Renormalization Theory of Beam-Beam Interaction

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This work has been done during my last 6 months at DESY, 30 years ago.

This work is an attempt to calculate analytically particle distributions under the beam-beam interaction using the renormalization technique of the quantum field theory, even when the particle motions are chaotic.

I thought that this work would be my best work (more than ABCI and TMCI).

But, it has been forgotten since I moved to LBNL and started new work here.

Outline

- Introduction
- Diagonalization of System
- Renormalization Procedure
- Direct Interaction Approximation
- Closed Set of Equations
- Comparisons with Simulations
- Conclusions

Introduction

- The beam-beam interaction has been extensively studied in terms of Hamiltonian analysis of single particle dynamics.
  - The Hamiltonian analysis may predict orbits of regular particle motion, and may give us some criterions (e.g. Chirikov's resonance overlap) for estimating the onset of chaotic behavior of particle orbit.
  - However, since the method is posed in terms of the behavior of a particle trajectory, it breaks down when the particle motion becomes chaotic.

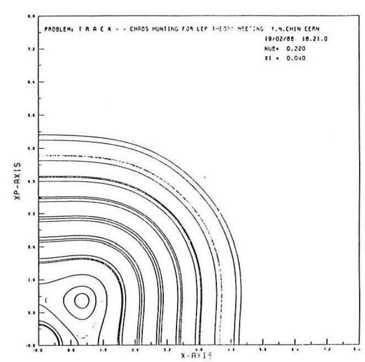


Fig. 2a One quarter of phase-space trajectories for $\nu_\beta = 0.22$ and $\xi = 0.04$.

Weak
Beam-Beam

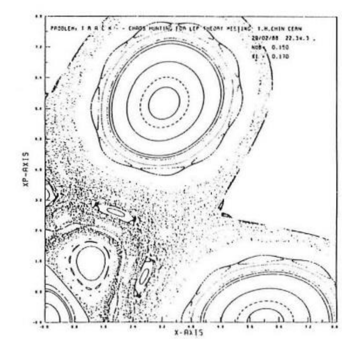


Fig. 4a One quarter of phase-space trajectories for $\nu_\beta = 0.15$ and $\xi = 0.17$.

Strong
Beam-Beam

Statistical Theory

- What is needed is a more statistical theory for dynamics of, not single particle, but ensemble of many particles where the chaos may be described by statistical terms.
    - That theory would allow us to calculate particle distributions in the presence of the beam-beam interaction.
    - These quantities are straight linked with a beam blowup and particle losses.

Three Premises

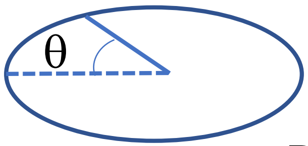
- Fokker-Planck equation for the evolution of the particle distribution
- “Strong-weak" beam-beam interaction
- One-dimensional

- Warning
  - This talk is quite theoretical due to its nature.
  - If you still have a fresh memory of what you have learned on the quantum field theory at school, you can follow it (I cannot anymore).
  - Only the outline of the theory is presented in this conceptual talk, but the concrete solutions exist for numerical evaluations.
  - At the end, I show some comparisons with simulations

Crux of the Problem

- Fokker-Plank equation for the particle distribution P:

$$(\partial_\theta + \Lambda)\,P = 0$$



Ring

  - θ= the azimuthal position in the ring
  - Λ=Fokker-Plank operator including all the effects (beam-beam, synchrotron radiation and so on)

- If we can find the Green function which satisfies

$$(\partial_\theta + \Lambda)\,G(x,p,t|x_o,p_o,t_o) = \delta(x-x_o)\delta(p-p_o) \quad \text{at } t = t_o$$

$$= 0 \qquad \text{at } t > t_0 \ ,$$

the solution is

Initial distribution

$$P(x,p,t,t_o) = \iint dx_o dp_o G(x,p,t|x_o,p_o,t_o) P(x_o,p_o,t_o)$$

Done! Simple!

Exact Green Function

- The exact Green function G includes all the orbit distortion effects and provides the exact transition probability of particle orbit, at any preceding moment, no matter whether the particle motion is chaotic or not.

- Too difficult to find it.

- Let us evaluate G with the perturbation method.

- One important rule in choice of the perturbation method.
  - The method has to guarantee that a perturbation solution of any order will be smaller than the lower-order ones so that the perturbation expansion series converges.
  - It is not so obvious.

Other View: Diagonalization of System

- The first step to solve unknow particle distribution P is to expand it by a complete set of known functions or modes f_n:
  - $P = P_0 + c_1 f + c_2 f + \cdots$,
- The second step is to make an "interaction matrix" for the expansion coefficients c_m:
  - $c_m = \sum_{k=-\infty}^{\infty} M_{mk}\, c_k$
- If we can diagonalize the interaction matrix, the problem is basically solved:
  - Eigenvalues, eigenfunctions, and all others follow.
- But, some systems are so complicated that it is not easy to diagonalize the interaction matrix.

