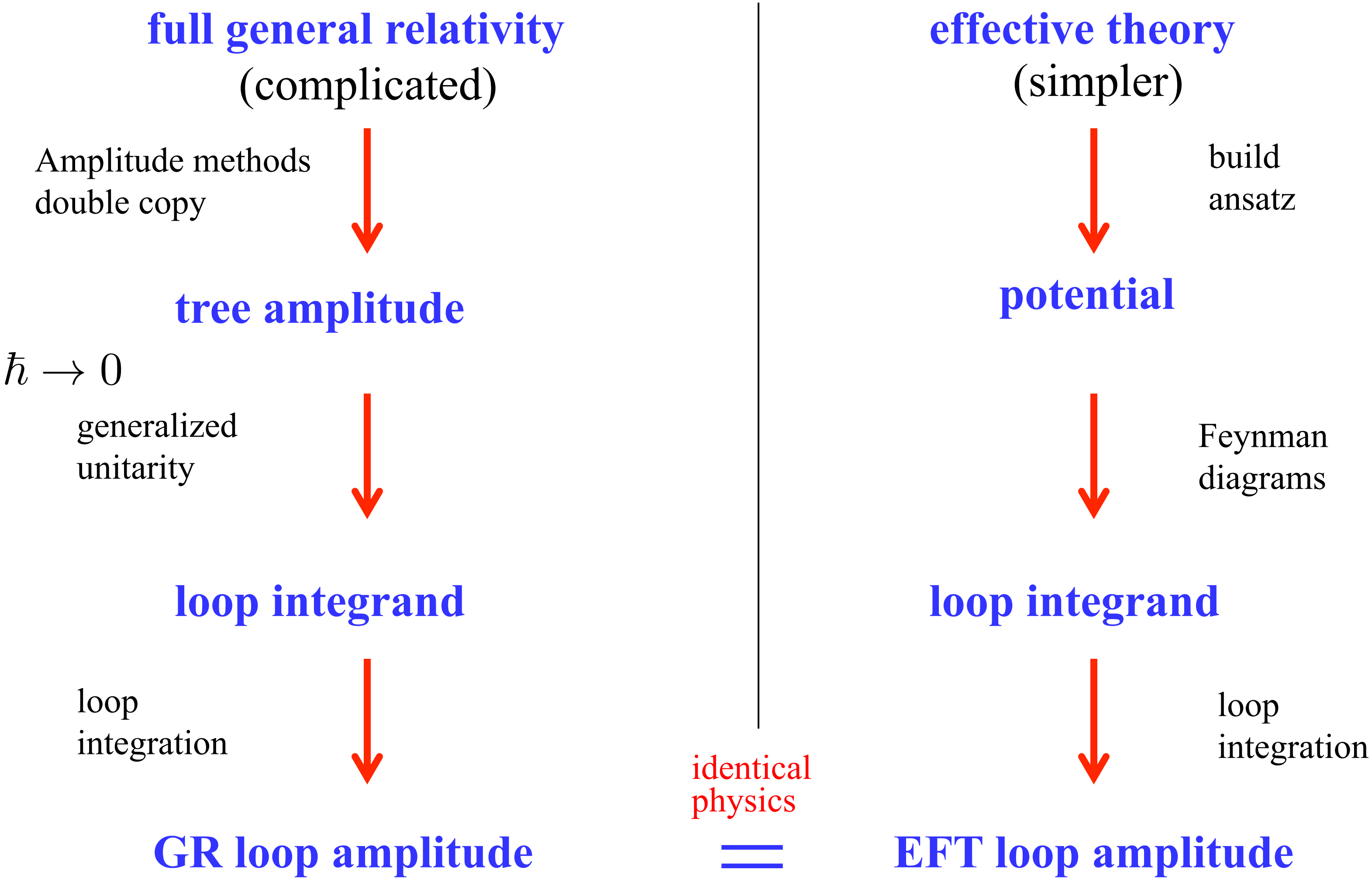
$$L_{\rm kin} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2} \right) A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2} \right) B(\boldsymbol{k})$$

A, B scalars represents spinless black holes

$$L_{\rm int} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}') A(\boldsymbol{k}) B^\dagger(-\boldsymbol{k}') B(-\boldsymbol{k})$$

Match amplitudes of this theory to the full theory in classical limit to extract a potential.

