Thermalization in SU(2) LGT

ETH and Entanglement dynamics

Commun Phys **8**, 368 (2025)
PRD **110**, 014505 (2024)
PRD **109**, 014504 (2024)

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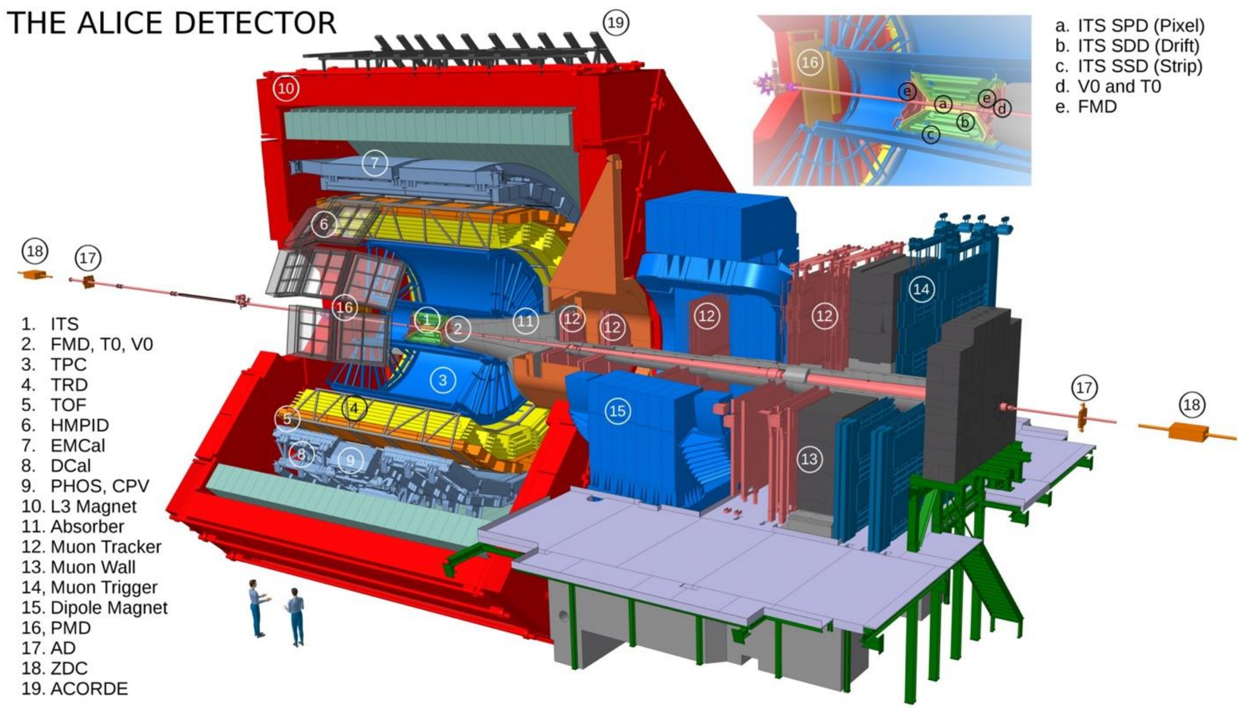
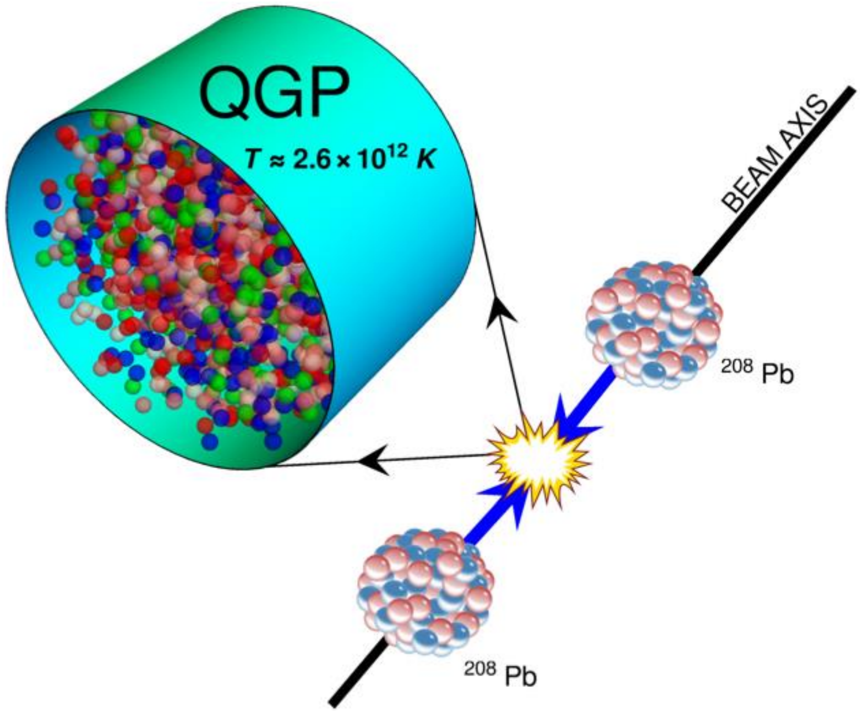
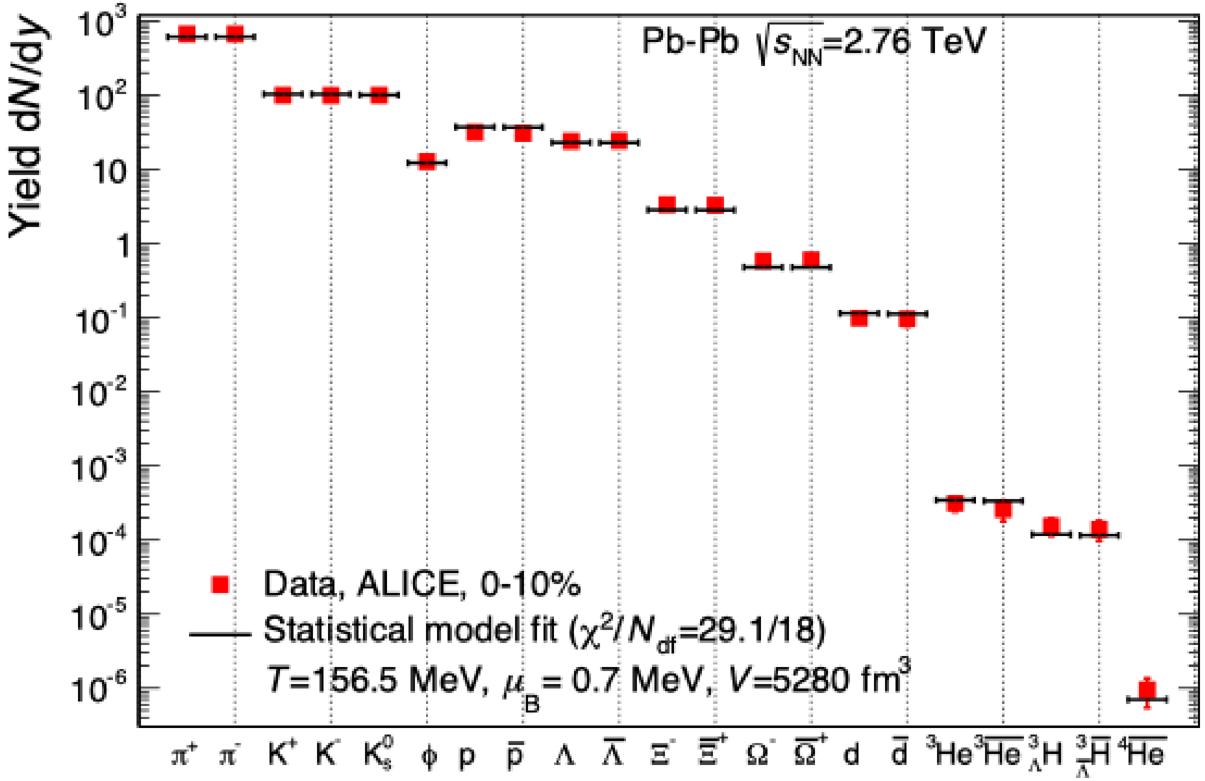
Big Picture





Munzinger, Dönigus, 1809.04681 (2018)
Gardim, Giacalone et. al., 1908.09728 (2020)

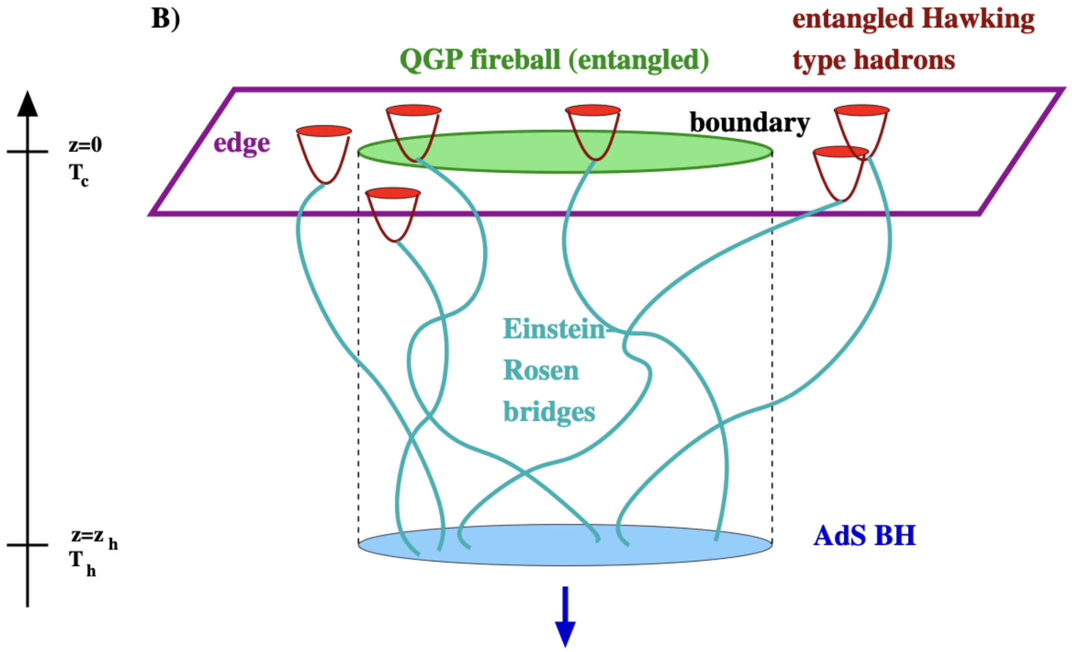
Big Picture



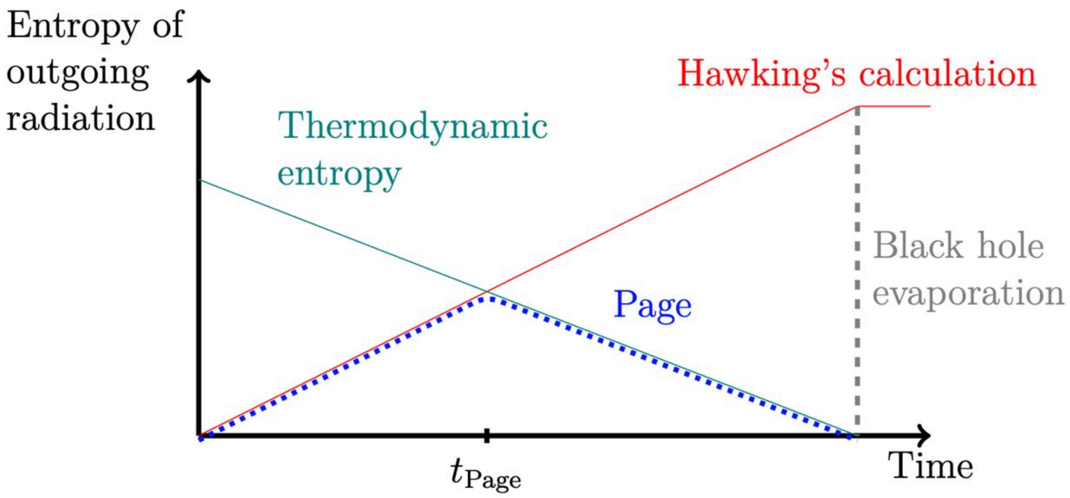
Andronic et al., 1611.01347 (2016)

- **Thermal model:**
  Hadron resonance gas with temperature T, baryochemical potential μ_B and volume V
- HIC is **nearly pure, isolated and highly entangled** quantum state that evolves **unitarily**
  → no entropy production
- How does apparent thermalization emerge in a closed quantum system, when energy is conserved?