Renormalization Theory

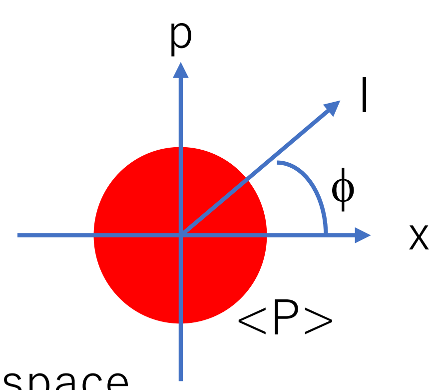
- Instead of pursuing the exact solution, let us find an approximate solution with good accuracy and the possibility to improve the accuracy by including more higher-order correction terms.
- The crux of the procedure is to move significant off-diagonal terms to diagonal terms in the matrix until remaining off-diagonal terms are all insignificant and thus negligible.
- The theory is originally motivated to avoid the small denominator singularities at the centers of resonances by including orbit distortion of resonant particles due to other resonances.
- But, it is most powerful when resonances strongly interact to each other, and the system can be no longer approximated by a collection of isolated resonances.

Renormalization Procedure

- Let us write down the Fokker-Planck eq. for the particle distribution P in a slightly explicit form:

$$(\partial_\theta + L)\, P = L_B P$$

  - L: Fokker-Plank operator except the beam-beam
  - L_B: Operator for beam-beam as a function of potential U
- Decompose P into



$$P = \langle P \rangle + \delta P$$

$$\langle \delta P \rangle = 0$$

  - $\langle P \rangle$: Average over the azimuthal angle ϕ in phase space
  - δP: Remaining part fluctuating around $\langle P \rangle$

Fourier Decomposition

- Due to the periodic boundary condition in ϕ,

$$\delta P = \sum_{m \neq 0} P_m(I,\theta) \exp(im\phi)$$

$$U = \sum_{\ell=-\infty}^{\infty} U_\ell(I) \exp(i\ell\phi) \propto \xi$$: the beam-beam parameter

- Averaging the Fokker-Plank Eq. over ϕ and Fourie decomposition lead equations for <P> and δP:

$$(\partial_o + L_o)\langle P \rangle = \sum_{k \neq o} M_{k,-k} U_k P_{-k}$$

$$(\partial_o + L_k) P_k = M_{ko} U_k \langle P \rangle + \underline{S_k}$$

$$S_k = \sum_{\ell \neq o} M_{\ell, k-\ell} U_\ell P_{k-\ell}$$

Beam-beam interaction matrix operator

Mode coupling term from other modes P_{k-l}

Unperturbed Green Function

- Equation for δP can be further Fourier composed in θ:

$$g^{0^{-1}}_{k\nu} P_{k\nu} = M_{ko} U_k \langle P \rangle + S_{k\nu}$$

- Here, the unperturbed Green function $g^{0^{-1}}_{k\nu}$ satisfies

$$(-i\nu - L_{ko})\, g^{o}_{k\nu} = \delta(I - I_o)$$

Mode Coupling Term S_{kn}

- The mode coupling term becomes important in two cases:

1. Very weak synchrotron radiation

   - The unperturbed green function is approximately given by

$$g^{o}_{k\nu} = \frac{i}{\nu - k\,(\nu_\beta + \Delta\nu\,(I))}$$

← Nonlinear detuning term

↑ Unperturbed betatron tune

   - If we ignore the mode coupling term $S_{k\nu}$,

$$P_{k\nu} = g^{o}_{k\nu}\, M_{ko}\, U_k \langle P \rangle = \frac{i\, M_{ko}\, U_k \langle P \rangle}{\nu - k\left(\nu_\beta + \Delta\nu(I)\right)}$$

← Diverges at the center of resonance $\nu - k\left(\nu_\beta + \Delta\nu(I)\right) = 0$

Resonance Singularity

- The singularity emerges since we have assumed that resonant particles receive only a part of beam-beam kick which creates the resonance.