EFT Matching



full general relativity

(complicated)

Amplitude methods
double copy

tree amplitude

$\hbar \to 0$

generalized
unitarity

loop integrand

loop
integration

GR loop amplitude

effective theory

(simpler)

build
ansatz

potential

Feynman
diagrams

loop integrand

loop
integration

EFT loop amplitude

identical
physics

$=$

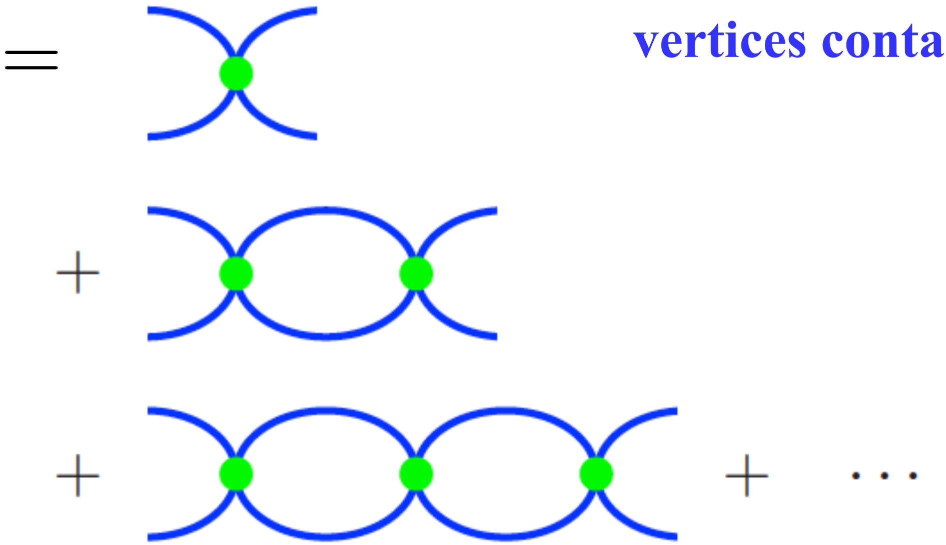
Roundabout but efficiently determines potential

Feynman diagrams for EFT

- **EFT scattering amplitudes easy to compute using Feynman diagrams.**
- **No need for advanced methods.**

Newton's constant

$$A_{\rm EFT} = \sum_{i=1}^{\infty} G^i A^{(i)}_{\rm EFT}$$



vertices contain all powers of *G*

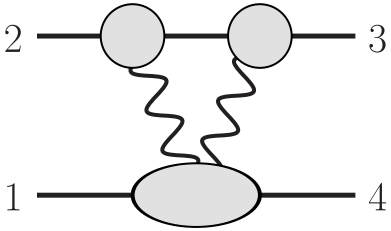
Match to Full Theory

Full Theory: Unitarity + Double Copy

- **Long range force: Two matter lines must be separated by cut propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

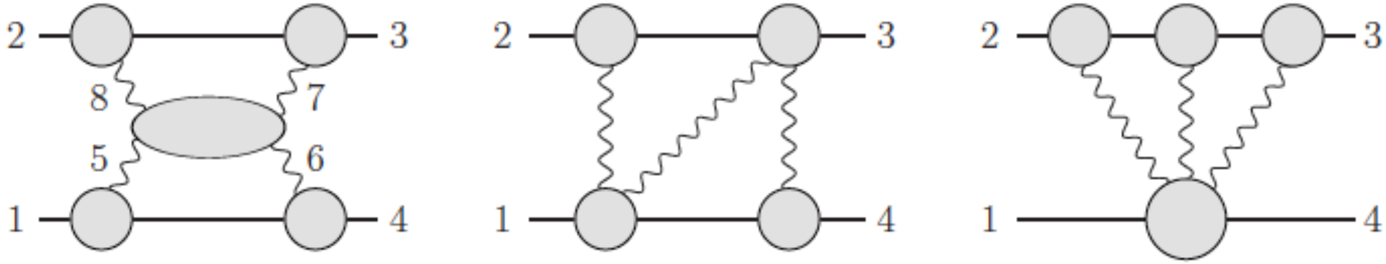
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM.



exposed lines on-shell (long range).
Classical pieces simple!

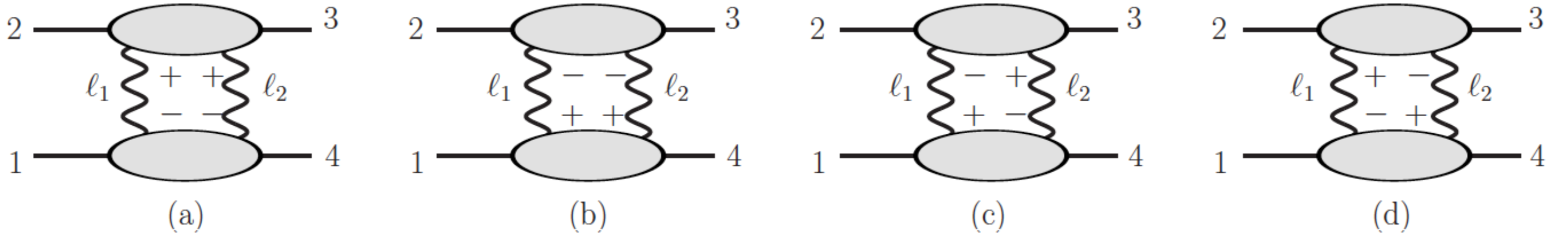
Independent generalized unitarity cuts for 3 PM.