Big Picture



Müller, Schäfer, 2211.16265 (2022)



Quantum entanglement is key to understanding apparent thermalization
→ Thermalization dynamics of nonabelian gauge theory

Outline

SU(2)

- Kogut-Susskind Hamiltonian
- Plaquette chain systems
- $j_{\max} = \frac{1}{2}$ truncation

ETH

- What is the ETH?
- Evidence for ETH

EE

- Basics and properties
- Page curve
- QMBSs
- Real time dynamics

AF

- Quantum complexity
- Anti-flatness
- AF barrier before thermalization

SU(2)

ETH

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AF

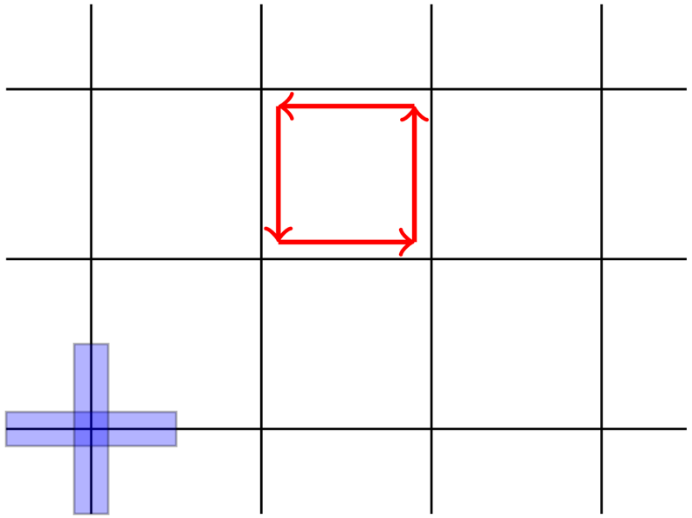
Kogut-Susskind for pure SU(2) LGT

KS Hamiltonian

$$H = \frac{g^2}{2} \sum_{\text{links}} (E_i^a)^2 - \frac{2}{a^2 g^2} \sum_{\text{plaquettes}} \Box(n)$$

with plaquette operator $\Box(n) \equiv \mathrm{Tr}[U^\dagger(\boldsymbol{n},\hat{y})U^\dagger(\boldsymbol{n}+\hat{y},\hat{x})U(\boldsymbol{n}+\hat{x},\hat{y})U(\boldsymbol{n},\hat{x})]$, lattice spacing a, coupling g and $U(\boldsymbol{n},\hat{i})$ is lattice version of Wilson line for path from $\boldsymbol{n}$ to $\boldsymbol{n}+\hat{i}$

- Electric basis on links:
  $$E^2|jm_Lm_R\rangle = j(j+1)|jm_Lm_R\rangle$$
- **Gauss law**: each **vertex** transforms as singlet
  $$(D_iE_i)^a|\psi\rangle = 0$$



Byrnes, Yamamoto, quant-ph/0510027 (2005)

Plaquette chain systems

SU(2)

ETH

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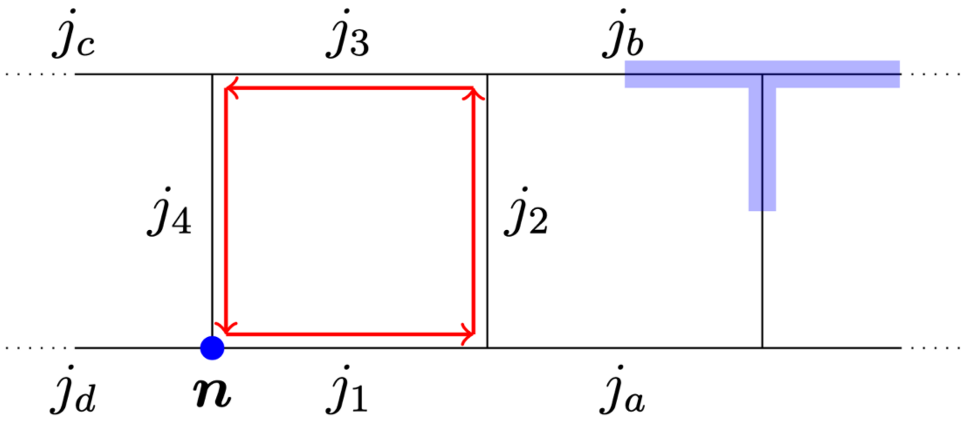
- Only three links per vertex
  → singlet uniquely determined by j values



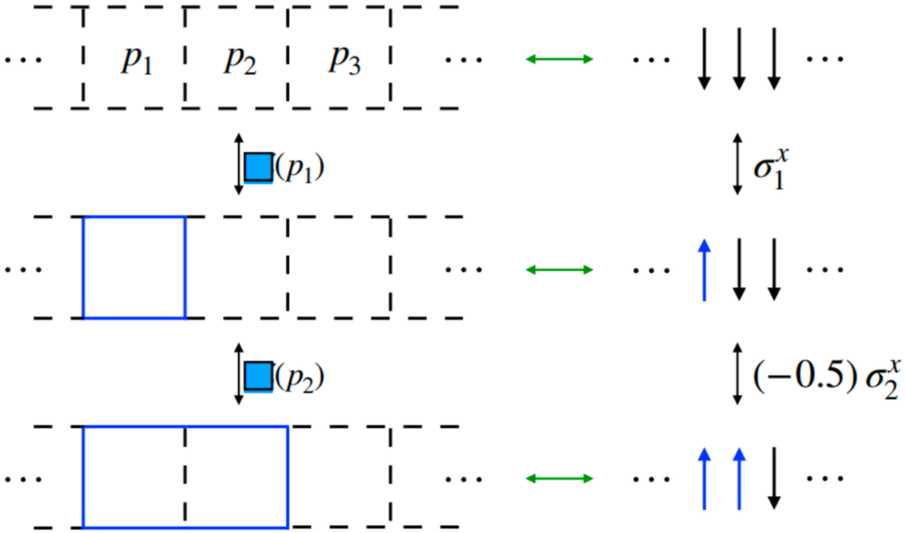
- Matrix elements of plaquette operator between physical states (j: initial, J: final):

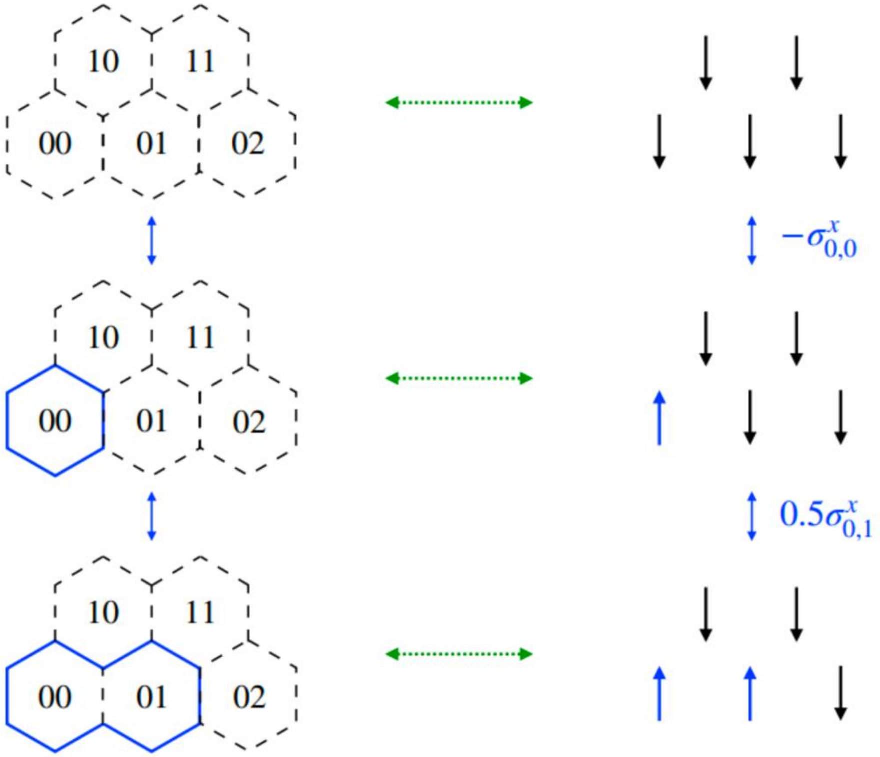
$$\langle J_1 J_2 J_3 J_4 | \Box | j_1 j_2 j_3 j_4 \rangle = \prod_{\alpha=a,b,c,d} (-1)^{j_\alpha} \prod_{\alpha=a,b,c,d} \left[(-1)^{j_\alpha + J_\alpha} \sqrt{(2j_\alpha+1)(2J_\alpha+1)} \right]$$
$$\begin{Bmatrix} j_a & j_1 & j_2 \\ \frac{1}{2} & J_2 & J_1 \end{Bmatrix} \begin{Bmatrix} j_b & j_2 & j_3 \\ \frac{1}{2} & J_3 & J_2 \end{Bmatrix} \begin{Bmatrix} j_c & j_3 & j_4 \\ \frac{1}{2} & J_4 & J_3 \end{Bmatrix} \begin{Bmatrix} j_d & j_4 & j_1 \\ \frac{1}{2} & J_1 & J_4 \end{Bmatrix}$$

Klco, Stryker, Savage, 1908.06935 (2019)

$j_{\max} = 1/2$ truncation

- SU(2)
- ETH
- EE
- AF





- SU(2) KS with $j_{\max} = 1/2$ can be mapped onto spin model
- project onto momentum eigenstates and symmetry sector

Yao, 2303.14264 (2023)
Müller, Yao, 2307.00045 (2023)

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Eigenstate thermalization hypothesis

- SU(2)
- ETH
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- How does apparent thermalization emerge in a closed quantum system, when energy is conserved?
- Time evolution of local operator expectation value

$$\langle O\rangle(t) = \mathrm{Tr}\{O\rho(t)\} = \sum_{n,m} \underbrace{\langle n| O |m\rangle}_{\displaystyle\downarrow} \langle m| \rho(0) |n\rangle\, e^{i(E_n - E_m)t}$$

$\langle O\rangle_{\mathrm{mc}}(E)$ after some time?

Eigenstate thermalization hypothesis

microcanonical ensemble average on diagonal
$E = (E_n + E_m)/2$

spectral function
$\omega = E_n - E_m$

$$\langle n | O | m \rangle = \langle O \rangle_{\mathrm{mc}}(E)\delta_{nm} + e^{-S(E)/2} f_O(E,\omega) R_{nm}$$

exponential suppression

Gaussian random matrix

Deutsch, PRA 43, 2046 (1991)
Srednicki, PRE 50, 888 (1994)
Rigol, Dunjko, Olshanii, Nature 452, 854 (2008)
D’Alessio, Kafri, Polkovnikov, Rigol, Adv. Phys. 65 (2016) 239

Eigenstate thermalization hypothesis

SU(2)

ETH

EE

AF

$$\langle n| O |m\rangle = \langle O\rangle_{\mathrm{mc}}(E)\delta_{nm} + e^{-S(E)/2} f_O(E,\omega) R_{nm}$$

For large system and initial state with small energy fluctuation, ETH leads to:

- Long time average $\bar{O} \approx$ thermal expectation value $\langle O\rangle_T$ → Ergodicity
- Fluctuations of $\langle O\rangle(t)$ around $\bar{O}$ decrease exponentially in system size
- Quantum fluctuations ≈ thermal fluctuations

The system, observed via O, is indistinguishable from a system in thermal equilibrium

SU(2)

ETH

EE

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Evidence for ETH

$$\langle n|O|m\rangle = \langle O\rangle_{\mathrm{mc}}(E)\delta_{nm} + e^{-S(E)/2} f_O(E,\omega) R_{nm}$$

To show:

1. Diagonal matrix elements are exponentially close to microcanonical ensemble
2. Off-diagonal matrix elements correspond to Gaussian random matrix
3. Identify smooth spectral function and show decay for large ω
4. Quantum chaos indicators: RMT properties (BGS conjecture → GOE)

