Threshold resummation at NLP: Drell-Yan $q\bar{q}$ channel and $gg \to H$

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SCET 2019
25-28 March
UC San Diego

Leading-logarithmic threshold resummation of the Drell-Yan process at next-to-leading power

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Robert Szafron, Leonardo Vernazza and Jian Wang

arXiv:1809.10631

- Operator basis and renormalization
  ↪ see talk by Martin
- Factorization theorem and evaluation of collinear functions
  ↪ see talk by Sebastian

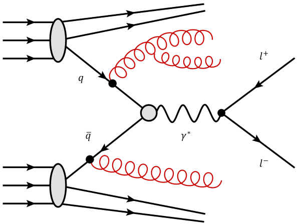
Outline

- Hard function
- Kinematic corrections
- Soft function
- Fixed order expansion
- Higgs threshold production

$$\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \alpha_s^n \Bigg[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \Bigg]$$

- Leading power
- Next-to-leading power
- Leading Log: $m = 2n-1 \longrightarrow \alpha_s \ln(1-z) + \alpha_s^2 \ln^3(1-z) + \dots$

DY cross-section



$$A(p_A)B(p_B) \to \mathrm{DY}(Q) + X$$

$z = \frac{Q^2}{\hat{s}}$ threshold $z \to 1$

$$\Omega \sim Q(1-z)$$

Factorization theorem valid at *LL accuracy*

$$\begin{aligned}\hat{\sigma}(z) &= H(\hat{s}) \times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}} \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\ &\quad \times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}\end{aligned}$$

Scales:

- hard $\mu_h \sim Q$
- collinear $\mu_c \sim \sqrt{Q\Omega}$ (no LL collinear contribution at NLP)
- soft $\mu_s \sim \Omega$

$Q \gg \Omega$

Hard Function

$$\begin{aligned}
\hat{\sigma}(z) &= \boxed{H(\hat{s})} \\
&\times \quad Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\
&\times \quad \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}
\end{aligned}$$

Hard function

When considering power corrections we have to be careful about kinematic factors.
Consider the hard function

$$\bar{\psi}\gamma_\mu\psi(0) = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, J_\mu^{A0}(t,\bar{t}), \qquad H(\hat{s},\mu_h) = |C^{A0}(-\hat{s})|^2$$

We can obtain power corrections from expansion of $\hat{s}$ around Q^2

$$H(\hat{s}) = H(Q^2) + Q^2(1-z)H'(Q^2) + \dots$$

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$$H(\hat{s}) = H(Q^2) + Q^2(1-z)H'(Q^2) + \ldots$$

This correction modifies the LP factorization

$$\hat{\sigma}(z) = H(Q^2)\, QS_{\rm DY}(Q(1-z)) + Q^2(1-z)H'(Q^2)\, QS_{\rm DY}(Q(1-z))$$

with

$$H(\hat{s}) = 1 + \mathcal{O}\left(\alpha_s \ln^2\left(\frac{\mu}{\mu_h}\right)\right) \quad \text{and} \quad S_{\rm DY}(\Omega) = \delta(\Omega) + \mathcal{O}\left(\alpha_s\right)$$

This contributions starts at $\alpha_s^2 \ln^2\left(\frac{\mu}{\mu_h}\right)$ so at the LL accuracy it is enough to replace $H(\hat{s})$ by $H(Q^2)$

Kinematic corrections

$$
\begin{aligned}
\hat{\sigma}(z) &= H(\hat{s}) \\
&\times \boxed{Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}} \\
&\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}
\end{aligned}
$$

Kinematic corrections I

At LP we only need the soft function at $x = x_0$ but for now consider the soft function for generic x

$$\widetilde{S}_0(x) = \frac{1}{N_c} \operatorname{Tr} \langle 0 | \bar{\mathbf{T}}(Y_+^\dagger(x) Y_-(x)) \, \mathbf{T}(Y_-^\dagger(0) Y_+(0)) | 0 \rangle$$