Generalized Unitarity Cuts

Primary means of construction uses BCJ, but KLT should have better scaling at high loops and easier to explain:

2PM



$$C_{\rm GR} = \sum_{h_1,h_2=\pm} M^{\rm tree}(3,\ell_2^{h_2},-\ell_1^{h_1},2)\times M^{\rm tree}(1,\ell_1^{-h_1},-\ell_2^{-h_2},4)$$

$$= -\sum_{h_1,h_2=\pm} s_{23}^2[A^{\rm tree}(3,\ell_2^{h_2},-\ell_1^{h_1},2)\times A^{\rm tree}(1,\ell_1^{-h_1},-\ell_2^{-h_2},4)]$$

$$\times\,[A^{\rm tree}(2,\ell_2^{h_2},-\ell_1^{h_1},3)\times A^{\rm tree}(4,\ell_1^{-h_1},-\ell_2^{-h_2},1)]$$

By correlating gluon helicities, removing dilaton is trivial.

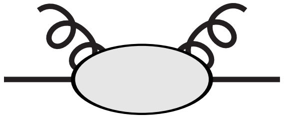
$h^-_{\mu\nu} \to A^-_\mu A^-_\mu$ $h^+_{\mu\nu} \to A^+_\mu A^+_\mu$ **Forbid:** $A^+_\mu A^-_\mu$

Problem of computing the generalized cuts in gravity is reduced multiplying and summing gauge-theory tree amplitudes.

Gauge-Theory Building Blocks for 2 PM Gravity

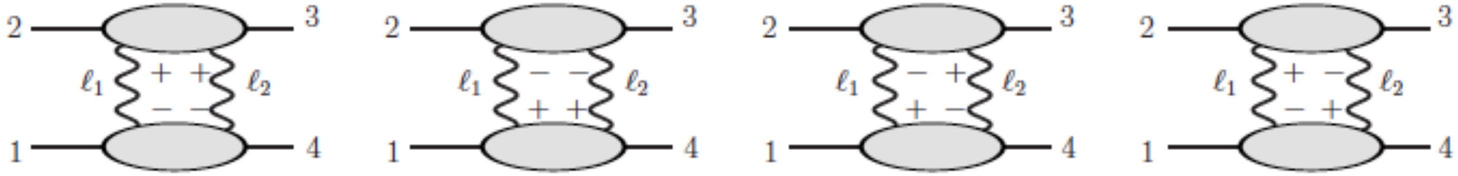
$$A^{\rm tree}(1^s,2^+,3^+,4^s) = i\frac{m_1^2[23]}{\langle 23\rangle t_{12}}$$

$$A^{\rm tree}(1^s,2^+,3^-,4^s) = -i\frac{\langle 3|1|2]^2}{\langle 23\rangle [23] t_{12}}$$



color-ordered gauge-theory tree amplitudes

- **This is all you need for 2 PM.**
- **Scaling with number of loops is very good.**



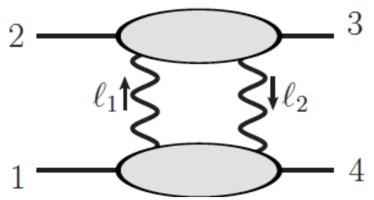
$$\boxed{C_{\rm YM} = 2\left(\frac{\mathcal{E}^2+\mathcal{O}^2}{s_{23}^2} + m_1^2 m_2^2\right)\frac{1}{t_{1\ell_1}\,t_{2\ell_1}}}$$

gauge theory

$$\mathcal{E}^2 = \frac{1}{4}\Big[-t_{12}s_{23} + s_{23}t_{1\ell_1} - s_{23}t_{2\ell_1} + 2t_{1\ell_1}t_{2\ell_1}\Big]^2$$

$$\mathcal{O}^2 = \mathcal{E}^2 - (s_{23}m_1^2 + s_{23}t_{1\ell_1} + t_{1\ell_1}^2)(s_{23}m_2^2 - s_{23}t_{2\ell_1} + t_{2\ell_1}^2)$$

One loop gravity warmup



Apply unitarity and KLT relations.
Import gauge-theory results.

$$\boxed{C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2\mathcal{O}^2\right) + m_1^4 m_2^4\right]\left[\frac{1}{t_{1\ell_1}} + \frac{1}{t_{4\ell_1}}\right]\left[\frac{1}{t_{2\ell_1}} + \frac{1}{t_{3\ell_1}}\right]}$$