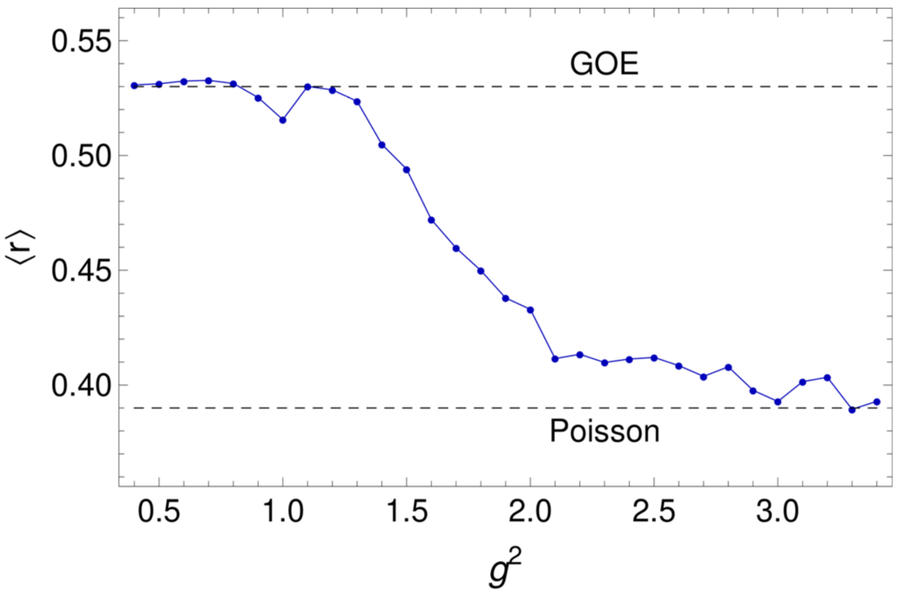
Evidence for ETH

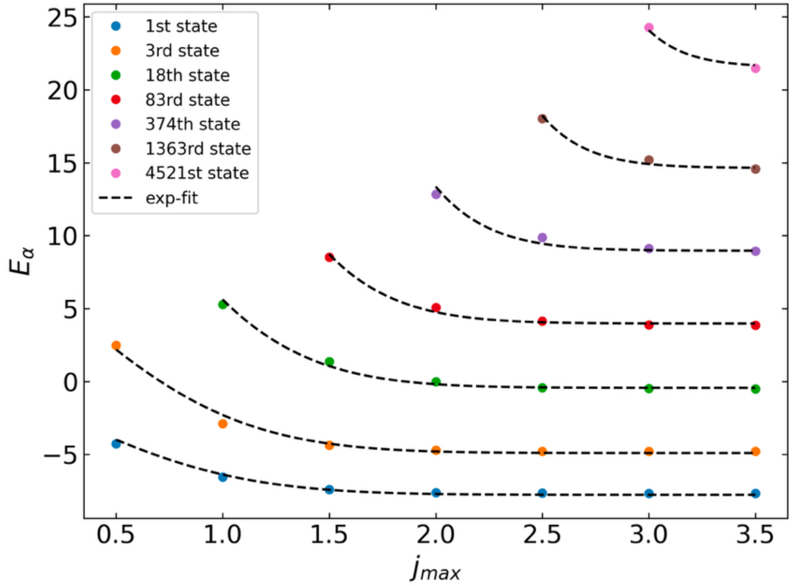
SU(2)

ETH

EE

AF





- Mean restricted gap ratio shows:
  - GOE for weak coupling
  - Poisson for strong coupling
- Smooth interpolation between non-ergodic and RMT behavior

- Only consider converged part of physical spectrum (upper bound)
- Truncate from below to account for finite size effects

Evidence for ETH

- SU(2)
- ETH
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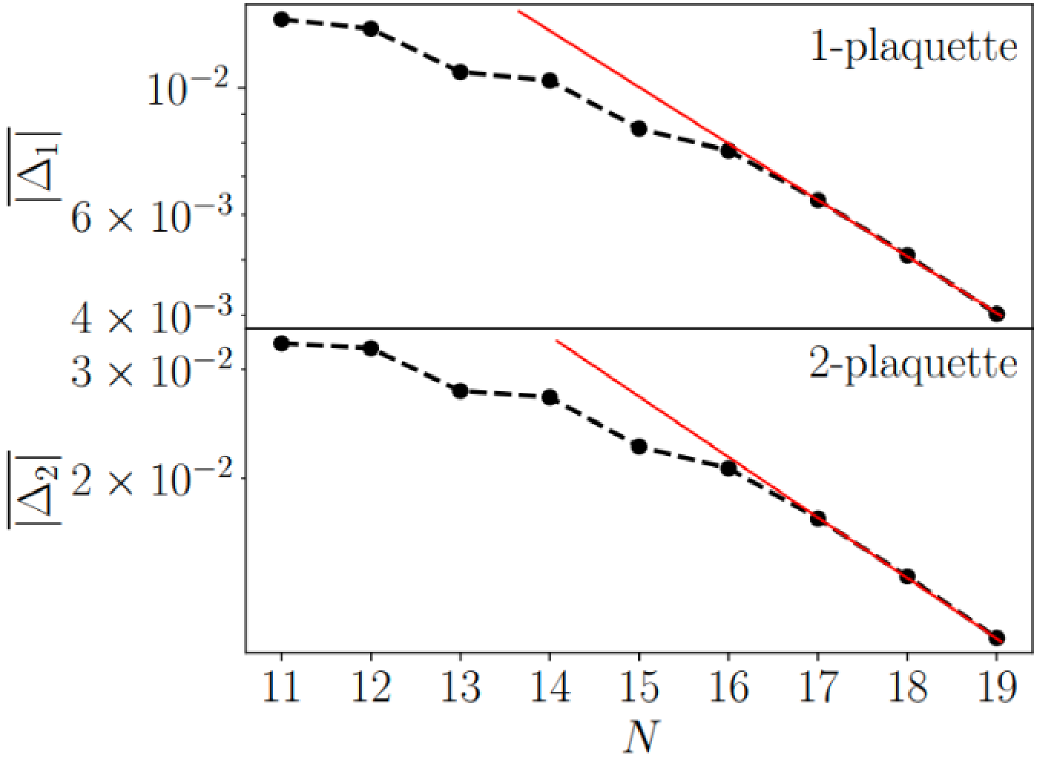
1. Diagonal matrix elements

- Consider 1-plaquette and 2-plaquette operators with ergodic coupling
- Proxy for MC ensemble:

$$\Delta_i(\alpha) \equiv \langle\alpha|O_i|\alpha\rangle - \frac{1}{21}\sum_{\beta=\alpha-10}^{\alpha+10}\langle\beta|O_i|\beta\rangle$$

- Matrix elements are exponentially close to ensemble average value

→ $e^{-S/2}$ scaling of fluctuations



SU(2)

ETH

EE

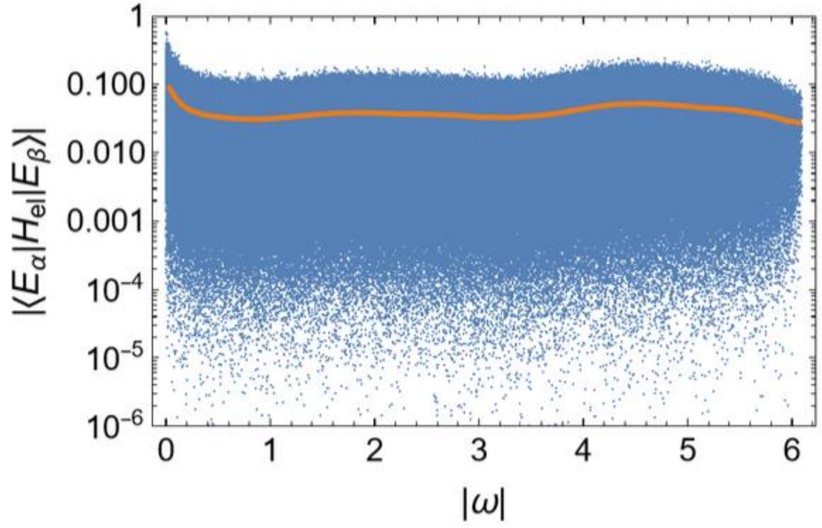
AF

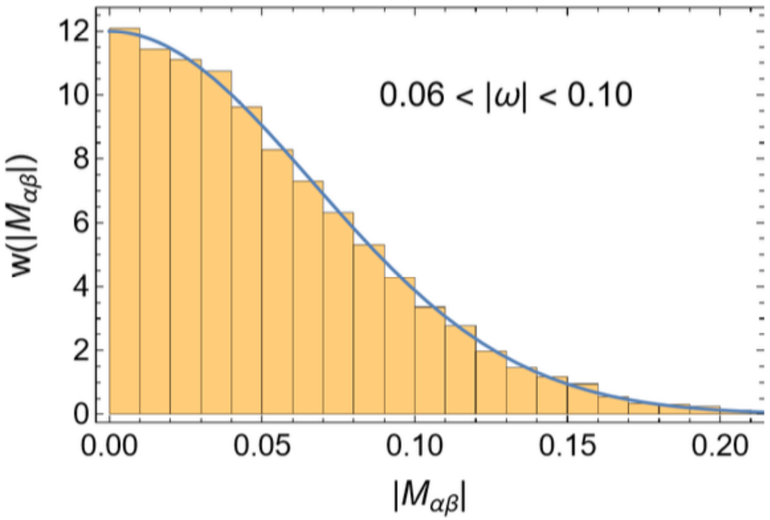
Evidence for ETH

2. Off-diagonal matrix elements

- Characteristic ω-regions:
  - Exponential decay at large ω
  - Bumpy intermediate region (quasiparticle contributions)
  - Diffusive plateau
  - Transport peak

- Off-diagonal matrix elements of H_{el} follow Gaussian distribution





Evidence for ETH

SU(2)

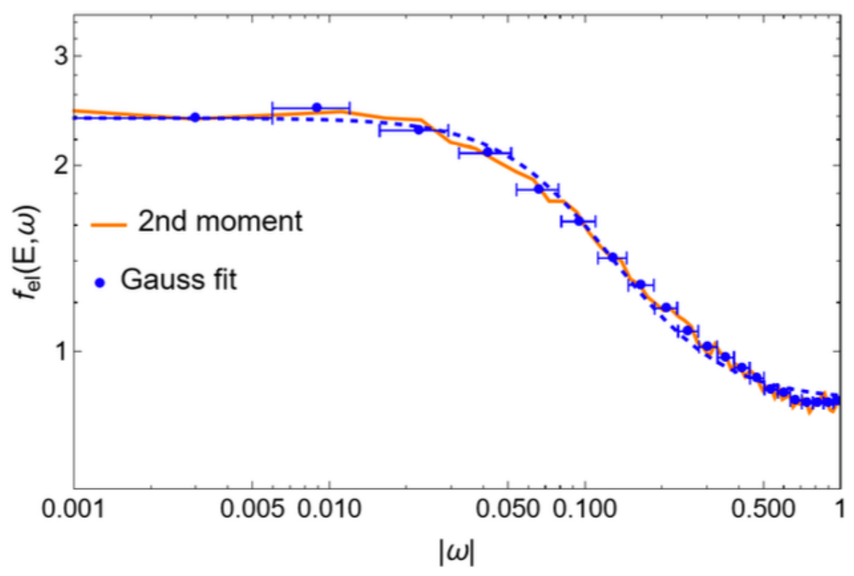
ETH

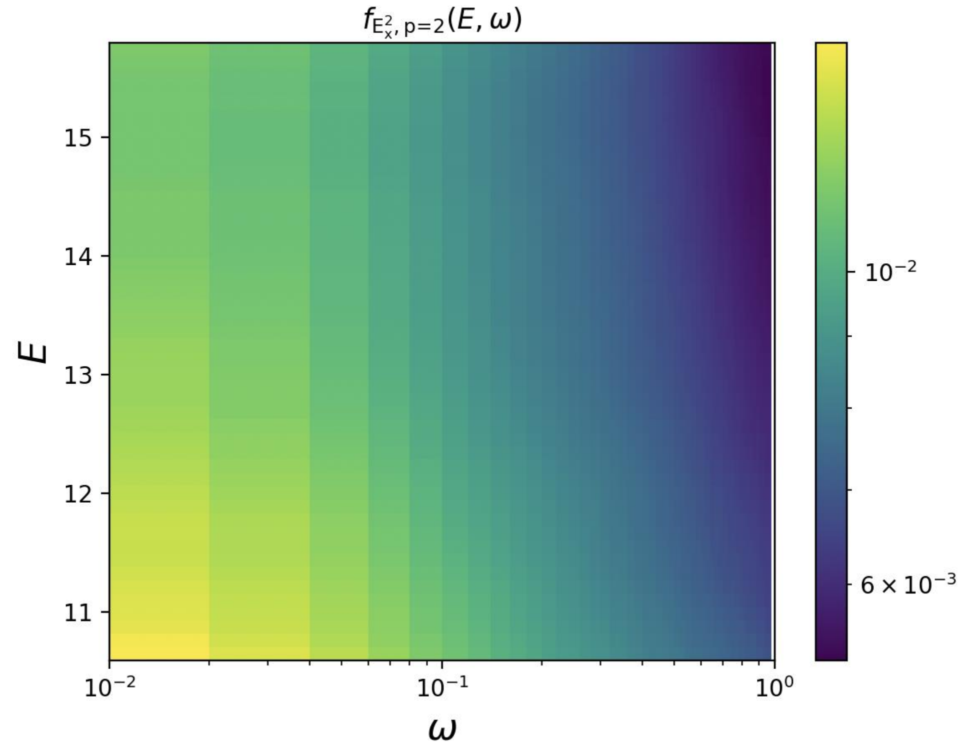
EE

AF

3. Smooth spectral function

- For small ω: diffusive transport peak with plateau
- Plateau disappears when system non-chaotic





Evidence for ETH

SU(2)

ETH

EE

AF

4. RMT properties

- Restricted gap ratio: $0 < r_\alpha = \dfrac{\min[\delta_\alpha, \delta_{\alpha-1}]}{\max[\delta_\alpha, \delta_{\alpha-1}]} \leq 1$

