Computing composite-particle mass spectra in the Hamiltonian formalism

Akira Matsumoto (Osaka Metropolitan U, RIKEN iTHEMS)
collaboration with
Etsuko Itou (YITP Kyoto U, RIKEN iTHEMS) and **Yuya Tanizaki** (YITP Kyoto U)

JHEP11 (2023) 231 [2307.16655]
JHEP09 (2024) 155 [2407.11391]

QuantHEP 2025, 29 September 2025 @Lawrence Berkeley National Lab

Osaka Metropolitan University

Background: mass spectrum of QCD

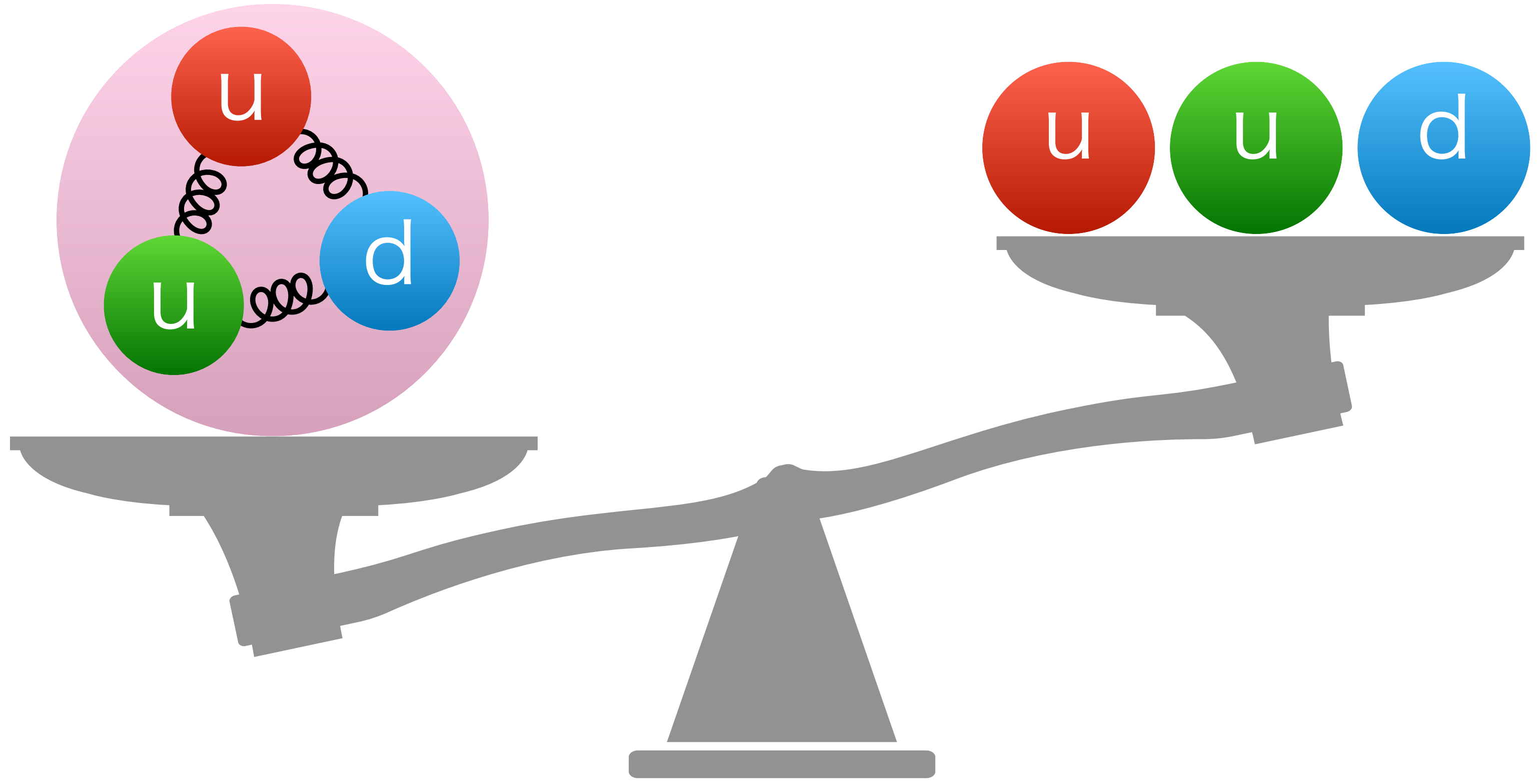
- quark confinement in Quantum ChromoDynamics (QCD)
  ⋯ low-energy d.o.f. are not quarks but composite particles (hadrons)

