Obtaining continuum physics from dynamical simulations of Hamiltonian lattice gauge theories

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Collaborators: Siddarth Hariprakash[3,4,5], Christian Bauer[4]

Based off work in: [arXiv:2506.16559]

QuantHEP 2025

Date: October 2, 2025

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Cost of a quantum simulation

- Calculating physical observables requires taking the continuum limit $a \to 0$
- Fair comparisons between methods must include this cost
- Error from approximate time evolution can spoil continuum limit
- Procedure to remove time evolution errors when using Trotter methods developed
  [Marcela Carena, Henry Lamm, Ying-Ying Li, Wanqiang Liu, PRD, arXiv:2107.01166]
  $\to$ not applicable to other algorithms
- **Our Goal:** Develop a general framework for controlling impact of time evolution errors on continuum limit applicable to any algorithm

Outline

1. Review continuum limit assuming exact time evolution

2. Review existing methods for treating Trotter errors using renormalization [Marcela Carena, Henry Lamm, Ying-Ying Li, Wanqiang Liu, PRD, arXiv:2107.01166]
→ Present simpler, alternative approach to treating Trotter errors

3. Present general procedure applicable to any time evolution algorithm
→ Statistically-Bounded Time Evolution Protocol

Hamiltonian LGT

Pure gauge Hamiltonian

$$H_{\mathrm{KS}} = \frac{1}{a}\left[g_t^2 \widetilde{H}_E - \frac{1}{g_s^2}\widetilde{H}_B\right]$$

Lorentz invariance broken $\longrightarrow$ gauge coupling for H_E and H_B different

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Speed of light $c(a) \neq 1$ changes overall scale of H:

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Hamiltonian LGT: Two Different Scales

Hamiltonian

$$\hat{H}_{\mathsf{KS}} = a_t H_{\mathrm{KS}}$$

Temporal scale: $a_t = \frac{a}{c}$

(Notation: hatted quantities are dimensionless)

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Different "rulers" for temporal and spatial quantities

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Dimensionless quantities measured on the lattice:

$$\hat{m} = a_t\, m\,, \qquad \hat{t} = \frac{1}{a_t} t\,, \qquad \hat{p} = ap, \qquad \hat{x} = \frac{1}{a} x$$

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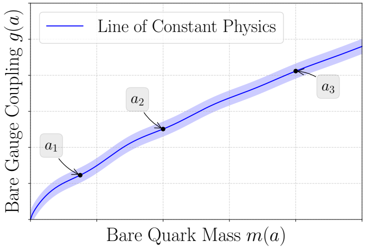
Renormalization: taking the continuum limit

1. **Determine renormalization trajectory**



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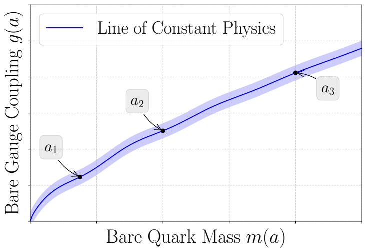

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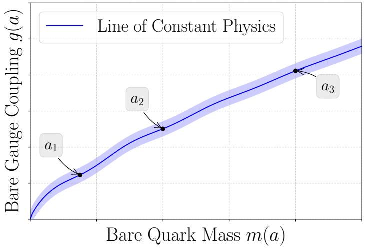

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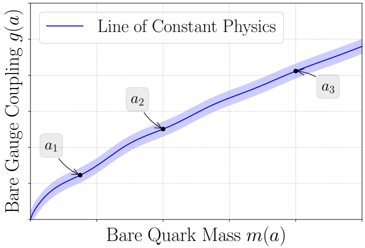

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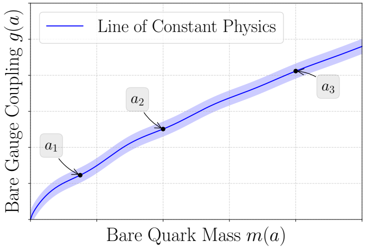

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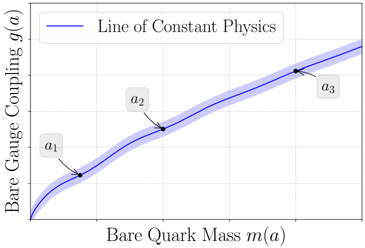