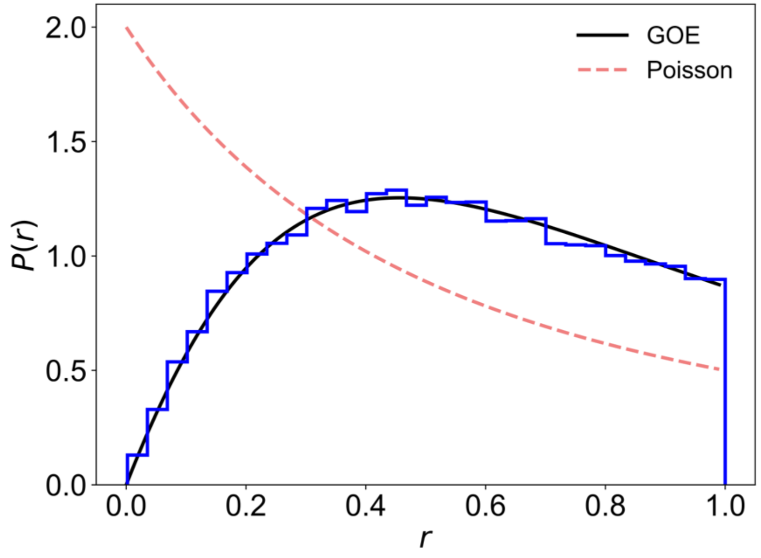
- GOE measure: $\Lambda^T = \dfrac{\left(\mathrm{Tr}[(\mathcal{O}_c^T)^2]\right)^2}{d\,(\mathrm{Tr}[(\mathcal{O}_c^T)^4])}$

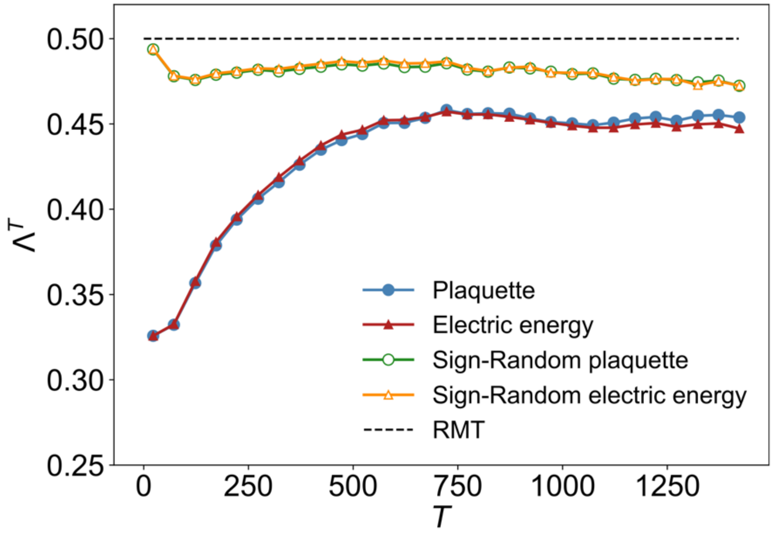
$$\mathcal{O}^T_{\alpha\beta} = \begin{cases} \langle\alpha|\mathcal{O}|\beta\rangle, & |E_\alpha - E_\beta| \leq \frac{2\pi}{T} \\ 0, & |E_\alpha - E_\beta| > \frac{2\pi}{T} \end{cases}$$

$$\mathcal{O}_c^T = \mathcal{O}^T - \mathrm{Tr}[\mathcal{O}^T]/d$$

- GOE prediction: $\Lambda^T \to \frac{1}{2}$

Wang, PRL 128, 180601 (2022)





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Basics and Properties

SU(2)

ETH

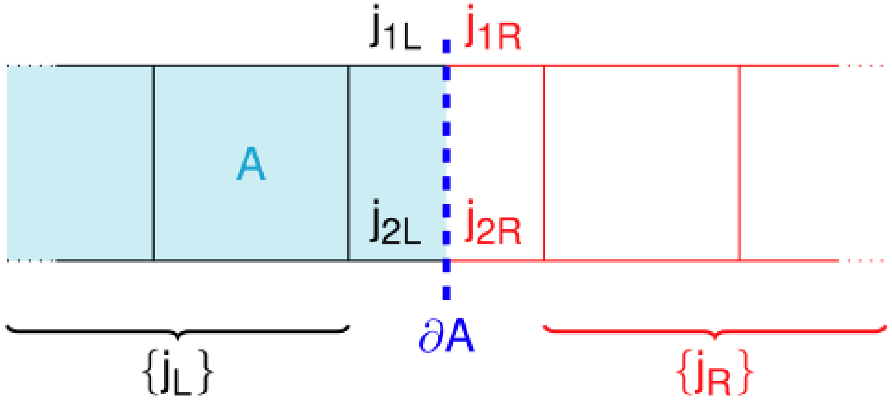
EE

AF

- Von Neumann entropy:

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

$$\rho_A = \mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$$



- EE measures degree of entanglement between a subsystem and its complement

- Ground state: $S_A \sim \mathrm{Area}(\partial A)$

- Highly excited states: $S_A \sim \mathrm{Vol}(A)$

- Crossover: sub-volume growth

- Thermal entropy:

$$\rho_{A,\mathrm{th}}(\beta) = \frac{\mathrm{Tr}_{A^c}(e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

$$E = -(\partial/\partial\beta) \log \mathrm{Tr}(e^{-\beta H})$$

SU(2)

ETH

EE

AF

Page Curve

- Scaling functions:

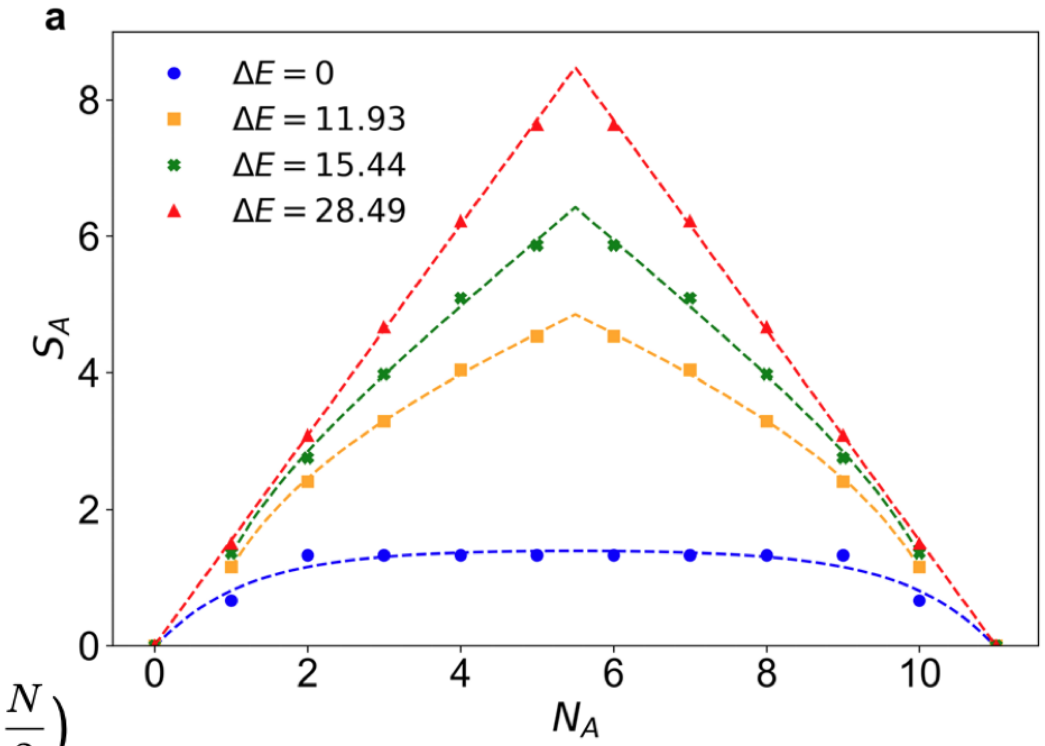
$$S_{\text{area}}(N_A) = b_0 - b_1\left(e^{-N_A/\ell_{\text{corr}}} + e^{-(N-N_A)/\ell_{\text{corr}}}\right)$$

$$S_{\text{vol}}(N_A) = sN_A\theta\left(\frac{N}{2} - N_A\right) + s(N-N_A)\theta\left(N_A - \frac{N}{2}\right)$$

- Crossover function from 2D CFT (Miao, Barthel, 10.1103/PRL.127.040603 (2021))

$$S_{\text{cross}}(N_A) = c_0 + \frac{c}{3}\ln[c_1\sinh(c_1^{-1}N_A)]\theta\left(\frac{N}{2} - N_A\right) + \frac{c}{3}\ln\{c_1\sinh[c_1^{-1}(N-N_A)]\}\theta\left(N_A - \frac{N}{2}\right)$$

- Crossover function can also be derived holographically in AdS3/CFT2



Page Curve

- SU(2)
- ETH
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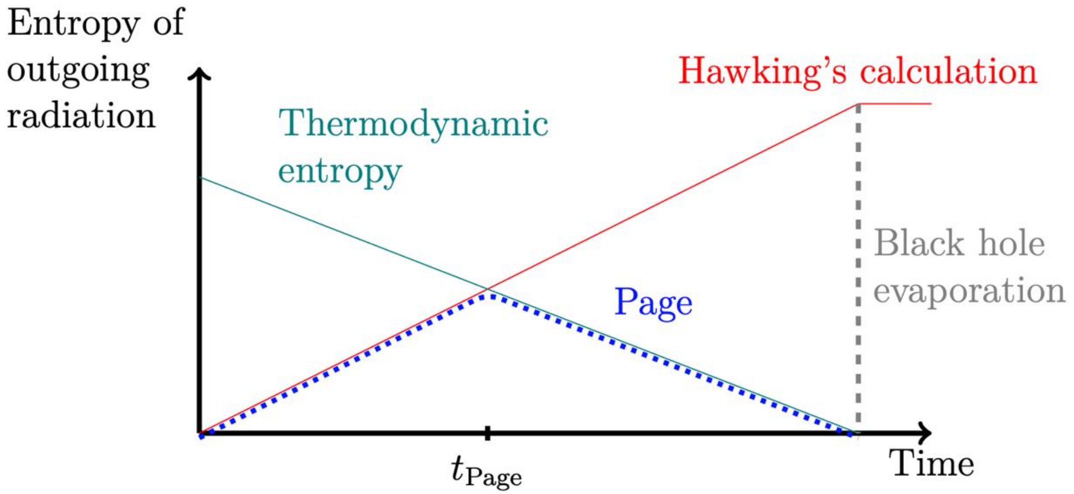
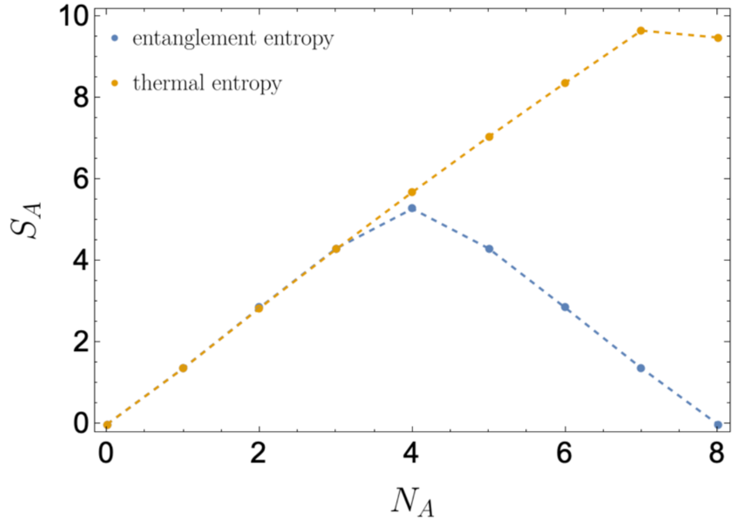




- Early time: system appears thermal
- After evaporation: full contained quantum information visible to observer

- Small subsystem: system appears thermal
- Large subsystem (more than half): quantum correlations become visible

Measurement of highly entangled state (like HIC) indistinguishable from thermal state