- **Same building blocks as gauge theory!**
- **Double copy is visible even though we have removed dilaton and axion.**

We can extract classical scattering angles or potentials following literature

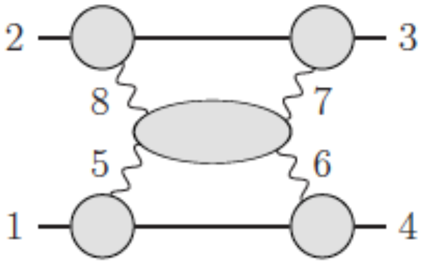
Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

This is 2nd PM order

$$s_{23} = (p_2 + p_3)^2$$
$$t_{ij} = 2p_i \cdot p_j$$

Two Loops for 3 PM

ZB, Cheung, Shen, Roiban, Solon, Zeng



- **Use KLT and sum over helicities**
- **Very similar to one loop**

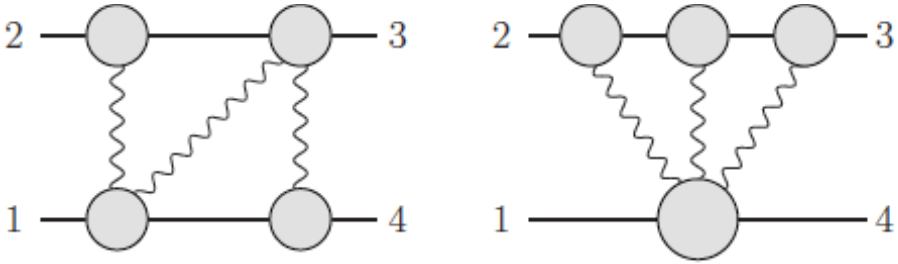
$$C^{\text{H-cut}} = 2i\left[\frac{1}{(p_5 - p_8)^2} + \frac{1}{(p_5 + p_7)^2}\right] \times \left[s_{23}^2 m_1^4 m_2^4 + \frac{1}{s_{23}^6} \sum_{i=1,2} \left(\mathcal{E}_i^4 + \mathcal{O}_i^4 + 6\mathcal{O}_i^2 \mathcal{E}_i^2\right)\right]$$

$$\mathcal{E}_1^2 = \frac{1}{4} s_{23}^2 (t_{18} t_{25} - t_{12} t_{58})^2, \qquad \mathcal{O}_1^2 = \mathcal{E}_1^2 - m_1^2 m_2^2 s_{23}^2 t_{58}^2,$$
$$\mathcal{E}_2^2 = \frac{1}{4} s_{23}^2 (t_{17} t_{25} - t_{12} t_{57} - s_{23}(t_{17} + t_{57}))^2,$$
$$\mathcal{O}_2^2 = \mathcal{E}_2^2 - m_1^2 m_2^2 s_{23}^2 t_{57}^2.$$

- **Double copy is visible.**
- **Remarkably simple, given it is two-loop gravity.**

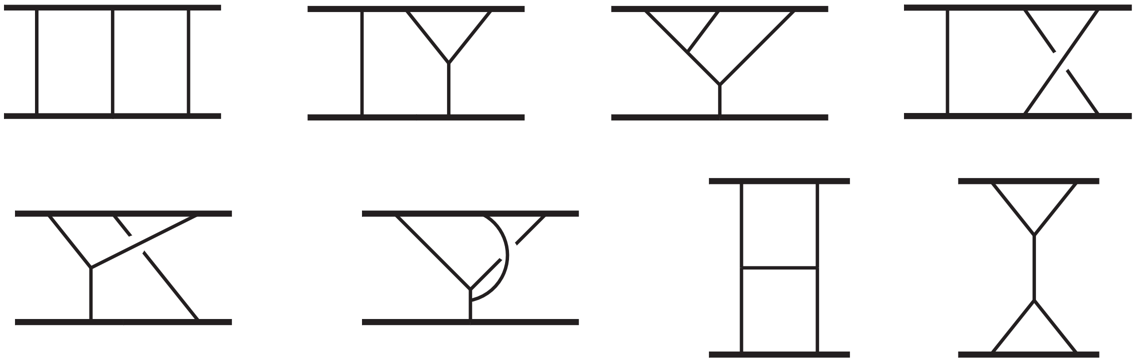
Two loops and 3 PM

Also need contributions from other cuts.



- **These unitarity cuts more complicated than previous cut.**
- **Evaluated using BCJ double copy, with KLT double copy as check.**
- **To interface easily with EFT approach merged unitarity cuts into diagrams to get integrand.**

Integrand organized into 8 independent diagrams that can contribute in classical limit:



Integration + Extraction of Potential

To integrate follow methods of Cheung, Rothstein and Solon.