Use partonic center-of-mass frame $x_a \vec{p}_A + x_b \vec{p}_B = 0$
Momentum $\vec{p}_{X_s}$ of the soft hadronic final state is balanced by the lepton-pair $\vec{q} + \vec{p}_{X_s} = 0$

$$\vec{q} \sim \lambda^2, \quad q^0 = \sqrt{\hat{s}} + \mathcal{O}(\lambda^2)$$

Energy of the soft radiation

$$[x_1 p_1 + x_2 p_2 - q]^0 = p_{X_s}^0 = \sqrt{\hat{s}} - \sqrt{Q^2 + \vec{q}^{\,2}} = \frac{\Omega_*}{2} - \frac{\vec{q}^{\,2}}{2Q} + \mathcal{O}\left(\lambda^6\right)$$

with

$$\Omega_* = 2Q \frac{1 - \sqrt{z}}{\sqrt{z}} = Q(1-z) + \frac{3}{4} Q (1-z)^2 + \mathcal{O}\left(\lambda^6\right)$$

Kinematic corrections II

Expansion of the kinematic factors leads to

$$Q \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}\, \widetilde{S}_0(x)$$

$$\to \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega_*/2} \left(1 + \frac{ix^0 \partial^2_{\vec{x}}}{2Q} + \mathcal{O}\left(\lambda^4\right)\right) \widetilde{S}_0(x^0,\vec{x})_{|\vec{x}=0}$$

$$\to S_{\mathrm{DY}}(Q(1-z)) + \frac{1}{Q} S_{K1}(Q(1-z)) + \frac{1}{Q} S_{K2}(Q(1-z)) + \mathcal{O}(\lambda^4)$$

NLP kinematic soft functions

$$\begin{aligned} S_{K1}(\Omega) &= \frac{\partial}{\partial\Omega}\partial^2_{\vec{x}} S_0\left(\Omega,\vec{x}\right)_{|\vec{x}=0} \\ S_{K2}(\Omega) &= \frac{3}{4}\,\Omega^2 \frac{\partial}{\partial\Omega} S_0(\Omega,\vec{x})_{|\vec{x}=0} \end{aligned}$$

Kinematic corrections III

It is more convenient to introduce

$$\Delta_{ab}(z) = \frac{\hat{\sigma}_{ab}(z)}{z}$$

$\Delta_{ab}^{\mathrm{LP}}(z) = \hat{\sigma}_{ab}^{\mathrm{LP}}(z)$ but $\Delta_{ab}^{\mathrm{NLP}}(z)$ receives additional NLP correction

$$(1-z) \times \hat{\sigma}_{\mathrm{LP}}(z)$$

which leads to

$$S_{K3}(\Omega) = \Omega\, S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Factorization theorem for $\Delta(z) = \Delta_{q\bar{q}}(z)$:

$$\begin{aligned} \Delta(z) &= H(Q^2) \\ &\times Q\left\{ S_{\mathrm{DY}}(Q(1-z)) + \sum_{i=1}^{3} \frac{1}{Q} S_{Ki}(Q(1-z)) \right. \\ &\left. +2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x, \omega) + \bar{c}\text{-term} \right\} \end{aligned}$$

No further expansion in λ is needed!

Example: expansion of the soft function RGE I

In position space, renormalization of the LP soft function is multiplicative

$$\frac{d}{d\ln\mu}\,\widetilde{S}_0(x) = \Big[2\Gamma_{\text{cusp}}L - 2\gamma_W\Big]\,\widetilde{S}_0(x)$$

$$L \equiv \ln\left(-\frac{1}{4}n_-x n_+x\mu^2 e^{2\gamma_E}\right)$$

$$\gamma_W = \mathcal{O}(\alpha_s^2)$$

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Expansion of the soft function, $x = (x^0, 0, 0, z)$

$$\widetilde{S}_0(x) = \widetilde{S}_0(x_0) + \ldots + \frac{1}{2}\vec{\partial}_z^{\,2}\,\widetilde{S}_0(x)_{|\vec{x}=0}\,z^2 + \ldots$$

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Expansion of the log generates inhomogeneous term

$$L = L_0 - \frac{z^2}{(x^0)^2} + \mathcal{O}\left(\frac{z^4}{(x^0)^4}\right)$$

$$L_0 \equiv \ln\left(-\frac{1}{4}(x^0)^2\mu^2 e^{2\gamma_E}\right)$$

Example: expansion of the soft function RGE II

Coefficient of z^2 gives

$$\frac{d}{d\ln\mu}\,\frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} = \Big[2\Gamma_{\text{cusp}}L_0 - 2\gamma_W\Big]\,\frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} - \frac{2}{(x^0)^2}\widetilde{S}_0(x_0)$$