Quantum many-body scars

SU(2)
ETH
EE
AF

QMBS

High-energy eigenstates that violate the Eigenstate Thermalization Hypothesis

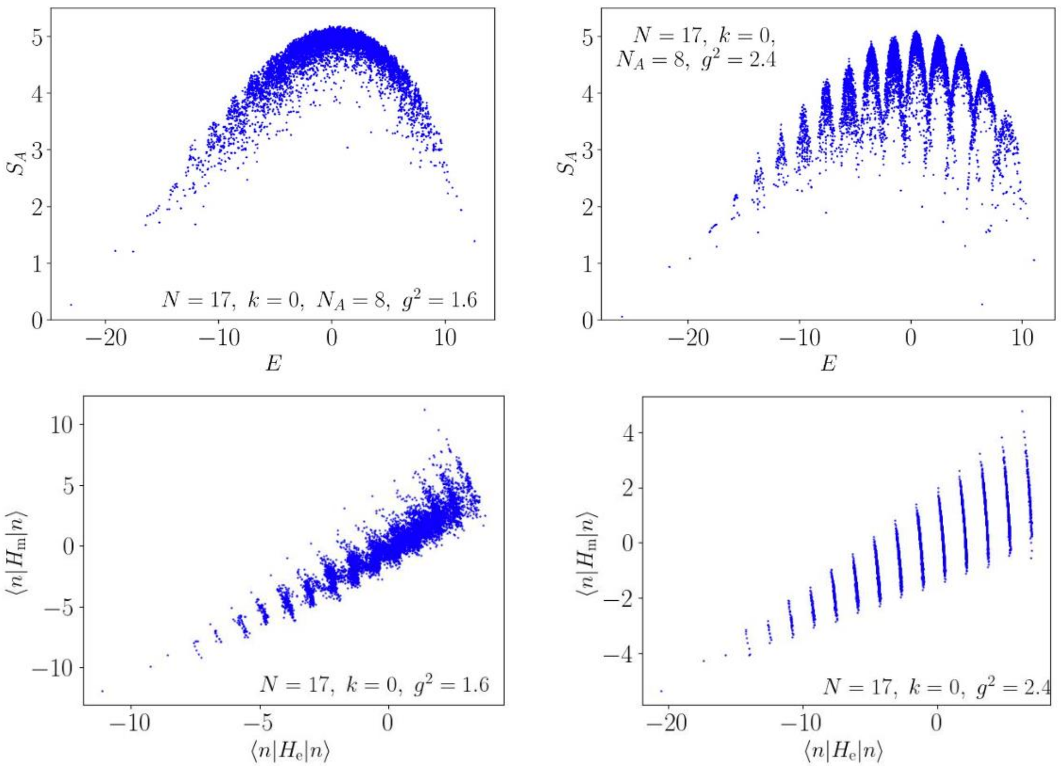
- weak ergodicity breaking mechanism
- subvolume entanglement law
- much lower EE than neighbouring energy eigenstates

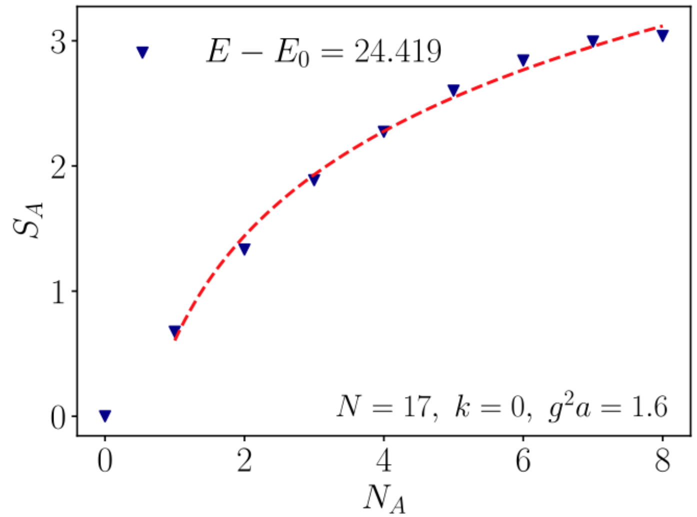
Banerjee, Sen, 2012.08540 (2020)
Aramthottil et al., 2201.10260 (2022)

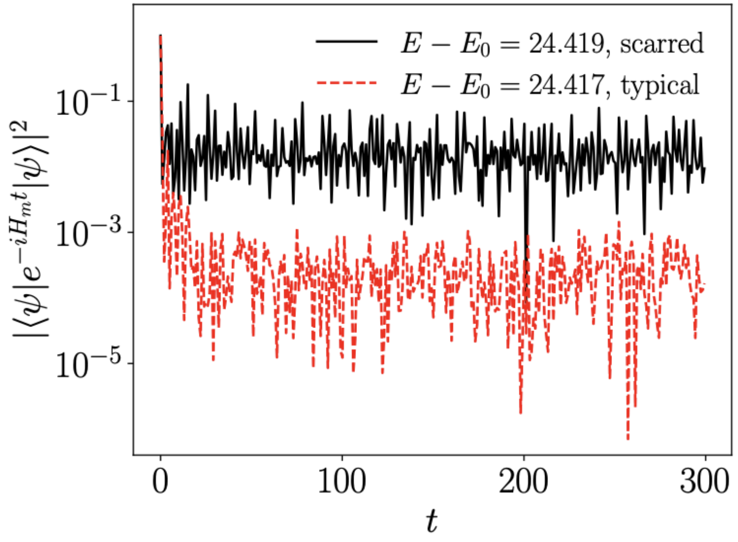
Does SU(2) LGT exhibit QMBSs in the ergodic coupling regime?

Quantum many-body scars: $j_{\max} = 1/2$

- SU(2)
- ETH
- **EE**
- AF







- Continuum property: magnetic energy = electric energy
- Many outliers in entanglement spectrum

Quantum many-body scars: $j_{\max} > 1/2$

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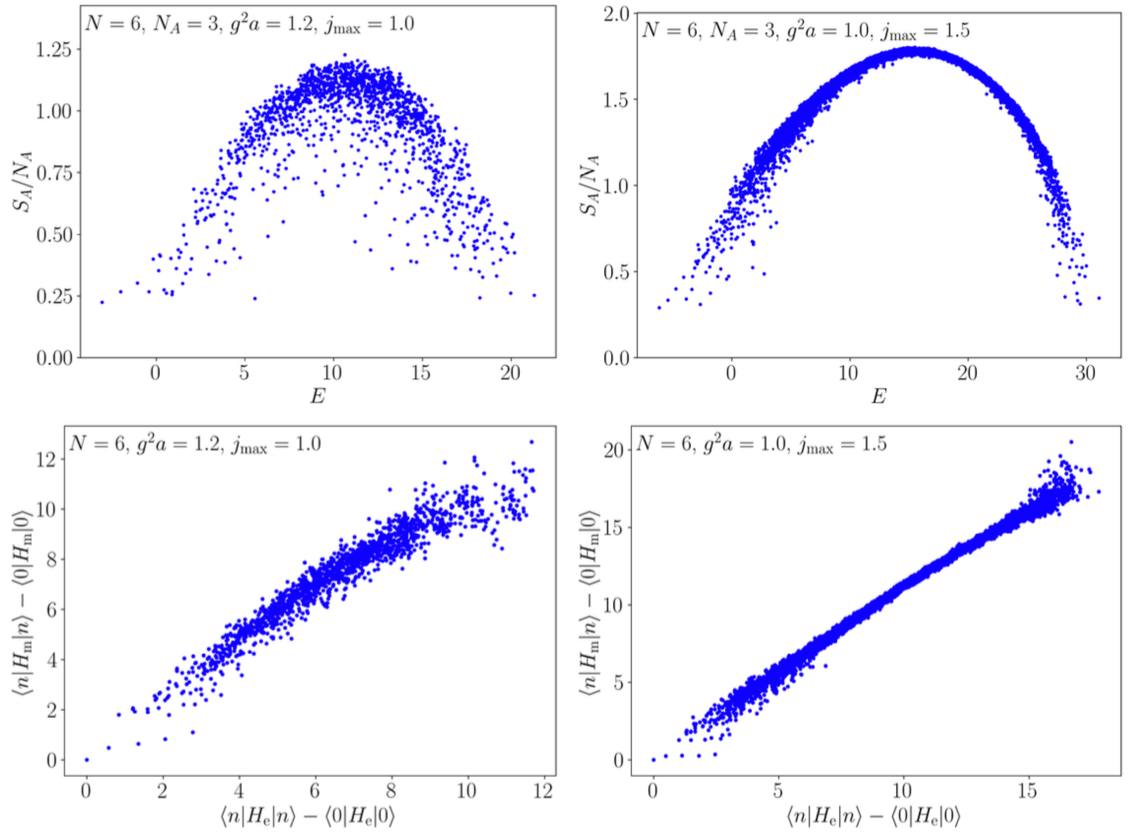

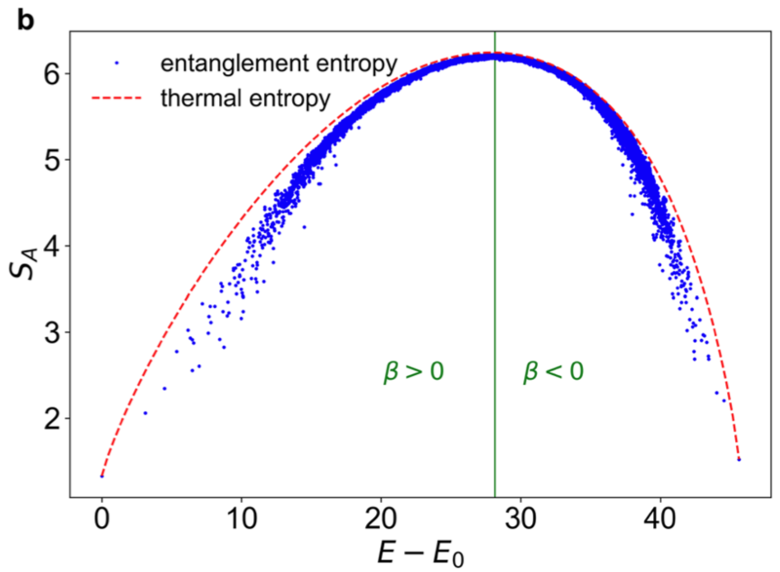

$N = 9,\ N_A = 4\ ,\ g^2 a = 1.2,\ j_{\max} = 1$

- No QMBS for sufficiently high $j_{\max}$
- No ETH-violating states in pure 2+1D SU(2) LGT

Real-time dynamics

- SU(2)
- ETH
- EE
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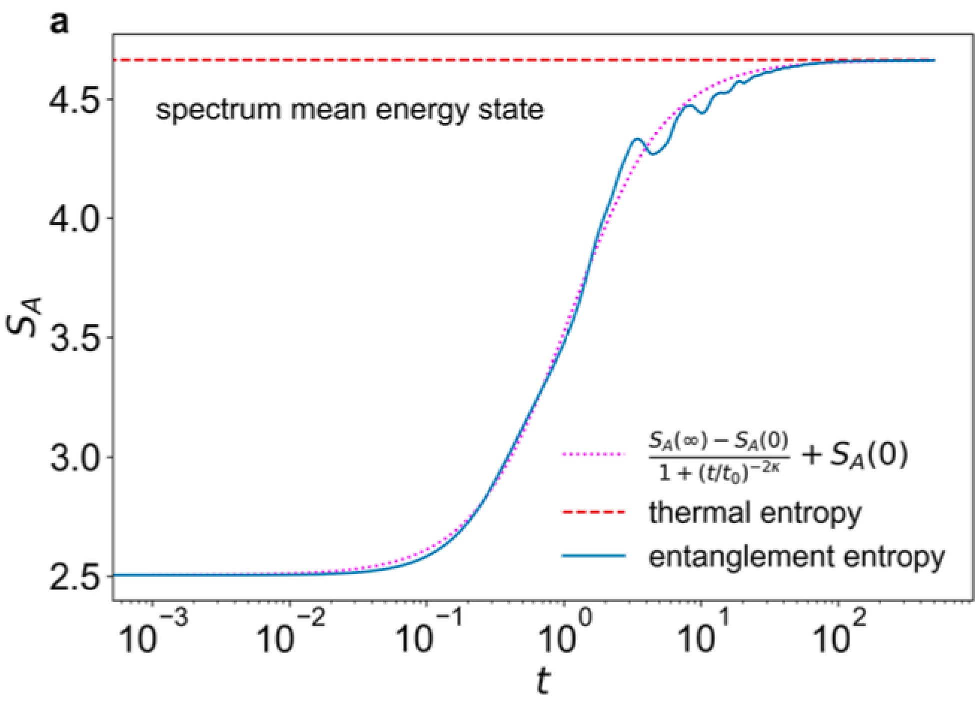
- Real-time evolution of non-eigenstates: electric/momentum basis states
- After thermalization: EE $\approx$ thermal entropy
- Two parameter fit: $$S_A(t) = S_A(0) + \frac{S_A(\infty) - S_A(0)}{1 + (t/t_0)^{-2\kappa}}$$
- t_0 controls thermalization time, κ controls entanglement growth rate

→ **Universal form of EE growth for highly excited states**

$\mathrm{N} = 9, N_A = 3\ , g^2 a = 1.2, j_{\max} = 1$



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Quantum Complexity

SU(2)

ETH

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- So far **FIRST LAYER OF QUANTUMNESS: ENTANGLEMENT**

!