- **Efficiently targets the classical pieces we want.**
- **Integrals reduce via residues to 3 dimensional integrals.**
- **Incorporates matching to effective field theory.**
- **Good scaling with perturbative order.**

Checks on integrals using standard tools of QCD:

- **Mellin-Barnes integration.** V. Smirnov; Czakon
- **Sector decompostion.** Binoth and Heinrich, A. Smirnov
- **Integration by parts.** K. G. Chetyrkin and F. V. Tkachov, Laporta; A. Smirnov; Maierhöfer, Usovitsch, Uwer
- **Differential equations.** ZB, Dixon, Kosower; Remiddi and Gehrmann
- **Method of regions.** Beneke, V. Smirnov; A. Smirnov.

Amplitude in Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Classical limit. The O(G^3) or 3PM terms are:

rapidity

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2\xi}\left[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \right.$$
$$\left. - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\right] + \frac{8\pi^3 G^3\nu^4 m^6}{\gamma^4\xi}\left[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\right]$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

F_1 and F_2 IR divergent iteration terms that don’t affect potential.

Result is very simple!

Conservative 3PM Potential

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Follow EFT strategy:

The 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\Bigg[& \frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ & - \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ & + \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg], \end{aligned}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

Checks

ZB, Cheung, Roiban, Shen, Solon, Zeng

Primary check:

Compare to 4PN Hamiltonian of Damour, Jaranowski, Schäfer

Need canonical transformation:

$$(\boldsymbol{r}, \boldsymbol{p}) \to (\boldsymbol{R}, \boldsymbol{P}) = (A\,\boldsymbol{r} + B\,\boldsymbol{p}, C\,\boldsymbol{p} + D\,\boldsymbol{r})$$

$$A = 1 - \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad B = \frac{G(1-2/\nu)}{4m|\boldsymbol{r}|}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots$$

$$C = 1 + \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad D = -\frac{Gm\nu}{2|\boldsymbol{r}|^3}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots\,,$$

For overlap terms of our Hamiltonian equivalent to 4PN Hamitonian. Explicit canonical transformation found.

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$\mathbf{n} = \hat{\mathbf{r}}$

$$\widehat{H}_{\mathrm{N}}\left(\mathbf{r},\mathbf{p}\right) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{1\mathrm{PN}}\left(\mathbf{r},\mathbf{p}\right) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{2\mathrm{PN}}\left(\mathbf{r},\mathbf{p}\right) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\left\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{1}{r}$$
$$+ \frac{1}{2}\left\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{3\mathrm{PN}}\left(\mathbf{r},\mathbf{p}\right) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\left\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3\right.$$
$$\left. + (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\right\}\frac{1}{r}$$
$$+ \left\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{1}{r^2}$$
$$+ \left\{\left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\right)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r^3} + \left\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\right\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$
\begin{aligned}
c^8\,\widehat{H}^{\text{local}}_{4\text{PN}}(\mathbf{r},\mathbf{p}) = {} & \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
& + \Bigg\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2 \\
& + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
& + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\Bigg\}\frac{1}{r} \\
& + \Bigg\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu \\
& + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
& + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\Bigg\}\frac{1}{r^2} \\
& + \Bigg\{\frac{105}{32}(\mathbf{p}^2)^2 + \Bigg(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu \\
& + \Bigg(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu^2 \\
& + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\Bigg\}\frac{1}{r^3} \\
& + \Bigg\{\frac{105}{32}\mathbf{p}^2 + \Bigg(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu \\
& + \Bigg(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu^2\Bigg\}\frac{1}{r^4} \qquad \longleftarrow \quad G^4 \\
& + \Bigg\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\Bigg\}\frac{1}{r^5}. \qquad \longleftarrow \quad G^5
\end{aligned}
$$