Define soft functions

$$\begin{aligned}\widetilde{S}_3(x_0) &= \frac{ix_0}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}\\ \widetilde{S}_{x_0}(x_0) &= \frac{-2i}{x^0 - i\varepsilon}\widetilde{S}(x_0)\end{aligned}$$

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Soft functions mix

$$\begin{aligned}\frac{d}{d\ln\mu}\widetilde{S}_3(x_0) &= \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\widetilde{S}_3(x_0) + \widetilde{S}_{x_0}(x_0)\\ \frac{d}{d\ln\mu}\widetilde{S}_{x_0}(x_0) &= \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\widetilde{S}_{x_0}(x_0)\end{aligned}$$

Note: $\widetilde{S}_3(x_0) = \mathcal{O}\left(\alpha_s L_0\right)$ and $\widetilde{S}_{x_0}(x_0) = 1 + \mathcal{O}\left(\alpha_s L_0^2\right)$

$\widetilde{S}_{x_0}(x_0)$ corresponds to θ-soft previously function defined in JHEP 1808 (2018) 013 by Ian Moult, Iain Stewart, Gherardo Vita, Hua Xing Zhu

RGE for kinematic soft functions

Proceeding like in the example we obtain

$$\begin{aligned}\frac{d}{d\ln\mu}\,\vec{S}(x^0) &= \left[2\Gamma_{\text{cusp}}L_0 - 2\gamma_W\right]\mathbf{1}\,\vec{S}(x)\\ &+ \Gamma_{\text{cusp}}\begin{pmatrix} 0 & 0 & 0 & +1\\ 0 & 0 & -6 & +3\\ 0 & 0 & 0 & -4\\ 0 & 0 & 0 & 0\end{pmatrix}\vec{S}(x^0)\end{aligned}$$

with $\vec{S}(x^0) = \left(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$

$$\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) = \left[2\Gamma_{\text{cusp}}L_0 - 2\gamma_W\right]\widetilde{S}_{K1+K2+K3}(x^0) - 6\,\Gamma_{\text{cusp}}\,\widetilde{S}_{K3}(x^0)\,,$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}(\alpha_s)$

No LL kinematic corrections to all orders!

Soft function

$$
\begin{aligned}
\hat{\sigma}(z) &= H(\hat{s}) \\
&\times \quad Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\
&\times \quad \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \boxed{\widetilde{S}_{2\xi}(x,\omega)} + \bar{c}\text{-term} \right\}
\end{aligned}
$$

Time-ordered products

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\,(x, z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x) Y_-(x)\right] \mathbf{T}\left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_-\partial} \mathcal{B}^+_{\perp\nu}(z_-)\right]$$

with decoupled soft fields

$$\mathcal{B}^\mu_\pm = Y_\pm^\dagger \left[iD_s^\mu Y_\pm\right]$$

Lagrangian is already multipole expanded $\rightarrow$ soft fields depend only on z_-

$$\mathcal{L}^{(2)}_{2\xi} = \frac{1}{2} \bar{\chi}_c z_\perp^\mu z_\perp^\nu \left[i\partial_\nu in_-\partial \mathcal{B}^+_\mu\right] \frac{\not{n}_+}{2} \chi_c$$

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In the factorization theorem we need only vacuum matrix element

$$S_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi}\, e^{ix^0\Omega/2 - i\omega(n_+z)/2}\, \frac{1}{N_c} \operatorname{Tr} \langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^0, z_-)|0\rangle$$

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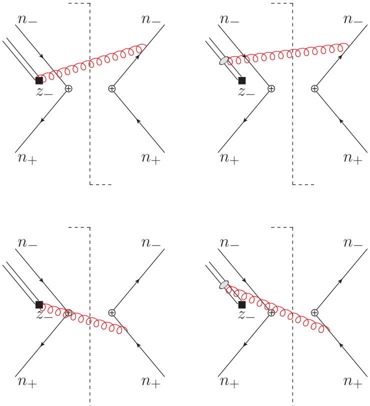

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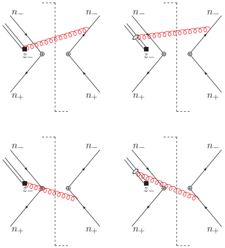


$$S_{2\xi}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\}$$

Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Renormalization through mixing with the same S_{x_0} as in the case of kinematic corrections

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega')$ at one loop?