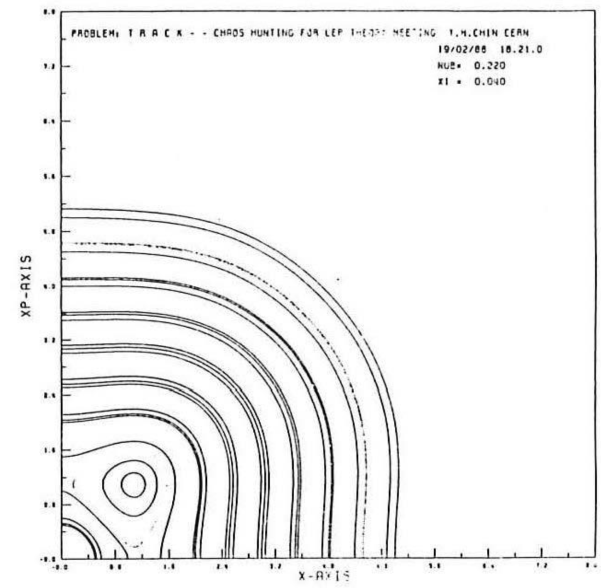
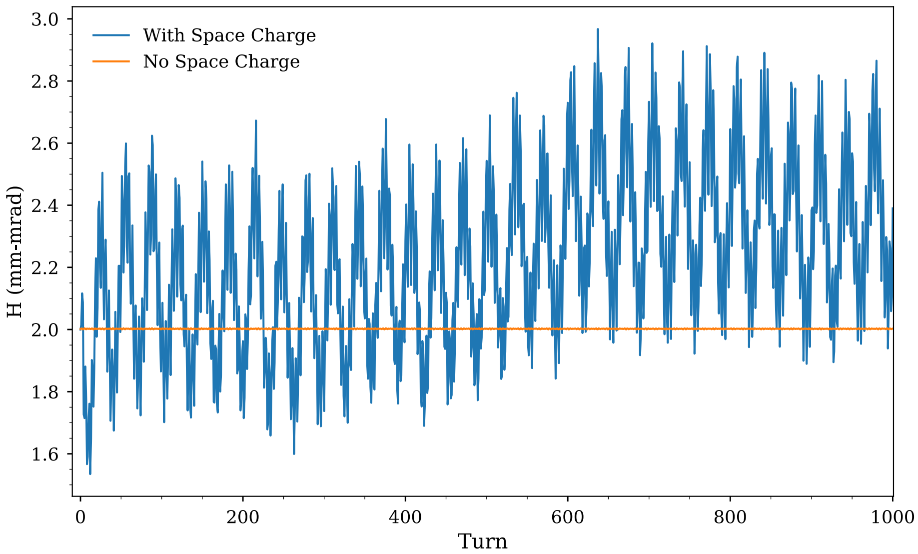


Fig. 2a One quarter of phase-space trajectories for $\nu_\beta = 0.22$ and $\xi = 0.04$.

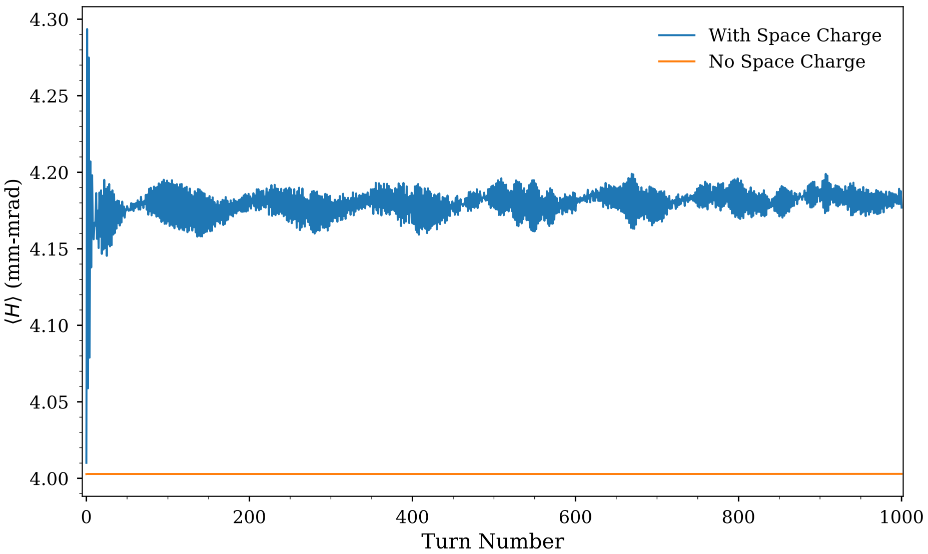
- In reality, particles receive the total kick of beam-beam force which generate all the resonances.
- By the random kicks from other resonances, the particle tunes are fluctuating and not strictly locked at the resonance tune.
- Therefore, the resonance singularity may be avoided in the real system even in the absence of the quantum fluctuation.