Gottesman-Knill theorem

→ States produced by Clifford gates may be very entangled but can be simulated efficiently with classical resources

- State can be **quantum** in the sense of entanglement, but **classical** in the sense of computation
- Need **SECOND LAYER OF QUANTUMNESS: MAGIC**
- Magic refers to the amount by which a quantum state departs from being a stabilizer (Clifford) one, quantified by **Stabilizer Renyi Entropy**

- **PROBLEM:** qudits instead of qubits for higher j representations

SU(2)

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EE

AF

Anti-flatness

- Instead, look at anti-flatness (AF) of entanglement spectrum: $$\mathcal{F}_A(\psi) := \mathrm{Tr}(\psi_A^3) - \mathrm{Tr}^2(\psi_A^2)$$

- Average over Clifford orbits is equal to magic:

$$\langle \mathcal{F}_A(\Gamma\,|\psi\rangle)\rangle_{C_n} = c(d, d_A) M_{\mathrm{lin}}(|\psi\rangle)$$

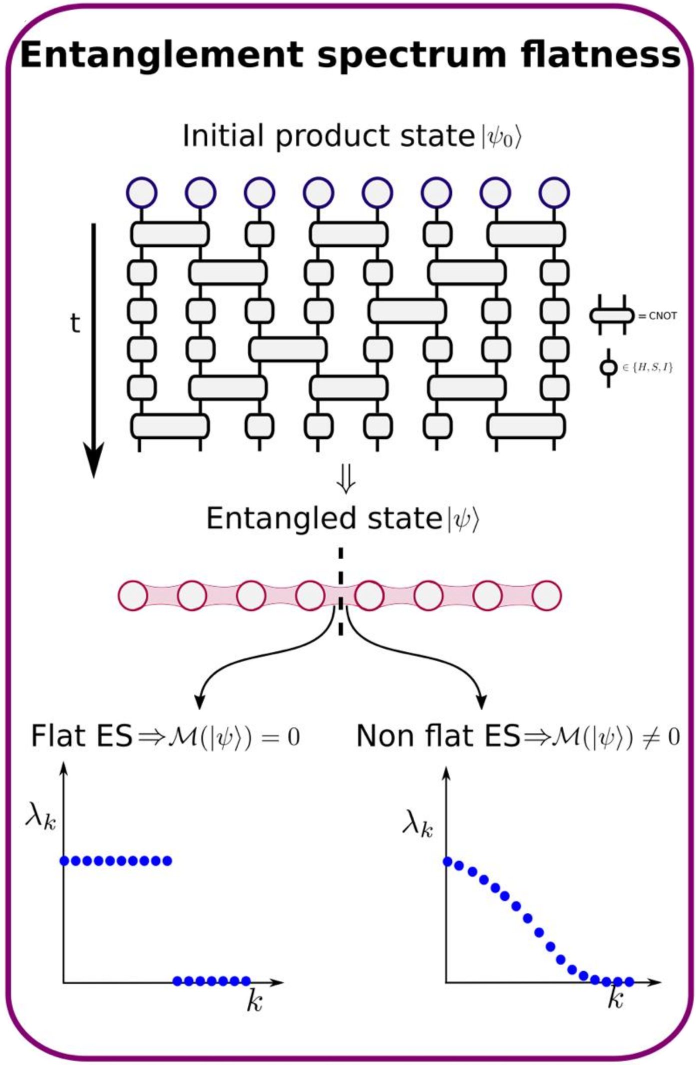
Tirrito et al., PRA 109, L040401 (2024)

- $\mathcal{F}_A(\psi) \neq 0 \Rightarrow$ state contains magic

Odavic et al., PRB 112, 104301 (2025)

- $\mathcal{F}_A(\psi)$ is lower bound for non-local magic

Cao et al., 2403.07056



Tirrito et al., PRA 109, L040401 (2024)

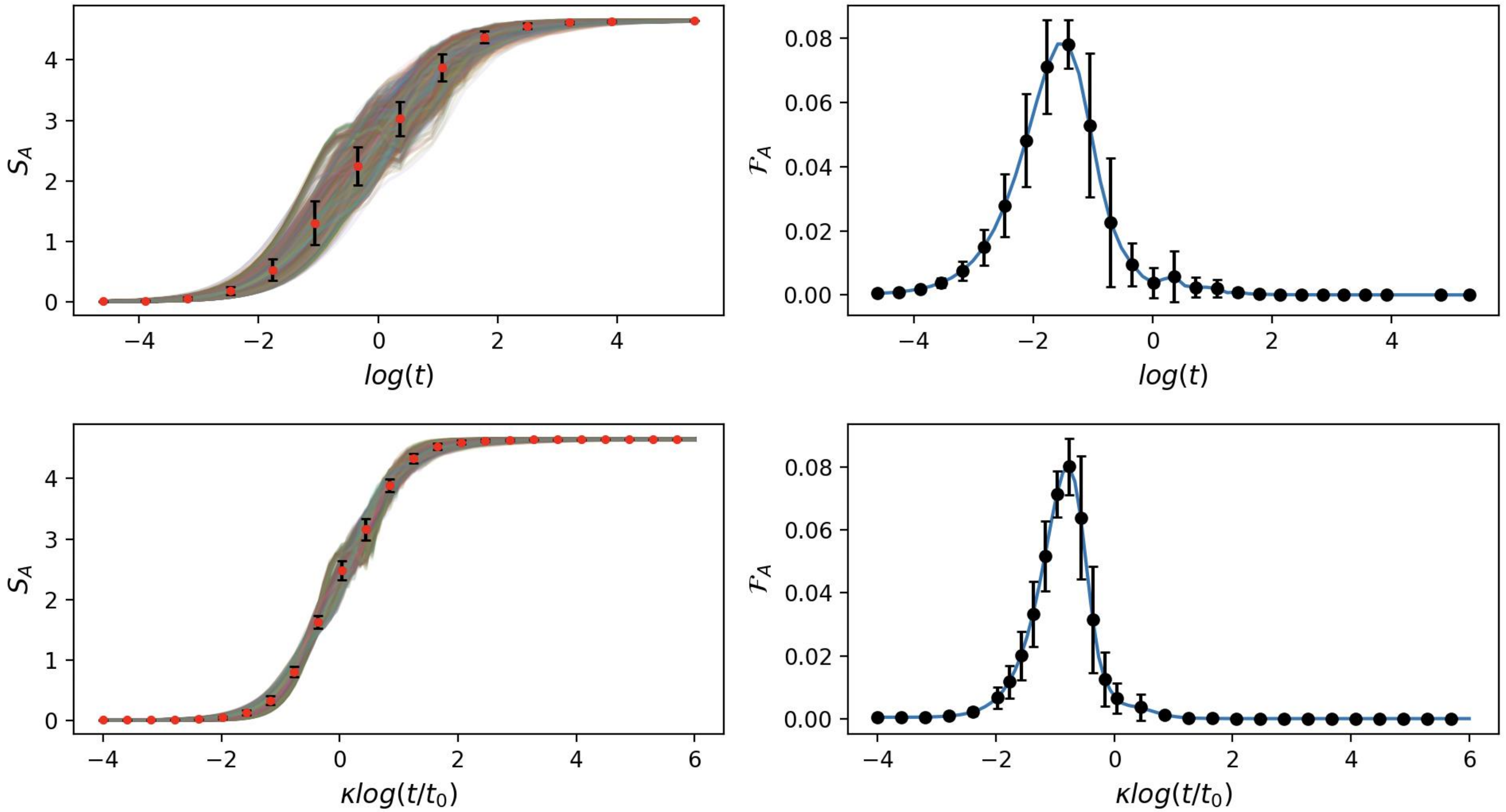
Anti-flatness barrier before thermalization

- Investigate ensemble of computational basis states in energy window
- Synchronize thermalization process using universal EE growth function

PRELIMINARY

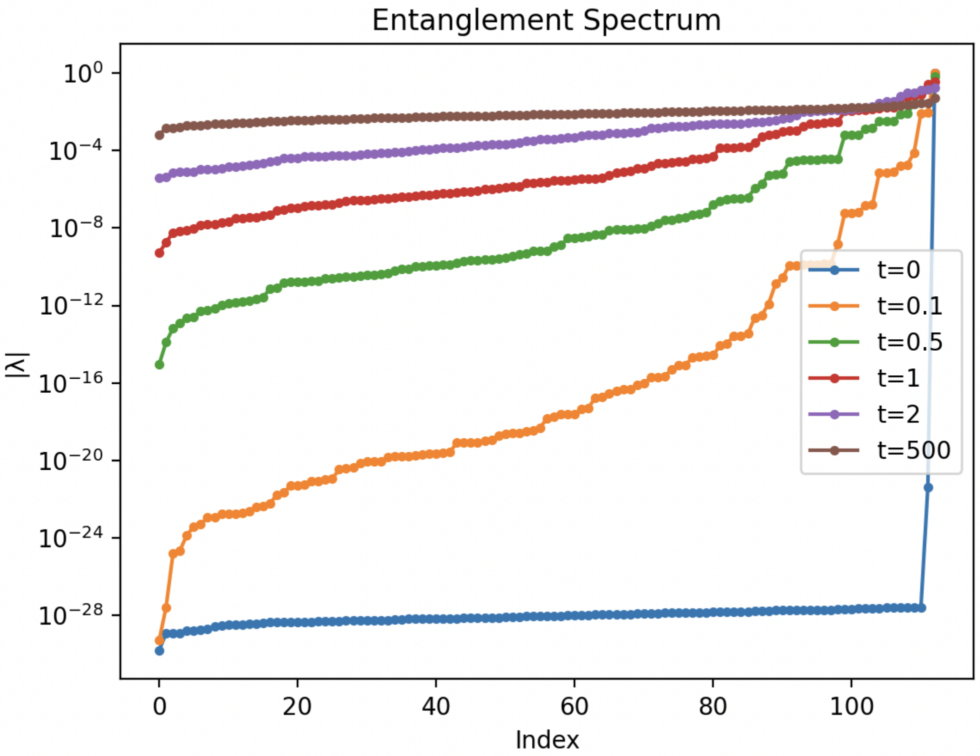
Anti-flatness barrier before thermalization

- Investigate ensemble of computational basis states in energy window
- Synchronize thermalization process using universal EE growth function

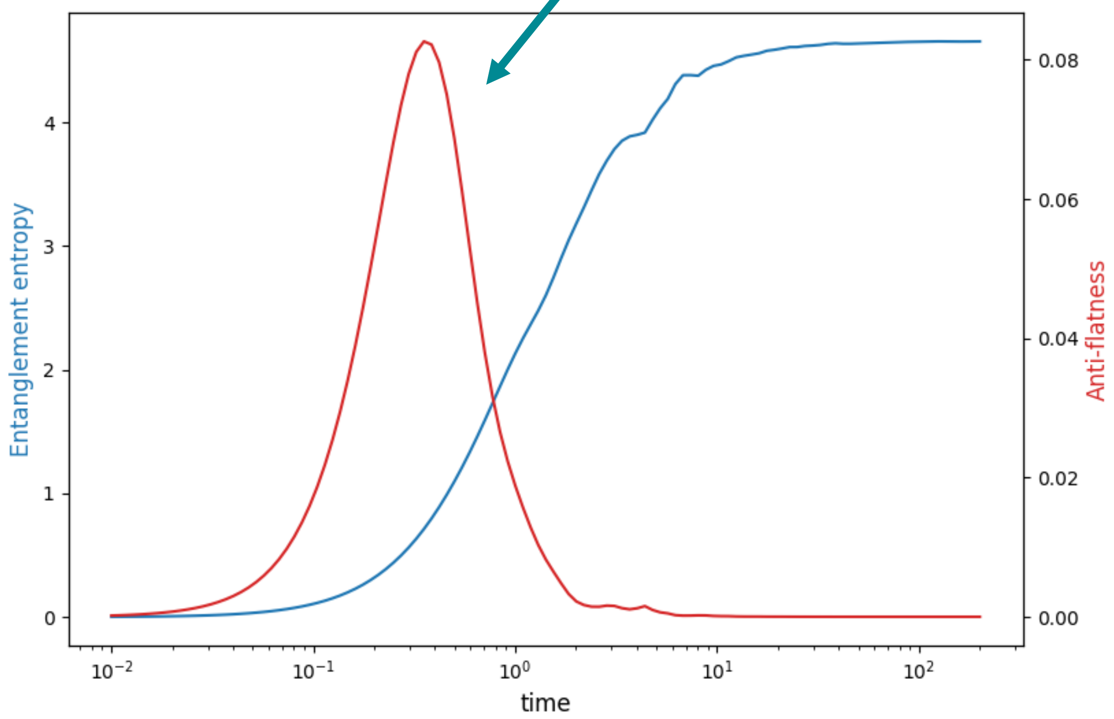


Anti-flatness barrier before thermalization

- Prepare highly excited low-AF state
- Entanglement spectrum flat for $t = 0$ and $t \gg 1$
- Peak correlated to time of maximum EE growth



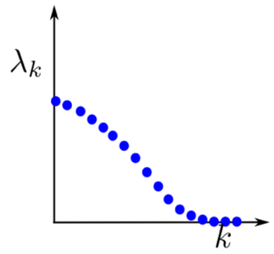
Magic barrier during thermalization process