After canonical transformation we match all but G^4 and G^5 terms

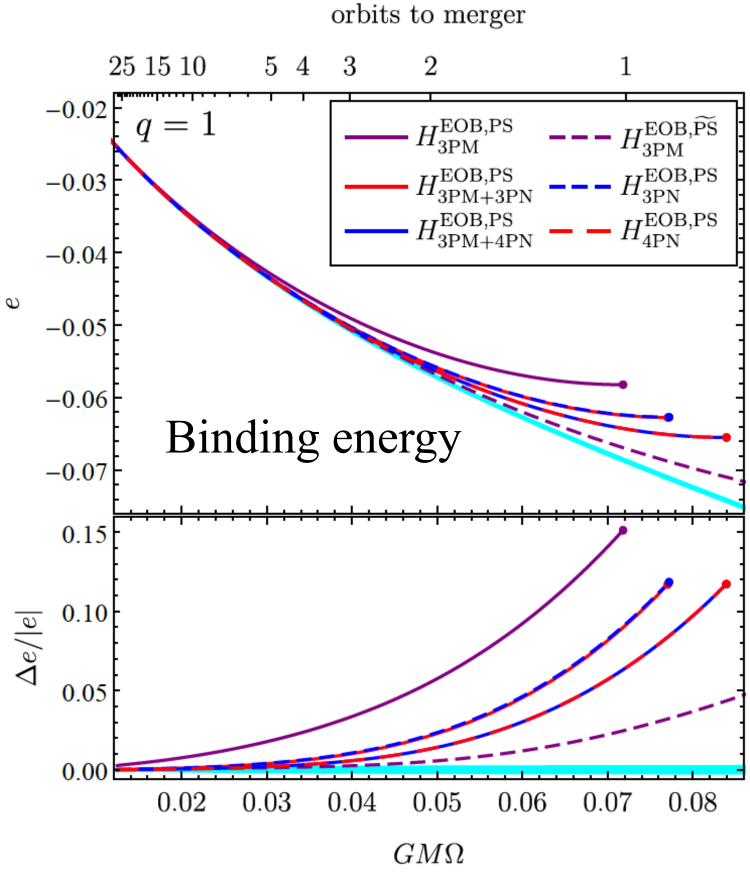
Mess is partly due to their gauge choice.

Ours is all orders in *p* at G^3

Tests of Our 3PM Hamiltonian for LIGO

Antonelli, Buonanno, Steinhoff, van de Meent, and Vines, arXiv:1901.07102

(8 days after our paper!)



Fed into EOB formalism.

Test against numerical relativity.

Note: Not conclusive, e. g. radiation not taken into accounted

Winning curve is based on 3PM.

numerical relativity taken as truth

"This rather encouraging result motivates a more comprehensive study…"

3PM + 4PN fed into EOB ➔ Most advanced 2 body Hamiltonian

Outlook for Gravitational Wave Physics

- **Methods are far from exhausted.**
- **Even more efficient methods seem likely.**
- **Methods should scale well to higher orders.**

Natural future questions to investigate:

- **Higher orders. Resummation in *G*.**
- **Radiation.**
- **Spin.**
- **Finite size effects.**

Summary

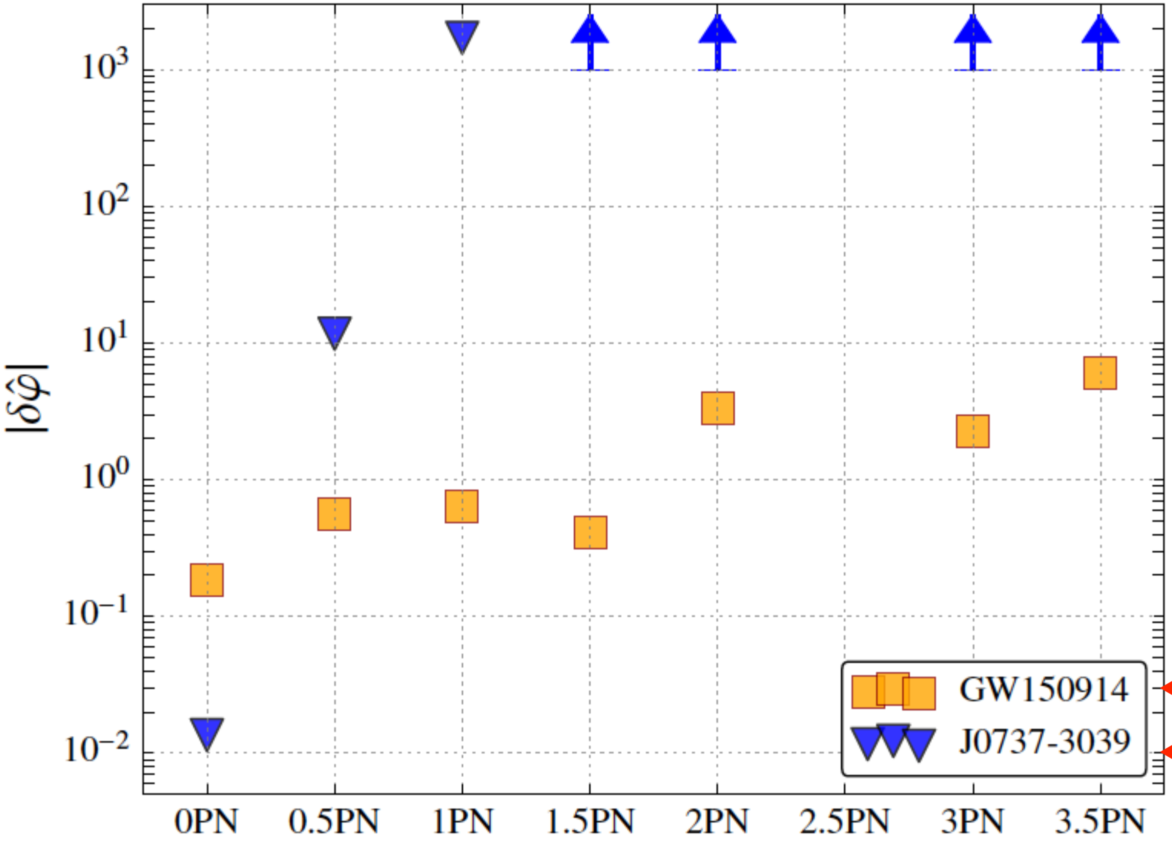
- Remarkable connection between gauge and gravity theories:
  - color ⟷ kinematics.
  - gravity ~ (gauge theory)^2
- Double-copy idea gives us a powerful new way to think about gravity. Unified framework for gravity and gauge theory.
- Obtained the 3PM conservative 2-body potential.
- Methods nowhere close to exhausted.
- Spin, finite size effects, radiation, and higher orders in G obvious possibilities to investigate.