Space Charge breaks Integrability

Singe-particle invariants are broken



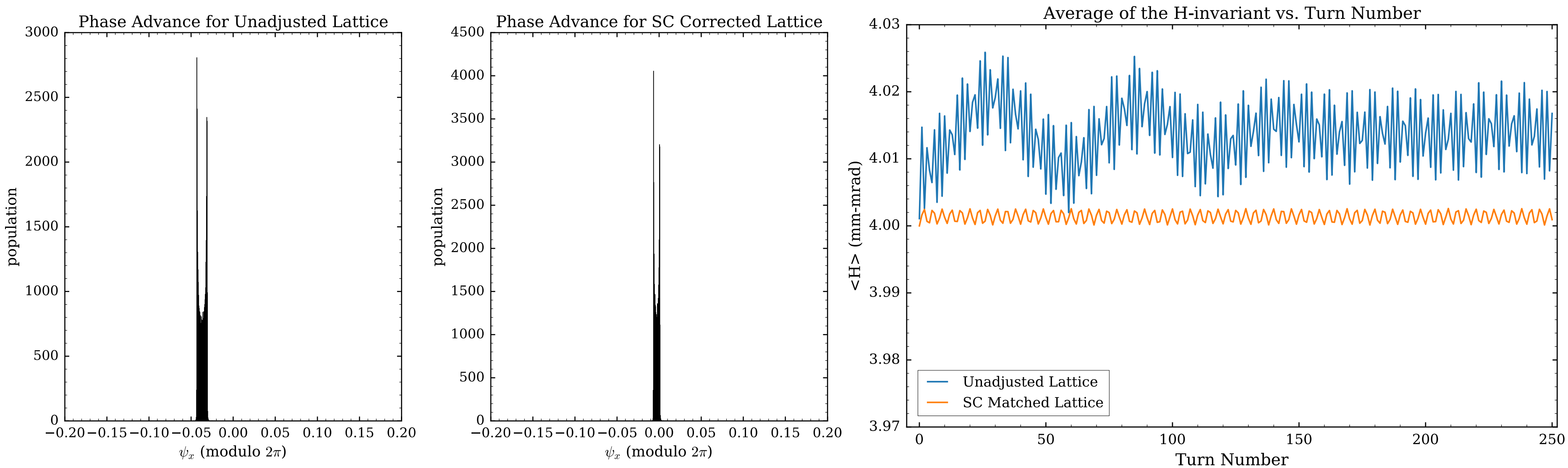
Ensemble average is better behaved



With space charge:

- 'Time independence' of Danilov & Nagaitsev theory is broken
- Both zero-current invariants now fluctuate significantly at 2 frequencies
- Some ensemble properties still appear to be approximately maintained
  - **we don't yet understand how meaningful this may be**
- Bounded motion + nonlinear decoherence may be all that is required

Results with the Compensated Lattice



- Phase advance around the ring is now corrected with the new compensated lattice.
- This leads to much better behavior of the first invariant (the Hamiltonian).

Strong Coupling between Resonances

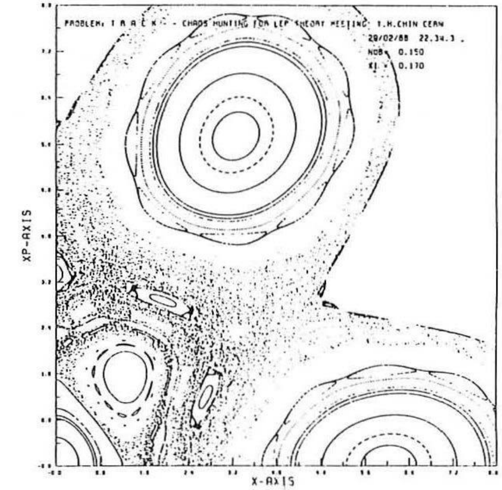


Fig. 4a One quarter of phase-space trajectories for $\nu_\beta = 0.15$ and $\xi = 0.17$.

- In this case, the particle motion between the resonances may be chaotic.

- Apparently, the exact Green's function will be very different from $g^{0}_{k\nu}$ which expresses regular orbits of resonant particles.

- It cannot be constructed in terms of $g^{0}_{k\nu}$ by calculating higher-order correction terms from the mode-coupling term $S_{k\nu}$, since the chaotic motion cannot be described by combination of regular motion.

- If one tries, then the expansion series will not converge.

Renormalized Green's Function

- Let us introduce the renormalized Green's function:

$$g_{k\nu} = \left[g^{0^{-1}}_{k\nu} + \Sigma_{k\nu}\right]^{-1}$$

  - $\Sigma_{k\nu}$: Renormalization correction operator to be determined

- Then,

$$g^{0^{-1}}_{k\nu} P_{k\nu} = M_{ko}\, U_k \langle P \rangle + S_{k\nu} \longrightarrow g^{-1}_{k\nu} P_{k\nu} = M_{ko}\, U_k \langle P \rangle + S_{k\nu} + \Sigma_{k\nu} P_{k\nu}$$

- Decompose the mode coupling term $S_{k\nu}$ to

$$S_{k\nu} = S^{d}_{k\nu} + S^{nd}_{k\nu}$$

$S^{d}_{k\nu}$: Proportional to $P_{k\nu}$

$S^{nd}_{k\nu}$: The rest

Incoherent Noise

- Identify (since both are proportion to $P_{k\nu}$)

$$S^{d}_{k\nu} = -\Sigma_{k\nu} P_{k\nu}$$

- Then, we have

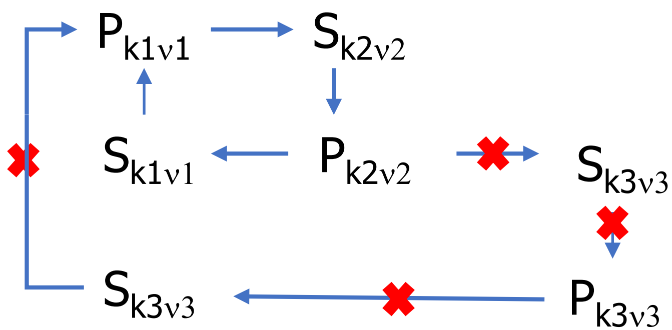
$$g^{-1}_{k\nu} P_{k\nu} = M_{ko} U_k \langle P \rangle + S^{nd}_{k\nu}$$