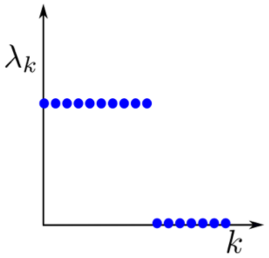


Anti-flatness barrier before thermalization

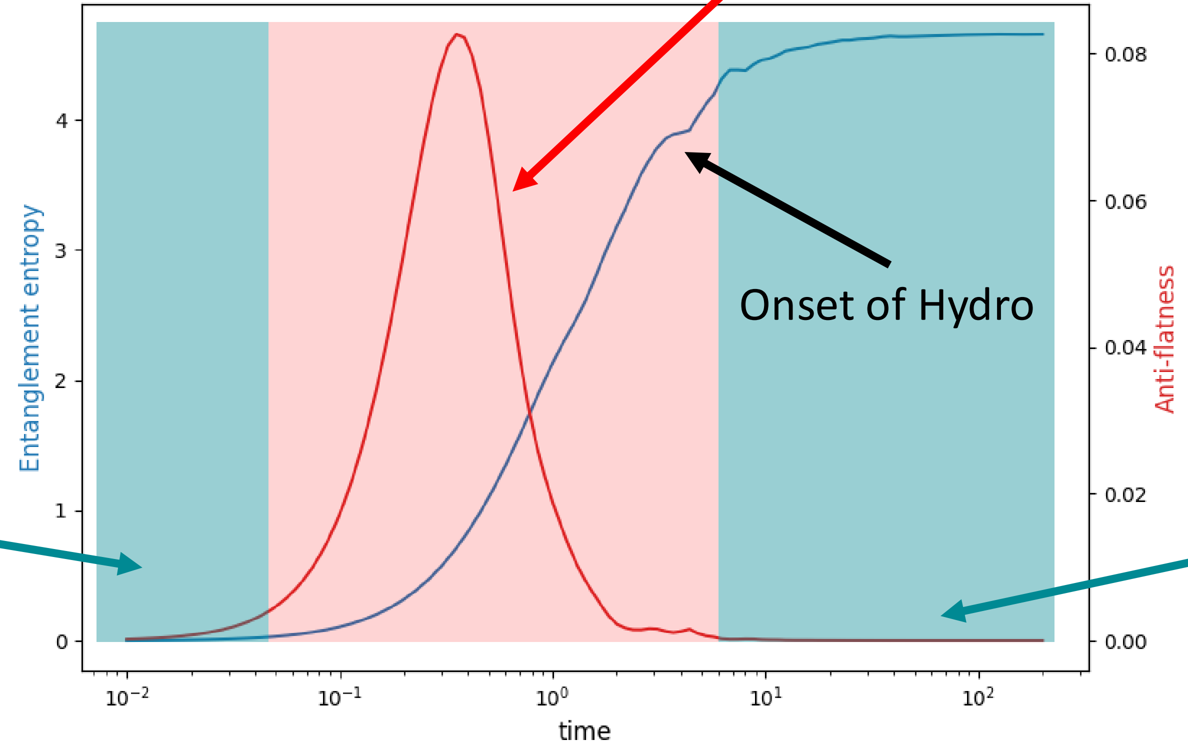
- Magic barrier before hydro reached
- **Need QC for time evolution until hydro models applicable**

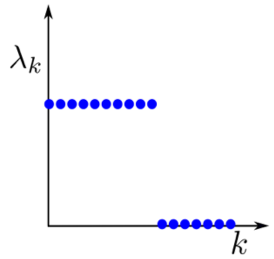
Highly quantum regime during EE growth
→ Demands full quantum computing





Highly excited, low-entangled initial product state

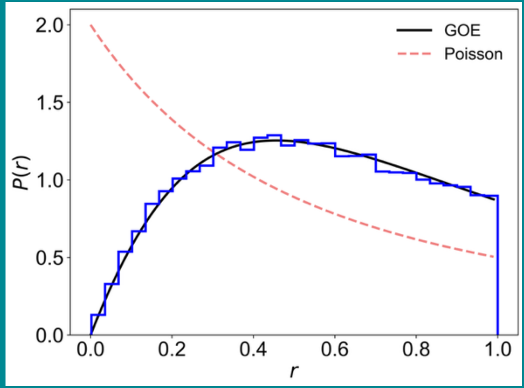




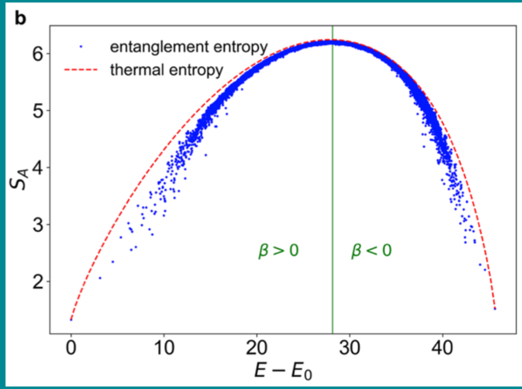
Equilibrium properties can be studied with thermal statistical models

Key Takeaways

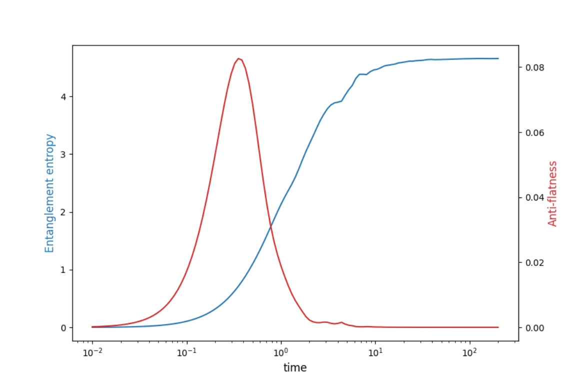
SU(2) LGT compatible with ETH



Absence of QMBSs in pure SU(2) LGT



AF/Magic barrier

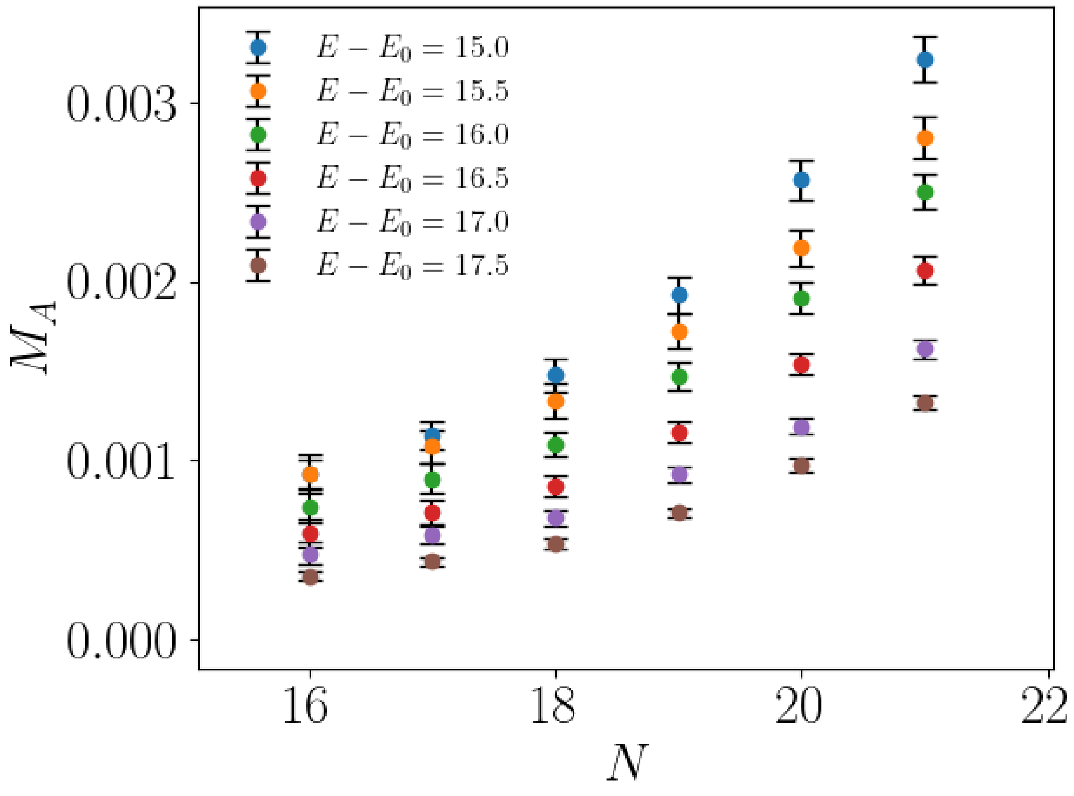


Next Steps

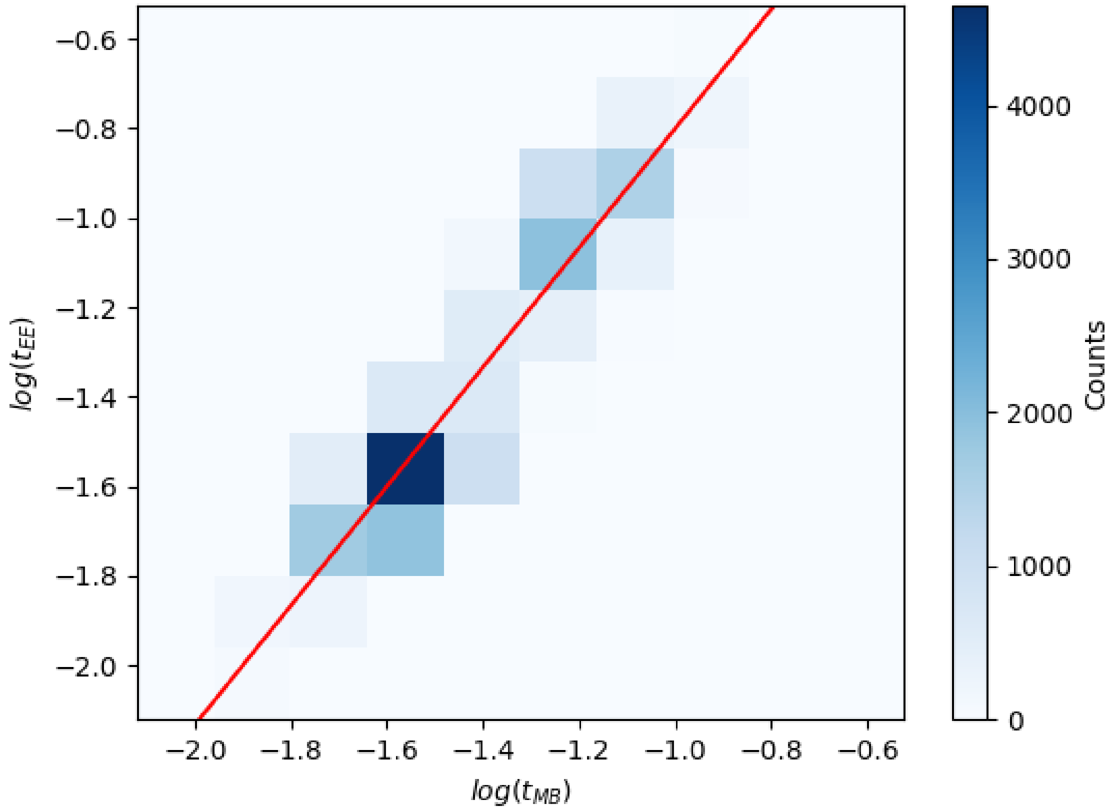
- Interpretation of different thermalization time scales
- Transport coefficients / diffusion constant
- Early-time evolution of EE on quantum device
- Physical limit and SU(3)

Thanks!