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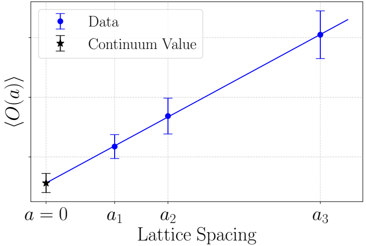

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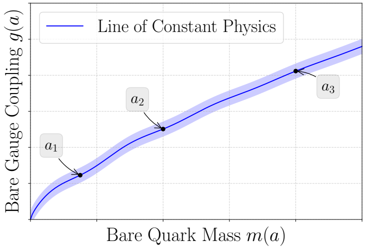

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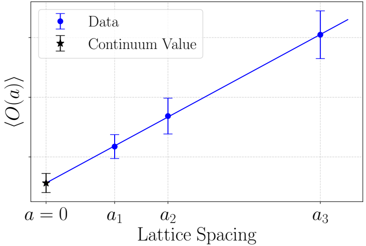

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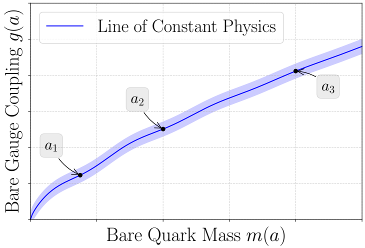

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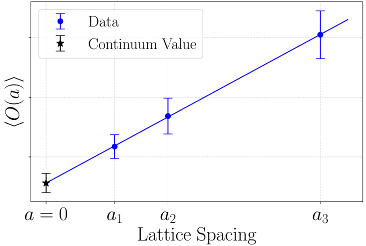
- **b** Renormalize operator $\hat{O}(t)$ (if necessary)
- **c** Extrapolate to continuum



Effective Hamiltonian with Trotter

*(hatted quantity $\leftrightarrow$ dimensionless)

Look at effective Hamiltonian simulated using Trotter [Marcela Carena, Henry Lamm, Ying-Ying Li, Wanqiang Liu, PRD, arXiv:2107.01166]

$$e^{-i\hat{\delta}_t(\hat{H}_E+\hat{H}_B)} \approx e^{-i\frac{\hat{\delta}_t}{2}\hat{H}_E} e^{-i\hat{\delta}_t\hat{H}_B} e^{-i\frac{\hat{\delta}_t}{2}\hat{H}_E}$$

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- View as “temporal lattice”, treat $\hat{\delta}_t$ as parameter in the effective Hamiltonian
- $\hat{\delta}_t \neq 0$ changes physics which changes values used in tuning and scale setting

$$g(a) \to g(a,\delta_t), \qquad m(a) \to m(a,\delta_t), \qquad \dots$$

Continuum physics achieved taking simultaneous limit $\lim_{a\to 0}\lim_{\delta_t\to 0}$

(Or, work at fixed anisotropy $\xi = a/\delta_t$ and extrap $a \to 0$)

Euclidean simulations to assist renormalization

Main idea: relate H_{eff} to Euclidean transfer matrix T on anisotropic lattice

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- Does not reduce size and quality of quantum device needed

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- Trotter errors go to zero as $\mathcal{O}(a^2)$ in continuum limit

Extension to other algorithms?

Can we apply a similar procedure to simulations done using other simulation algorithms?

Consider Quantum Signal Processing as test case

Quantum Signal Processing: high level review

Hamiltonian input model: given $|\psi\rangle \to H|\psi\rangle$, one can implement $f(H)|\psi\rangle$

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Using Jacobi-Anger expansion for time evolution:

$$e^{-iHt} \approx J_0(\lambda t) + 2\sum_{k>0 \text{ even}}^{d} (-i)^{k/2} J_k(\lambda t) T_k(H/\lambda) - 2i \sum_{k \text{ odd}}^{d} (-i)^{(k-1)/2} J_k(\lambda t) T_k(H/\lambda)$$

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Provably optimal scaling with t and ϵ [G. Low, I. Chuang, Quantum, arXiv:1606.02685]

$$\text{Calls to } U_H = \mathcal{O}\big(\lambda t + \log\frac{1}{\epsilon}\big)$$

Breakdown of previous approach to QSP

Can we apply previous approaches to control time evolution errors as $a \to 0$ for QSP?