Soft operator

Let us consider an operator rather than its matrix element

$$\mathcal{S}_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi}\int\frac{d(n_+z)}{4\pi} e^{i\left(x^0\Omega - n_+z\omega\right)/2}\,\overline{\mathbf{T}}\left[Y_+^\dagger(x_0)\,Y_-(x_0)\right] \times \mathbf{T}\left[Y_-^\dagger(0)\,Y_+(0)\,\frac{i\partial_{\perp\mu}}{in_-\partial}\mathcal{B}_+^\mu(z_-)\right]$$

Generalize renormalization equation to

$$[\mathcal{S}_A(\Omega,\omega_i)]_{\text{ren}} = \sum_B \int d\Omega' d\omega_j' \mathcal{Z}_{AB}\left(\Omega,\omega_i;\Omega',\omega_j'\right)\left[\mathcal{S}_B\left(\Omega',\omega_j'\right)\right]_{\text{bare}}$$

$$Z_{2\xi\,2\xi} = \frac{1}{N_c}\sum_{a,c}(\mathcal{Z}_{2\xi\,2\xi})_{aa,cc}$$

For the leading $1/\epsilon^2$ pole we find that

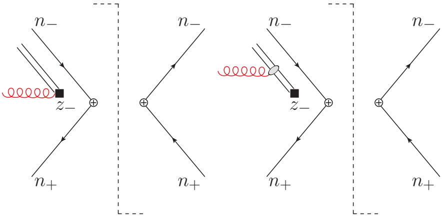
$$(\mathcal{Z}_{2\xi\,2\xi})_{ab,cd} \equiv \delta_{ac}\delta_{bd}\mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

hence

$$Z_{2\xi\,2\xi} = \mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

Soft matrix elements

Problem of finding Z-factor reduced to operator renormalization

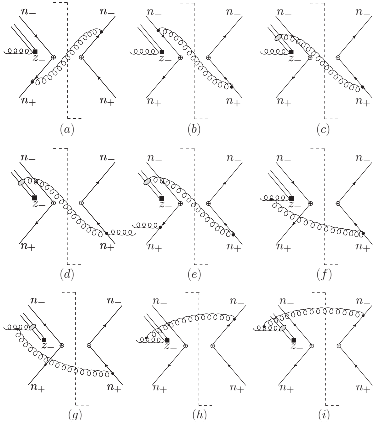


Tree level matrix element is not zero

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}} = g_s T^A \left(\frac{p_\perp \cdot \epsilon_\perp^*}{n_- p} - \frac{p_\perp^2 n_- \epsilon^*}{(n_- p)^2}\right)\delta(\Omega)\delta(\omega - n_- p).$$

Dependence on the external momentum allows to determine full dependence on ω'

One loop “real” diagrams

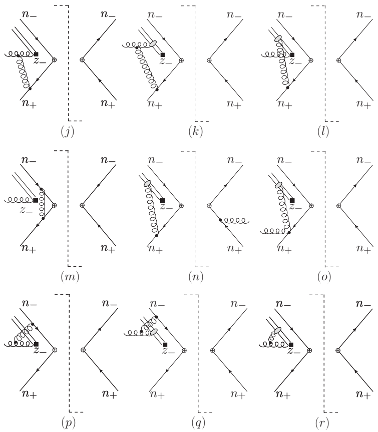


$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{a)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{b)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{c)}_{\text{1-loop}} = \left[-\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

One loop "virtual" diagrams



$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{j)+k)}_{\text{1-loop}} =$$

$$\left[\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

Diagonal part of the anomalous dimension

We find the sum of virtual and real contribution to give a result exactly equal to the corresponding cusp anomalous dimension of the leading power soft function

$$Z^{(1)}_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

$$\Gamma_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = 4\frac{\alpha_s C_F}{\pi}\ln\frac{\mu}{\mu_s}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

- C_A part cancels!
- leading pole is diagonal in color indices
- result is proportional to the tree level but the dependence on Ω' must be extrapolated from the LP result

LL soft function RGE

We checked our result by explicit two-loop computation of the soft function. Both methods lead to the same AD matrix → non-trivial check of

- the choice of S_{x_0}
- the correctness of our procedure to extract leading poles
- the relation between soft operator and soft function renormalization

At the LL we have

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s} \end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix}$$

with a solution

$$\begin{aligned} S_{2\xi}^{\mathrm{LL}}(\Omega,\omega,\mu) &= \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \\ &= C_F\frac{\alpha_s}{\pi}\ln\frac{\mu_s}{\mu}\exp\left[-2C_F\frac{\alpha_s}{\pi}\ln^2\frac{\mu_s}{\mu}\right]\theta(\Omega)\delta(\omega) \end{aligned}$$