- By the definition, $S^{nd}_{k\nu}$ does not depend on $P_{k\nu}$ and thus acts as an incoherent noise to $P_{k\nu}$

- Resonances can still cause changes in other resonances through $S^{nd}_{k\nu}$, but they are not coupled by $S^{nd}_{k\nu}$

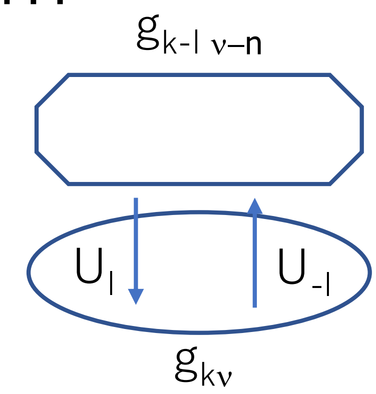
Direct Interaction Approximation

- A resonance $P_{k1\nu1}$ can cause a change in another resonance $P_{k2\nu2}$ through the mode-coupling term $S_{k2\nu2}$.
- The change in $P_{k2\nu2}$ can act back to the resonance $P_{k1\nu1}$ through the mode-coupling $S_{k1\nu1}$ and $P_{k1\nu1}$ will be changed.



- This self-interaction should be identified as $S^{d}_{k\nu}$, since its strength depends on $P_{k1\nu1}$ proportionally.
- Only the direct interaction between resonances is considered in the present theory.

Renormalization Correction Term



- The explicit form

$$\Sigma_{k\nu} = -\sum_{\ell \neq 0} \sum_{n} M_{\ell,k-\ell} U_{\ell}\, g_{k-\ell,\nu-n} M_{-\ell,k} U_{-\ell}$$

The second order in the beam-beam parameter

- The physical interpretation
  - The particle subject to the Green's function $g_{k\nu}$ in the (k,v) resonance is scattered by the field U_{-l} and is effected by the resonance $g_{k-l\ \nu-n}$
  - Then, it is scattered again by the field U_l to emerge at the initial resonance $g_{k\nu}$.

- Since the particle comes back to the initial resonance, the above trajectory going through other resonance should be included in the transition probability of the particle orbit subject to the (k,v) resonance, namely the renormalized Green's function $g_{k\nu}$ for the renormalized resonance.

Feynman Diagrams

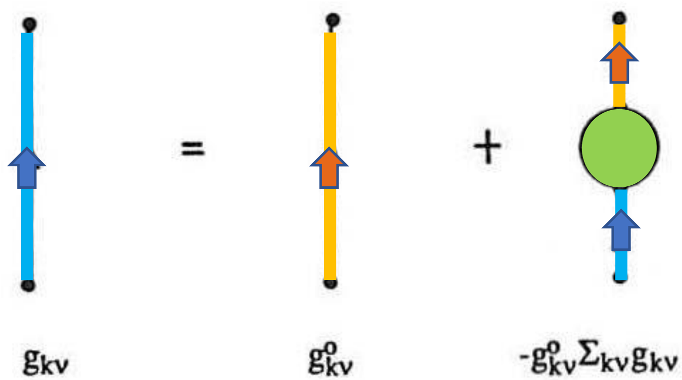


Fig. 1a Feynman diagram of Dyson's equation for the renormalized Green's function

$$g_{kv} = \left[g^{o^{-1}}_{kv} + \Sigma_{kv}\right]^{-1}$$

$$= g^o_{kv} - g^o_{kv}\,\Sigma_{kv}\,g_{kv} \quad .$$

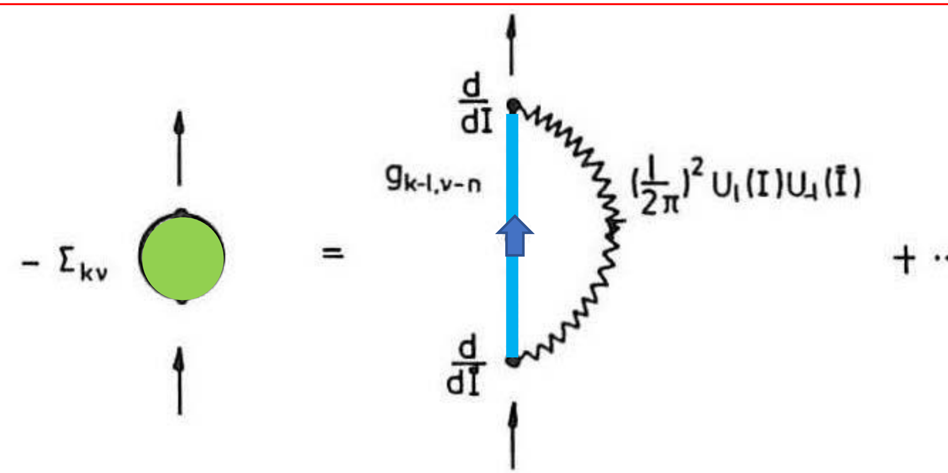


Fig. 1b Feynman diagram of the renormalization correction Σ_{kv}.