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Previous picture relies on effective Hamiltonian formalism

Unclear how to proceed:
$\to$ approximate evolution is not unitary, not clear what H_{eff} is

We need an alternative more general approach

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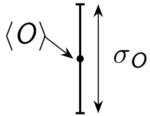
$\to$ renormalization procedure simplifies to exact time evolution case

Statistically-Bounded Time Evolution (SBTE) Protocol

Calculate time-dependent observable $\langle \hat{O}(t, a) \rangle$ to uncertainty σ_O
$\rightarrow$ sources of uncertainty from shot noise, device noise, etc.



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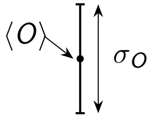

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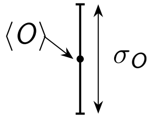

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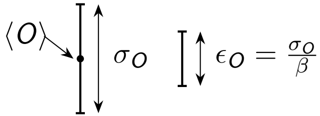

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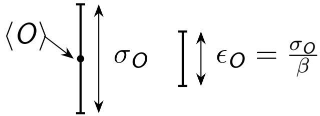

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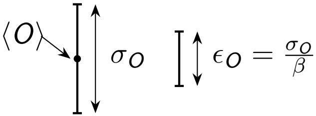

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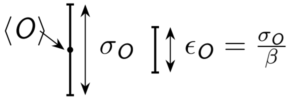
Key question: what is the additional computational cost to ensure this?

Examples: Product formula and Quantum Signal Processing

Set simulation error $\epsilon_{\text{sim}} = \left(\frac{2\beta\|O(0,a)\|}{\sigma_O}\right)^{-1}$



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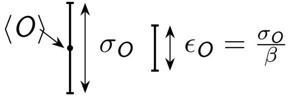

SBTE applied to Product Formulas:

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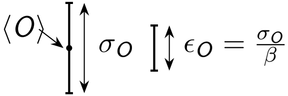

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[A. Childs, Y. Su, M. Tran, S. Zhu, PRX arXiv:1912.08854]

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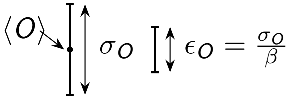

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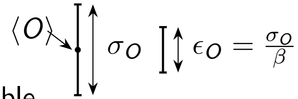

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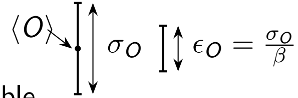

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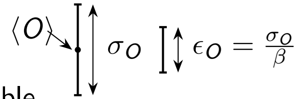

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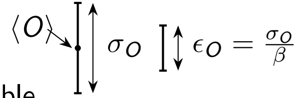

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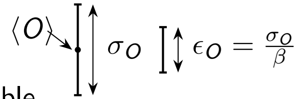

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SBTE applied to QSP: extra cost negligible *a priori*

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Summary and conclusions

- Full cost of a quantum simulation requires cost of $a \to 0$ limit
- Previously, controlling impact of approximate time evolution errors only understood for Trotter methods
- We introduced a simple, general approach applicable to any quantum algorithm
  $\to$ Statistically-Bounded Time Evolution Protocol
- Opens the door to performing end-to-end cost comparisons between different algorithms

Backup slides

Hamiltonian LGT

Kogut-Susskind Hamiltonian

$$H_{\text{KS}} = \left[\frac{g_t^2}{a}\widetilde{H}_E - \frac{1}{ag_s^2}\widetilde{H}_B + \frac{\kappa}{a}\widetilde{H}_{\text{hop}} + m\widetilde{H}_M\right]$$

Lorentz invariance broken $\longrightarrow$ gauge coupling for H_E and H_B different

- Speed of light: $c = \frac{g_t}{g_s}$
- Gauge coupling $g = \sqrt{g_s g_t}$
- Hopping coefficient κ
  $\rightarrow$ Euclidean simulations on anisotropic lattices have bare fermionic anisotropy factor γ_f
  $\rightarrow$ need γ_f so physical anisotropy $\xi = a/a_0$ "seen" by gluons and fermions is the same
  $\rightarrow$ keeping track of γ_f in Transfer matrix implies need for $\kappa(a) \neq 1$

Speed of light $c(a) \neq 1$ changes overall scale of H:

$$H_{\text{KS}} = \frac{c}{a}\left[g^2\widetilde{H}_E - \frac{1}{g^2}\widetilde{H}_B + \frac{\kappa}{c}\widetilde{H}_{\text{hop}} + (\frac{a}{c}m)\widetilde{H}_M\right] \equiv \frac{1}{a_t}\hat{H}_{\text{KS}}$$