LL resummation

The resummed collinear function does not contribute to the LL result, we only need tree level result

$$J^{\mu\rho}_{2\xi;\alpha\beta,abde}(n_+p, n_+p'; \omega) = -\frac{g_\perp^{\mu\rho}}{n_+p}\delta(n_+p - n_+p')\delta_{\alpha\beta}\delta_{ad}\delta_{eb} + \mathcal{O}\left(\alpha_s \ln\left(\frac{\mu}{\mu_c}\right)\right)$$

The resummed cross-section is

$$\begin{aligned}\Delta^{\mathrm{LL}}(z) &= \Delta^{\mathrm{LL}}_{\mathrm{LP}}(z) \\ &\quad - \exp\left[4S^{\mathrm{LL}}(\mu_h, \mu) - 4S^{\mathrm{LL}}(\mu_s, \mu)\right] \times \frac{8C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\end{aligned}$$

where at LL accuracy

$$S^{\mathrm{LL}}(\mu_1, \mu_2) = -\frac{\alpha_s C_F}{2\pi}\ln^2\frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0}\ln\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi}\ln\frac{\mu_2}{\mu_1}$$

Fixed order expanded result

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014

$$
\begin{aligned}
\Delta^{\rm LL}_{\rm NLP}(z,\mu) \quad = \quad & -\theta(1-z)\Bigg\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z)-L_\mu\Big] \\
& + 8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z)-3L_\mu\ln^2(1-z)+2L_\mu^2\ln(1-z)\Big] \\
& + 8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z)-5L_\mu\ln^4(1-z)+8L_\mu^2\ln^3(1-z)-4L_\mu^3\ln^2(1-z)\Big] \\
& + \frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z)-7L_\mu\ln^6(1-z)+18L_\mu^2\ln^5(1-z)-20L_\mu^3\ln^4(1-z) \\
& \qquad +8L_\mu^4\ln^3(1-z)\Big] \\
& + \frac{8}{3}C_F^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z)-9L_\mu\ln^8(1-z)+32L_\mu^2\ln^7(1-z)-56L_\mu^3\ln^6(1-z) \\
& \qquad +48L_\mu^4\ln^5(1-z)-16L_\mu^5\ln^4(1-z)\Big]\Bigg\} + \mathcal{O}(\alpha_s^6\times(\log)^{11}),
\end{aligned}
$$

$L_\mu = \ln(\mu/Q)$.

Higgs threshold production I

$$A(p_A)B(p_B) \to H(q) + X(p_X)$$

Threshold variable

$$z \equiv \frac{m_H^2}{\hat{s}}$$

$$\mathcal{L}_{\text{eff}} = \frac{\alpha_s(\mu)}{3\pi} C_t(m_t, \mu) \; \frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} \; \ln\left(1 + \frac{H}{\nu}\right)$$

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LP current

$$F^A_{\mu\nu} F^{\mu\nu}_A \to 2 g^{\perp}_{\mu\nu} n_- \partial \mathcal{A}^{\nu A}_{\bar{c}\perp} n_+ \partial \mathcal{A}^{\mu A}_{c\perp}$$

The derivation of the factorization is similar like in the DY case, with Wilson lines in the adjoint representation

$$Y_\pm(x) \to \mathcal{Y}^{AB}_\pm = \mathcal{P} \exp\left\{ g_s \int_{-\infty}^0 ds f^{ABC} n_\mp A^C_s(x + s n_\mp) \right\}$$

Higgs threshold production II

Important differences with respect to the Drell-Yan case are:

- Derivatives in the current produce additional factor of $\hat{s}$ compared to DY case

$$S_{K3}(\Omega) = 2\,\Omega\, S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Kinematic corrections do not cancel for the Higgs production!

Higgs threshold production II

Important differences with respect to the Drell-Yan case are:

- Derivatives in the current produce additional factor of $\hat{s}$ compared to DY case

  $$S_{K3}(\Omega) = 2\,\Omega\,S_0(\Omega,\vec{x})_{|\vec{x}=0}$$

  Kinematic corrections do not cancel for the Higgs production!