Hartree-Fock approximation for self-energy of a fermion

Solution

- The solution $P_{k\nu}$ can be decomposed to two terms:

$$P_{k\nu}(I) = P_{k\nu}^{coh}(I) + P_{k\nu}^{inc}(I)$$

with

$$P_{k\nu}^{coh} = g_{k\nu} M_{ko} U_k \langle P \rangle$$

and

$$P_{k\nu}^{inc} = g_{k\nu} S_{k\nu}^{nd} \ .$$

- The renormalized Green's function $g_{k\nu}$ has no small denominator problem anymore.
- The incoherent part is formally one –order of magnitude smaller in ξ than the coherent part.
- Let us neglect the incoherent part hereafter.

Closed Set of Equations

- Now, each mode $P_{k\nu}$ couples only with <P>, no longer with other modes.
- Thus, the equations for <P> and $P_{k\nu}$ are closed:
  - Once we know <P>, we can calculate $P_{k\nu}$, and vice versa.

$$(\partial_o + L_o)\langle P\rangle = \sum_{k\neq o} M_{k,-k}\, U_k\, P_{-k}$$

$$(\partial_o + L_k)\, P_k = M_{ko}\, U_k\, \langle P\rangle$$

- These equations can be solved with rough approximations.
- I skip what follows, but we can derive concrete solutions.

Comparison with Simulations Relevant to LEP at 50GeV

- No chaotic case for ξ=0.04

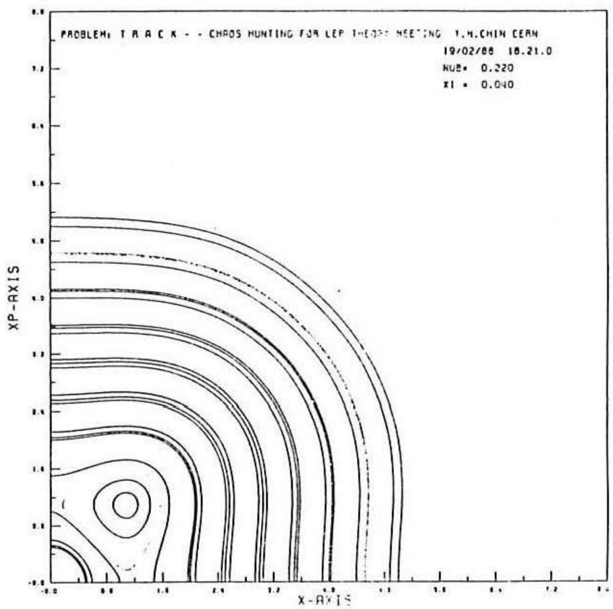


Fig. 2a One quarter of phase-space trajectories for $\nu_\beta = 0.22$ and $\xi = 0.04$.

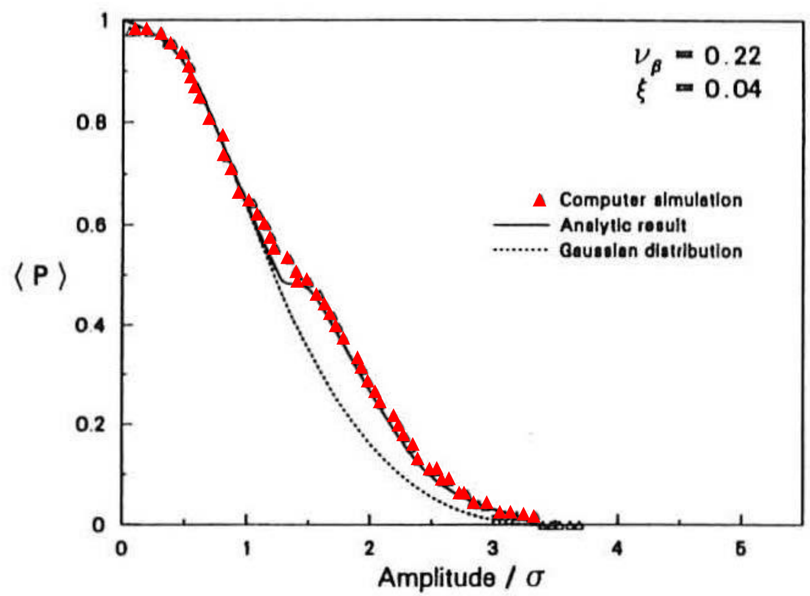


Fig. 2b Averaged particle distributions as a function of amplitude in polar coordinate.