Backup

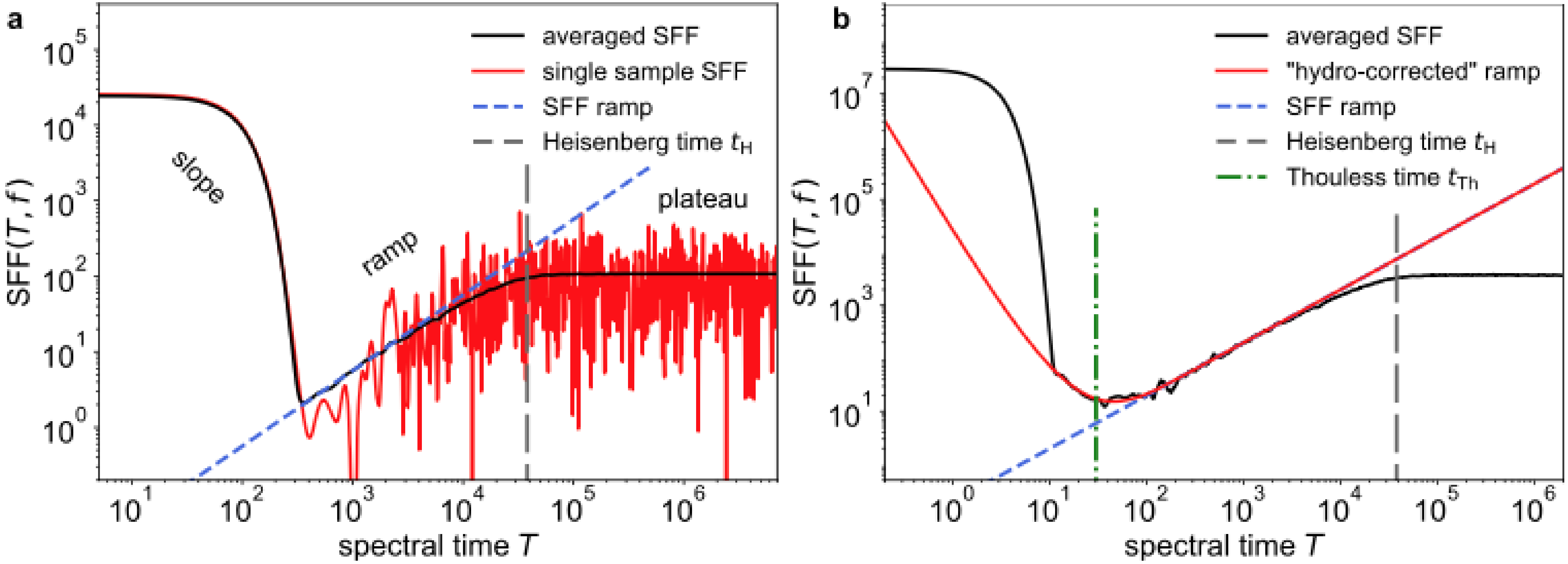


$E - E_0 = 15.0$
$E - E_0 = 15.5$
$E - E_0 = 16.0$
$E - E_0 = 16.5$
$E - E_0 = 17.0$
$E - E_0 = 17.5$
M_A
0.000
0.001
0.002
0.003
16
18
20
22
N



$log(t_{EE})$
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
−1.8
−2.0
−2.0
−1.8
−1.6
−1.4
−1.2
−1.0
−0.8
−0.6
$log(t_{MB})$
Counts
0
1000
2000
3000
4000

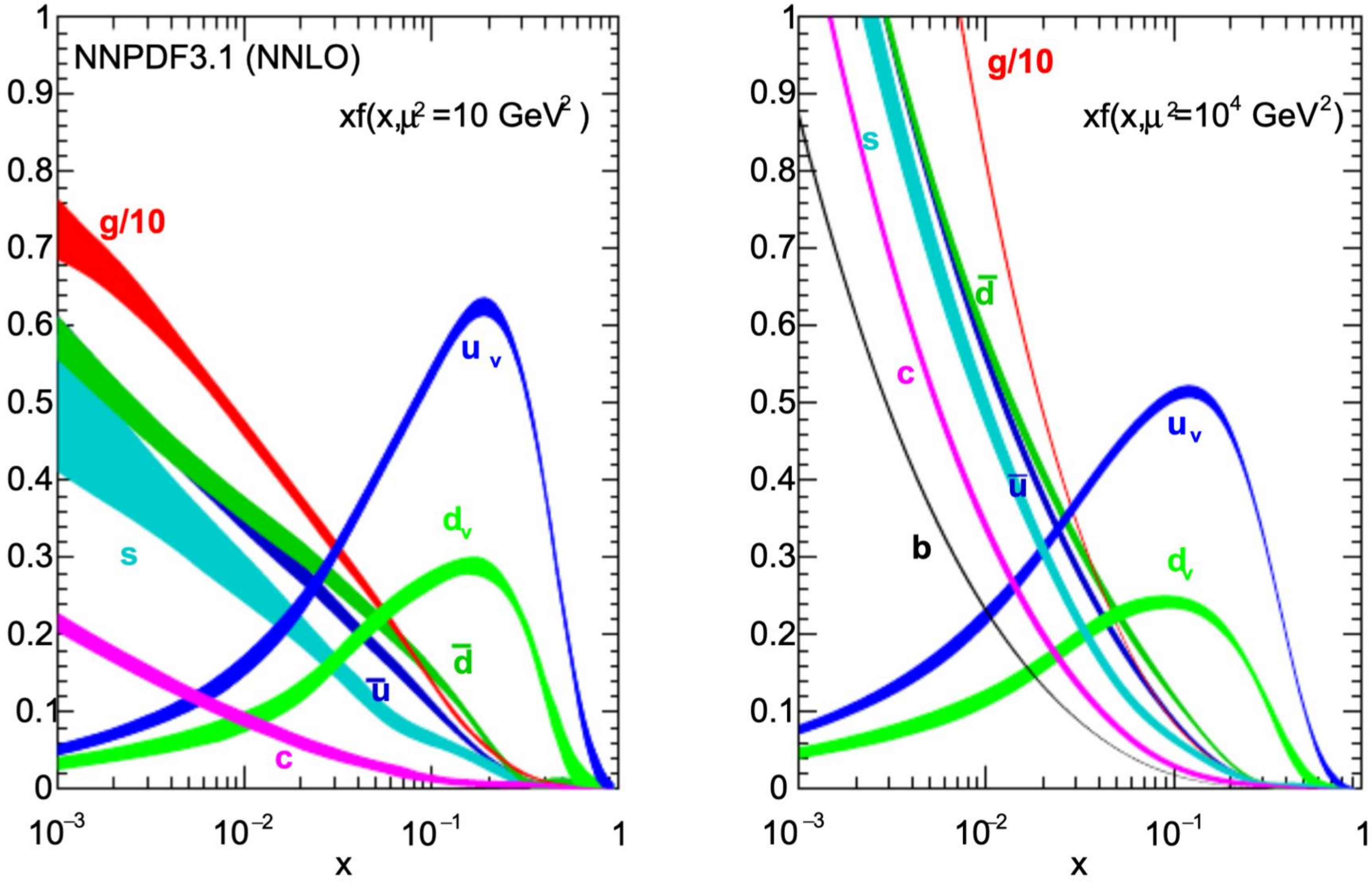
Backup



$$\mathrm{SFF}\,(T,f) := \overline{\sum_{i,j} f(E_i)f(E_j)e^{i(E_i-E_j)T}} = \overline{\left|\int_{-\infty}^{\infty} dE f(E)\rho(E)e^{-iET}\right|^2}$$

$$\mathrm{SFF}_{\text{hydro-ramp}}(T,f) = \frac{T}{\pi\mathfrak{b}}\left[\prod_{\lambda\in\mathrm{spec}(-\Delta)} \frac{1}{1-\mathrm{e}^{-D\lambda T}}\right]\int dE f^2(E)$$

Backup



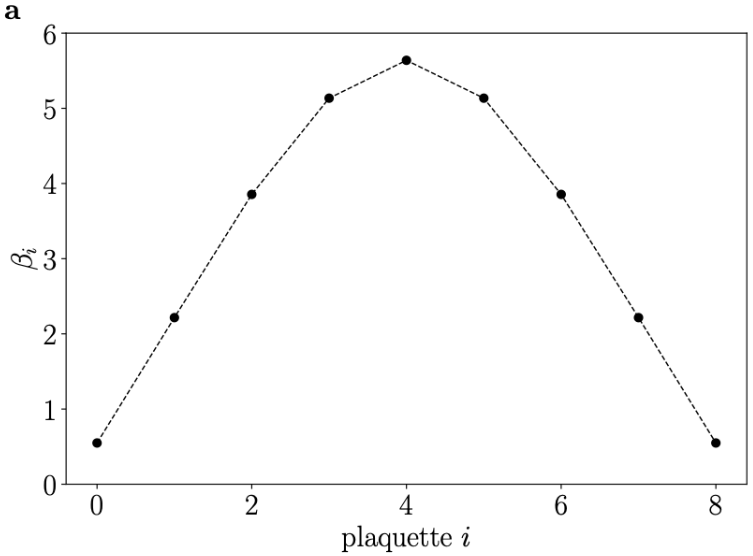
Ball et al., 1706.00428 (2017)

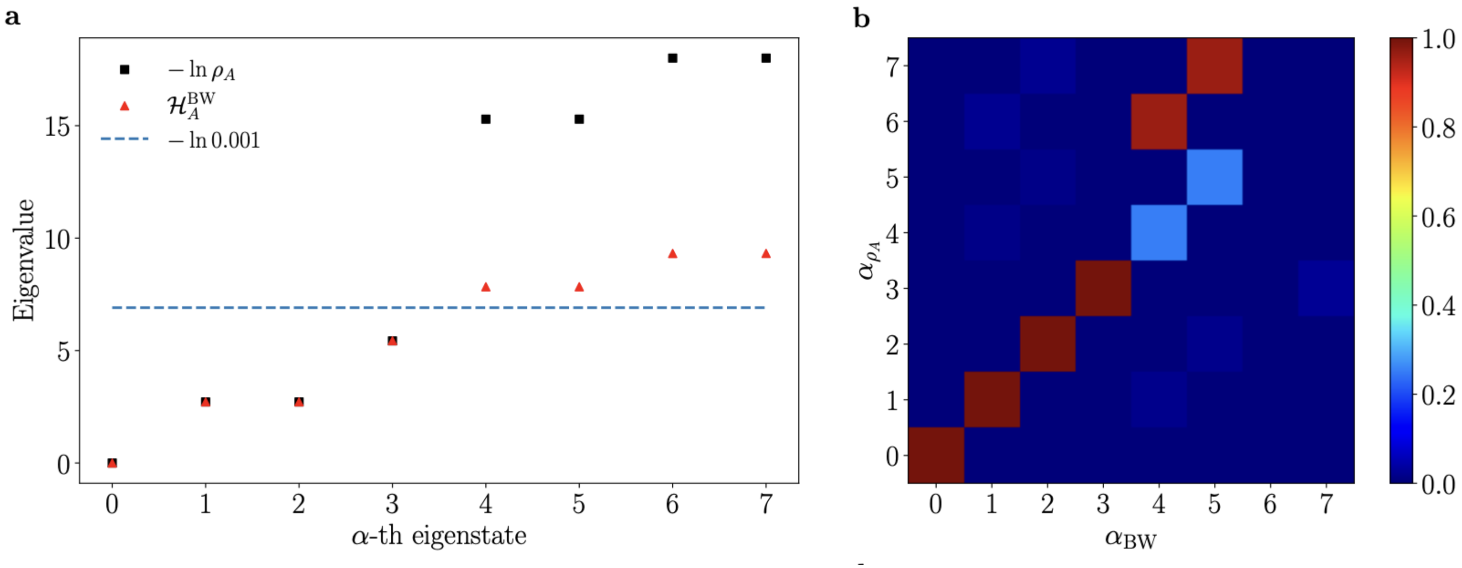
Backup

$$\mathcal{H}_A = \int_0^\infty dx_1 \beta(x_1) H(x_1)$$

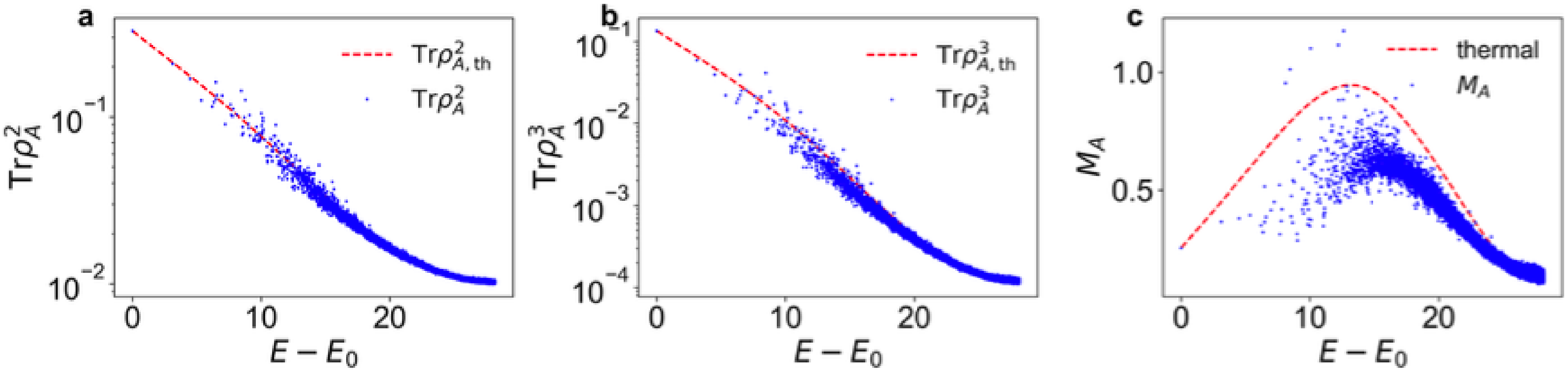
$$\rho_A \equiv e^{-\mathcal{H}_A} / \mathrm{Tr}(e^{-\mathcal{H}_A})$$

Compare to RDM of ground state





Backup



a
$\mathrm{Tr}\rho^2_{A,\mathrm{th}}$
$\mathrm{Tr}\rho^2_A$
10^{-1}
10^{-2}
0
10
20
$E - E_0$
b
$\mathrm{Tr}\rho^3_{A,\mathrm{th}}$
$\mathrm{Tr}\rho^3_A$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
10
20
$E - E_0$
c
thermal
M_A
1.0
0.5
0
10
20
$E - E_0$