Expect many more advances in coming years, not only for gravitational waves but also for understanding gravity and its relation to the other forces via double copy.

Extra Slides

Importance of higher orders for LIGO

LIGO/Virgo Collaboration arXiv:1602.03841



Binary pulsar confirms quadrupole radiation formula and not much else.

LIGO

Binary pulsar

Newton
(1686)

Einstein
(1938)

Ohta (1974)
Damour (1981)

Blanchet (2000)
Damour (2000)

Blanchet (2015)
Damour (2014)

LIGO/Virgo sensitive to high PN orders.

Gravity vs Gauge Theory

Consider the Einstein gravity Lagrangian

curvature

Flat-space metric

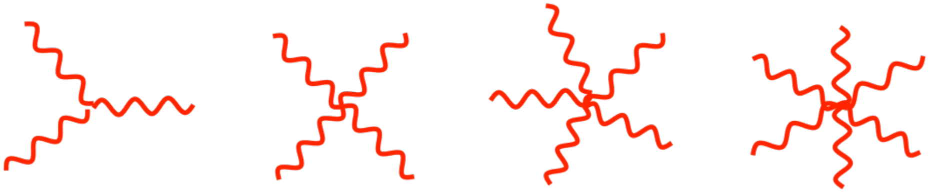
$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

graviton field

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions

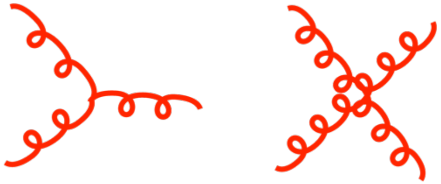


$+ \cdots$

terrible mess

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



Only three and four point interactions

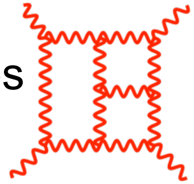
Gravity seems so much more complicated than gauge theory.

Theories do not look related!

Feynman Diagrams for Gravity

Spectacularly poor scaling in GR

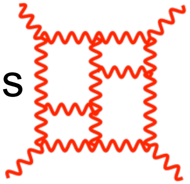
3 loops



$\sim 10^{20}$ TERMS

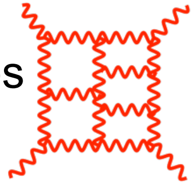
No surprise it has never been calculated via Feynman diagrams.

4 loops



$\sim 10^{26}$ TERMS

5 loops



$\sim 10^{31}$ TERMS

More terms than atoms in your brain!

- Such calculations seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

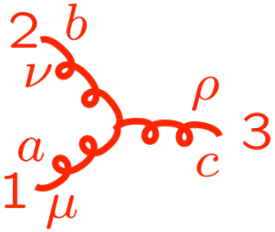
Modern methods make such calculations routine, but challenging.

Simplicity of Gravity Amplitudes

**People were looking at gravity amplitudes the wrong way.
On-shell viewpoint much more powerful.**

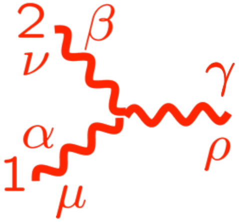
On-shell **three vertices contains all information:** $k_i^2 = 0$

Yang-Mills gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times\ (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

“square” of Yang-Mills vertex.

Very simple!