No Chaotic Case for $\xi = 0.06$

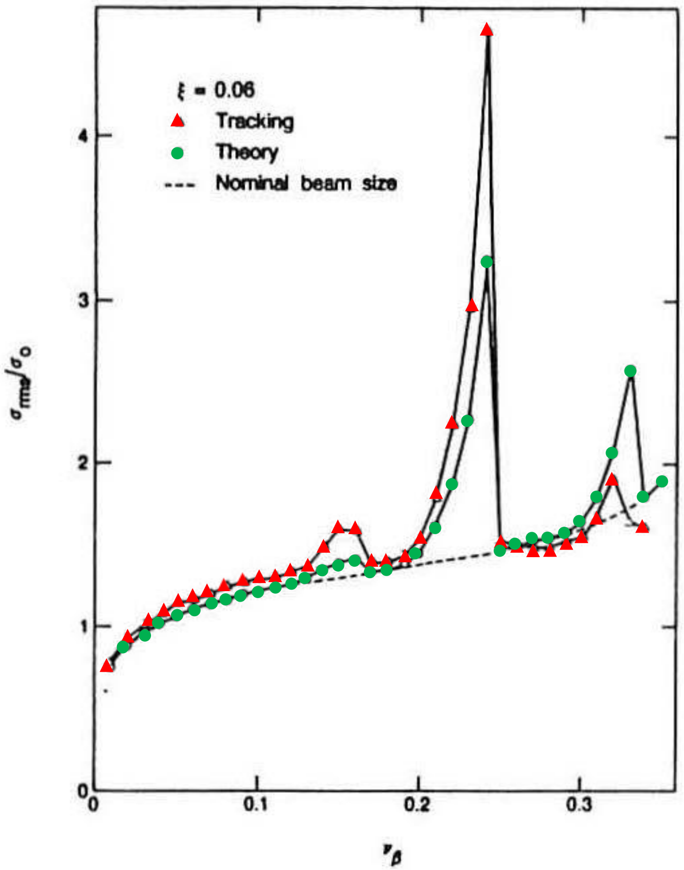
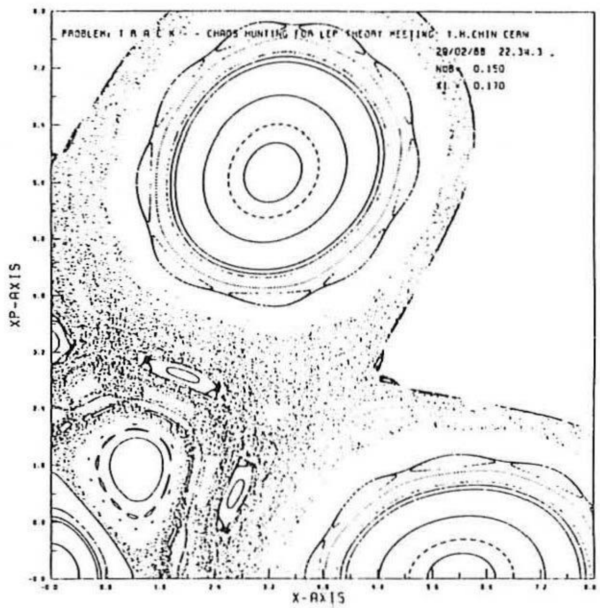


Fig. 3 The rms beam sizes as a function of ν_β for $\xi = 0.06$.

Very Chaotic Case for ξ=0.17



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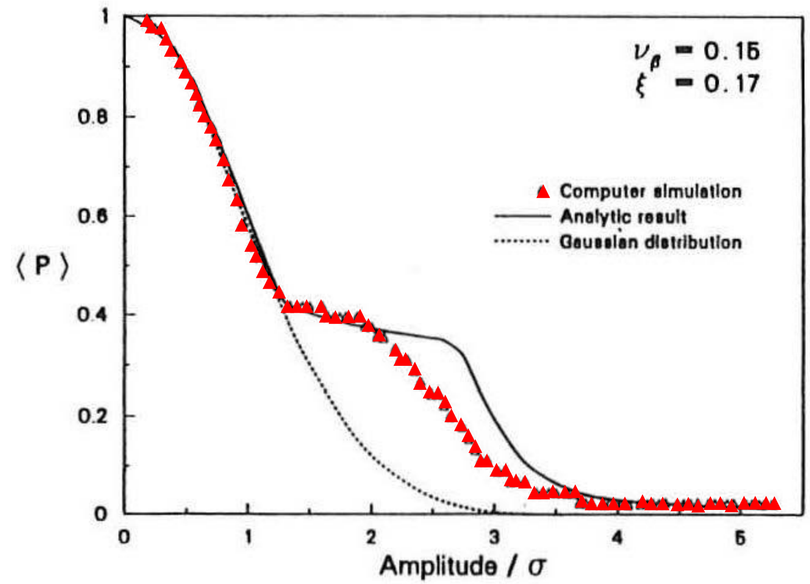