- After using EOM for soft fields, the tree-level collinear function takes form

  $$J^{ABC}_{2\xi\;\mu\rho}(n_+p, n_+p';\omega) = -2iT_R f^{ABC} g^{\perp}_{\mu\rho}\left[2 - 2n_+p'\frac{\partial}{\partial n_+p}\right]\delta(n_+p - n_+p')$$

  The derivative term does not contribute to the DY LL but it contributes to the Higgs case due a factor $\hat{s}$ in the current

NLP Resummation for Higgs threshold production

$$S_{2\xi}^{\mathrm{LL}}(\Omega,\omega,\mu) = \frac{2C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega)$$

$$S_{\mathrm{K}}^{\mathrm{LL}}(\Omega,\omega,\mu) = \frac{8C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega)$$

Convoluting $S_{2\xi}$ with tree-level $J_{2\xi}$ and adding kinematic correction we obtain the LL resummed cross-section

$$\begin{aligned}\Delta^{\mathrm{LL}}(z) &= \Delta^{\mathrm{LL}}_{\mathrm{LP}}(z)\\ &\quad -\exp\left[4S^{\mathrm{LL}}(\mu_h,\mu)-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\times\frac{8C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\end{aligned}$$

where at LL accuracy

$$S^{\mathrm{LL}}(\mu_1,\mu_2) = -\frac{\alpha_s C_A}{2\pi}\ln^2\frac{\mu_2}{\mu_1}$$

The result has the same form as Drell-Yan with $C_F \leftrightarrow C_A$

Summary and Conclusions

- NLP LL threshold resummation for Drell-Yan and Higgs production is completed, simple relation $C_F \leftrightarrow C_A$ holds to all orders
- Renormalization of new, generalized soft functions is understood at LL accuracy but we must better understand its renormalization properties
- Work in progress: Extension to the quark gluon channel
- Next step: extension of factorization theorem to NLL and resummation

Auxiliary slide: Hard function running

Well known RGE for two-jet operator

$$\frac{d}{d\ln\mu}H(Q^2,\mu) = \left(2\Gamma_{\rm cusp}\ln\frac{Q^2}{\mu^2} + 2\gamma\right)H(Q^2,\mu)$$

$$\Gamma_{\rm cusp} = \frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2), \qquad \gamma = -\frac{3}{2}\frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2),$$

The general solution RGE reads

$$H(Q^2,\mu) = \exp\left[4S(\mu_h,\mu) - 2a_\gamma(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)} H(Q^2,\mu_h)$$

where

$$S(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')},$$

$$a_\Gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}, \qquad a_\gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

Auxiliary slide: Soft function in position space

At the one-loop order in dimensional regularization with $d = 4 - 2\epsilon$, the bare soft function must have a simple dependence

$$\tilde{S}_{0,\mathrm{bare}}(x) = 1 + \frac{\alpha_s}{\pi}\left(-n_- x n_+ x \mu^2\right)^{\epsilon} f\left(\epsilon, \frac{x^2}{n_+ x n_- x}\right)$$

Explicit evaluation gives

$$\begin{aligned}\widetilde{S}_{0,\mathrm{bare}}(x) &= 1 + \frac{\alpha_s C_F}{\pi}\,\frac{\Gamma(1-\epsilon)}{\epsilon^2} e^{-\epsilon\gamma_E} \\ &\quad \times \left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right)^{\epsilon} \left(\frac{x^2}{n_- x n_+ x}\right)^{1+\epsilon} {}_2F_1\left(1, 1, 1-\epsilon; 1 - \frac{x^2}{n_- x n_+ x}\right) \\ &= 1 + \frac{\alpha_s C_F}{\pi}\left(\frac{1}{\epsilon^2} + \frac{L}{\epsilon} + \frac{L^2}{2} + \frac{\pi^2}{12} + \mathrm{Li}_2\left(1 - \frac{x^2}{n_- x n_+ x}\right) + \mathcal{O}(\epsilon)\right)\end{aligned}$$

where we defined

$$L \equiv \ln\left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right).$$

Auxiliary slide: Kinematic soft functions at $\mathcal{O}(\alpha_s)$

Expanding the kinematic factors in the factorization formula we obtain further corrections related to the LP soft function

$$
\begin{aligned}
S_{K1}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\ \frac{1}{\epsilon}+2\ln\frac{\mu}{\Omega}-2\right)\theta(\Omega)\\
S_{K2}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\ \frac{3}{\epsilon}+6\ln\frac{\mu}{\Omega}+6\right)\theta(\Omega)\\
S_{K3}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(-\frac{4}{\epsilon}-8\ln\frac{\mu}{\Omega}\right)\theta(\Omega)
\end{aligned}
$$

$$
\sum_{i=1}^{3} S_{Ki}(\Omega) = 2\,\frac{\alpha_s C_F}{\pi}\,\theta(\Omega)
$$

At $\mathcal{O}(\alpha_s)$ no LL kinematic corrections!