Duality Between Color and Kinematics

Consider five-point tree amplitude:

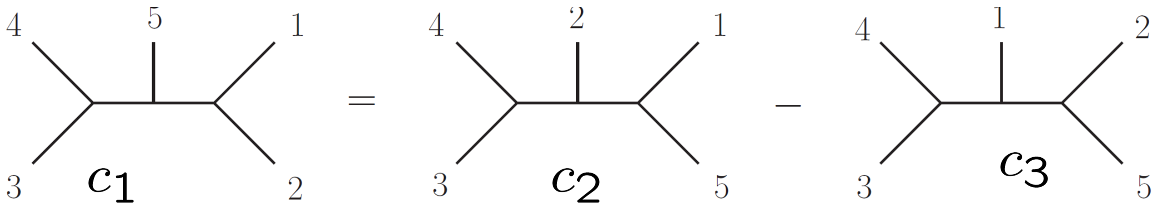
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 + c_2 + c_3 = 0 \;\; \Leftrightarrow n_1 + n_2 + n_3 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Progress on unraveling relations.

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer;
Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer
O'Connell and Montiero; Bjerrum-Bohr, Damgaard, O'Connell and Montiero; O'Connell, Montiero, White;
Du, Feng and Teng, Song and Schlotterer, etc.

Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

color factor

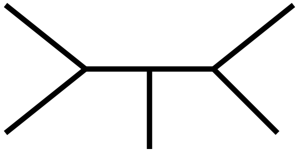
kinematic numerator factor

gauge theory: $\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$

Feynman propagators

Einstein gravity: $\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$



sum over diagrams with only 3 vertices

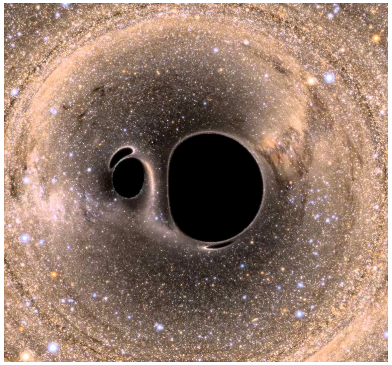
Gravity and QCD kinematic numerators are the same!

Cries out for a unified description of gravity with gauge theory.

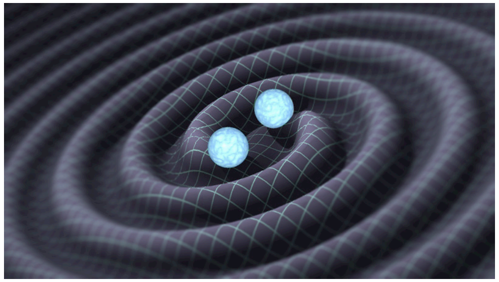
Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:



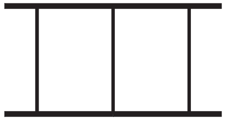
- **Schwarzschild and Kerr black holes.**
- **Taub-NUT space.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**



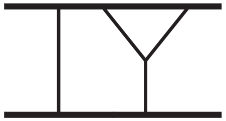
Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;
Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell, etc

Still no general understanding.
But plenty of examples.

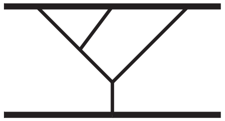
Two-Loop Diagram Numerators



$$(t_{12}^2 - 2m_1^2 m_2^2)^3$$



$$2m_2^3 t_{47}^2 (t_{12}^2 - 2m_1^2 m_2^2)$$



$$\begin{aligned}
&2m_2^4(s_{23}^4 + s_{23}^3(2t_{12} + 2t_{15} - t_{47} - 2t_{67}) - 2m_1^2 m_2^2(s_{23} - t_{67})^2 + (t_{15}t_{56} + (t_{12} - t_{47})t_{67})^2 \\
&+ s_{23}^2(t_{12}^2 + t_{15}^2 + t_{47}^2 - t_{47}t_{56} + t_{12}(4t_{15} - 2t_{47} + t_{56} - 4t_{67}) + t_{15}(-2t_{47} + t_{56} - 2t_{67}) \\
&+ 2t_{47}t_{67} + t_{67}^2) + s_{23}(t_{15}(t_{56}^2 + 2(-2t_{12} + t_{47})t_{67} - t_{56}t_{67}) \\
&+ t_{67}(-2t_{12}^2 + t_{47}(-2t_{47} + t_{56} - t_{67}) + t_{12}(4t_{47} - t_{56} + 2t_{67}))))
\end{aligned}$$

etc. Remaining 5 diagrams somewhat more complicated but not a big deal.

- **Very simple compared to the usual Feynman diagram explosion.**
- **Higher-loop integrand constructions definitely possible!**

Additional Tests

Additional (somewhat redundant) tests:

1. Calculated classical scattering angle (ignoring radiation reaction). Match overlap terms in known 4PN result.

Bini and Damour

2. Calculated amplitude using potentials. Match overlap terms using known 4PN Hamitonian.

Damour, Jaranowski, Schaefer

3. In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian.

Wex and Schaefer

Gives us confidence that our Hamiltonian is correct and lines up with LIGO's template construction.