Fig. 4b Averaged particle distributions for $\nu_\beta = 0.15$ and

The RMS Beam Size for Large ξ

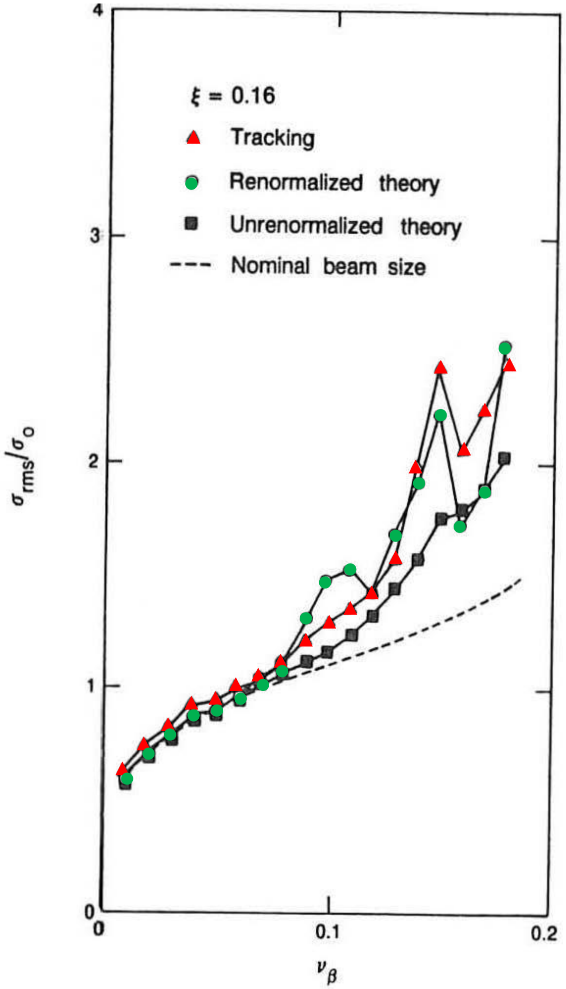

Fig. 5 The rms beam sizes as a function of ν_β for $\xi = 0.16$.

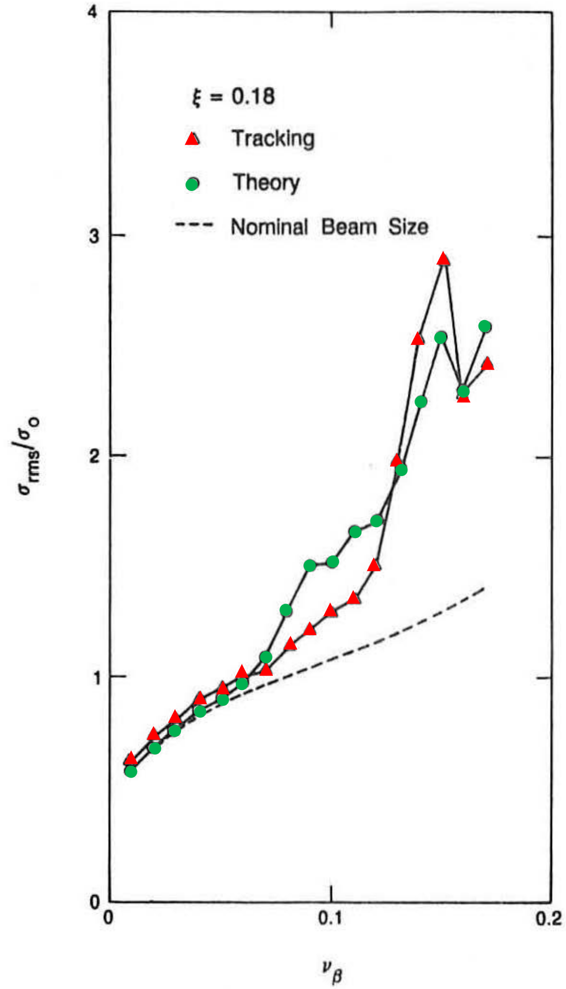

Fig. 6 The rms beam sizes as a function of ν_β for $\xi = 0.18$.

Conclusions

- Despite of some rough approximations for the renormalized Green's function, the theory exhibits reasonably good agreements with computer simulations.
- To try to explain a beam blowup by looking at the distortion of particle orbit lies on the same line as the Hamiltonian analysis.
- However, by describing the orbit distortion in terms of the Green's function, we gain more capacity in the theory where statistics comes in.
- At the same time, the physical mechanism of a beam blowup, due to either chaos or regular resonances, is explicit in the theory.
- The present one-dimensional strong-weak beam picture is still unpractical for application to real machines, but the extension to the two dimensional strong-strong case is nearly impossible.
- Hopefully, somebody will solve this problem to advance the theory.