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} &= \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} \\ &\quad + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}} \end{aligned}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Aside:

Is the convolution assumption too strong?

- Dependence of Z on Ω' cannot be uniquely determined - at LP we determine it from the known properties of Wilson loop renormalization in position space – multiplicative renormalization in position space
- Dependence on ω' can by determined under additional assumptions

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} &= \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} \\ &\quad + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}} \end{aligned}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $\mathcal{O}(\alpha_s)$ of the diagonal Z-factor?

Auxiliary slide: Alternative approach without operator renormalization

Renormalization condition for the two-loop soft function $S^{(2)}_{2\xi}$

$$
\begin{aligned}
S^{(2)}_{2\xi} + Z^{(1)}_{2\xi\, x_0} S^{(1)}_{x_0} + Z^{(2)}_{2\xi\, x_0} S^{(0)}_{x_0} + Z^{(1)}_{2\xi\, 2\xi} S^{(1)}_{2\xi} &= \text{finite} \\
S^{(1)}_{x_0} + Z^{(1)}_{x_0\, x_0} S^{(0)}_{x_0} &= \text{finite} \\
S^{(1)}_{2\xi} + Z^{(1)}_{2\xi\, x_0} S^{(0)}_{x_0} &= \text{finite}
\end{aligned}
$$

Following structure

$$
\Gamma = \alpha_s(\mu) \begin{pmatrix} \Gamma_{AA} \ln \frac{\mu}{\mu_s} + \gamma_{AA} & \gamma_{AB} \\ \gamma_{BA} & \Gamma_{BB} \ln \frac{\mu}{\mu_s} + \gamma_{BB} \end{pmatrix}
$$

implies

$$
Z^{(2)}_{AB} = \frac{1}{4} Z^{(1)}_{AB} \left(Z^{(1)}_{AA} + 3 Z^{(1)}_{BB} \right) + \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \quad A \neq B.
$$

$$
S^{(2)}_{2\xi} - \frac{1}{4} Z^{(1)}_{2\xi\, x_0} \left(3 Z^{(1)}_{2\xi\, 2\xi} + Z^{(1)}_{x_0\, x_0} \right) S^{(0)}_{x_0} = \mathcal{O}\left(\frac{1}{\epsilon^2}\right)
$$

Two loop result agrees with one-loop operator renormalization

Auxiliary slide: Fixed order check

For arbitrary μ we then find

$$\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) \quad = \quad \exp\left[4S^{\mathrm{LL}}(\mu_h,\mu) - 4S^{\mathrm{LL}}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\, \theta(1-z)$$

Note $\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu_c)$ has the same form $\rightarrow$ *no LL in collinear function!*

$$S^{\mathrm{LL}}(\mu_1,\mu_2) = -\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0} \ln \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi} \ln \frac{\mu_2}{\mu_1}$$

Our result

$$\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) = \frac{\hat{\sigma}^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu)}{z} \quad = \quad \exp\left[2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_s} - 2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_h}\right] \times(-4)\frac{\alpha_s C_F}{\pi} \ln \frac{\mu_s}{\mu}\, \theta(1-z)$$

agrees with

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991, full fixed order NNLO computation
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014 approximate results for $\mu = \mu_h$ up to N^4LO

Auxiliary slide: RGE for kinematic soft functions – Higgs case

Proceeding like in the example we obtain

$$
\begin{aligned}
\frac{d}{d\ln\mu}\,\vec{S}(x^0) \quad &= \quad \Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\
&+ \quad \Gamma_{\mathrm{cusp}}\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0)
\end{aligned}
$$

with $\vec{S}(x^0) = \left(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$

$$
\begin{aligned}
\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) = &\Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) \\
&-4\Gamma_{\mathrm{cusp}}\widetilde{S}_{x_0}(x^0) - 6\,\Gamma_{\mathrm{cusp}}\,\widetilde{S}_{K3}(x^0)\,,
\end{aligned}
$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}\left(\alpha_s \ln\frac{\mu}{\mu_s}\right)$