- **hadrons are much heavier than quarks**

  u/d quark: $m_u \sim 2$ MeV, $m_d \sim 5$ MeV

  π^+ meson (u, d): $140 \text{ MeV} \gg m_u + m_d$

  proton (u, u, d): $938 \text{ MeV} \gg 2m_u + m_d$



- nonperturbative calc. is essential to understand the properties of hadrons

our motivation:
Numerically investigate low-energy spectra of gauge theories such as QCD

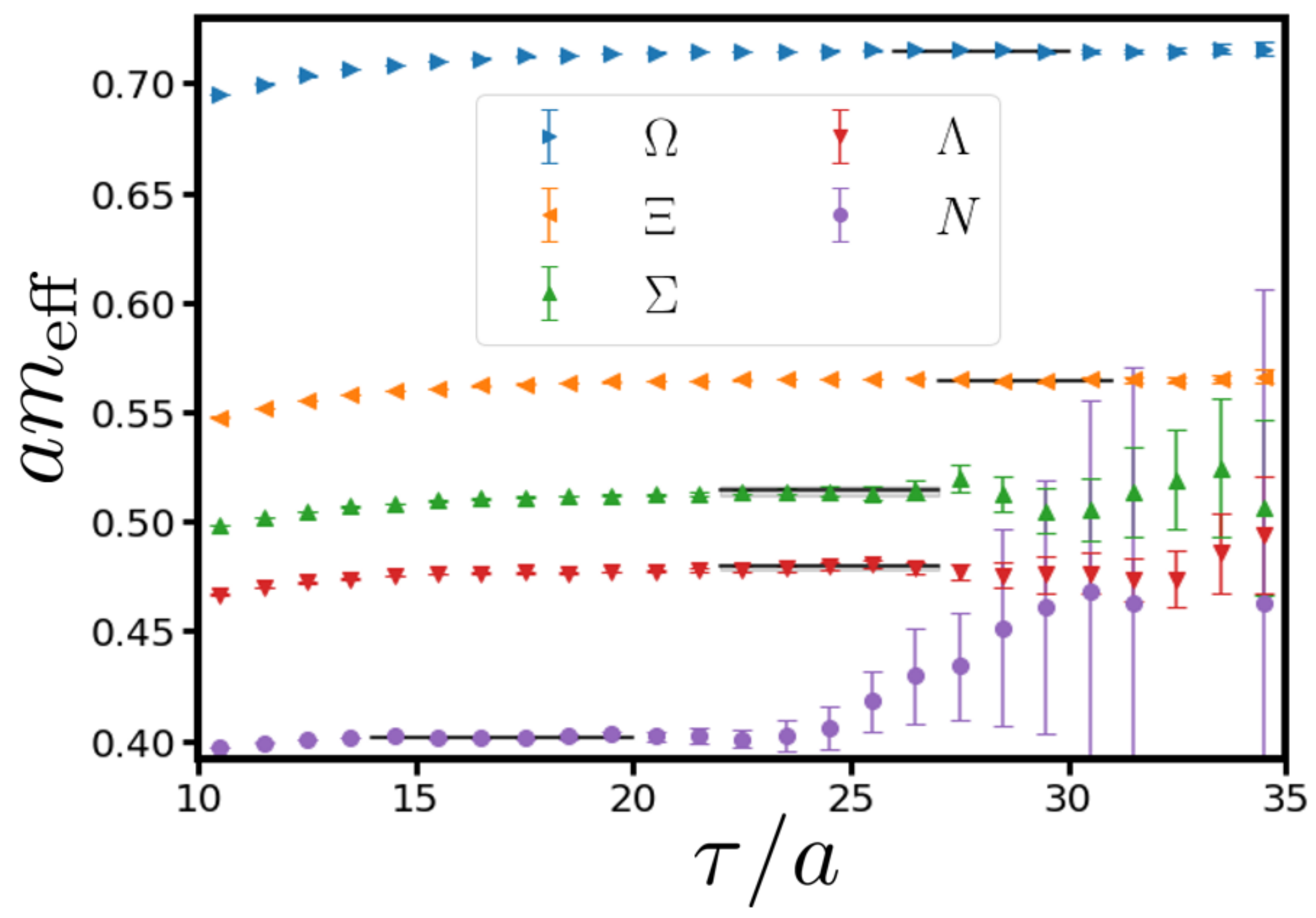
Mass spectrum by lattice QCD

- conventional method:
  Monte Carlo simulation of the lattice gauge theory in Lagrangian formalism

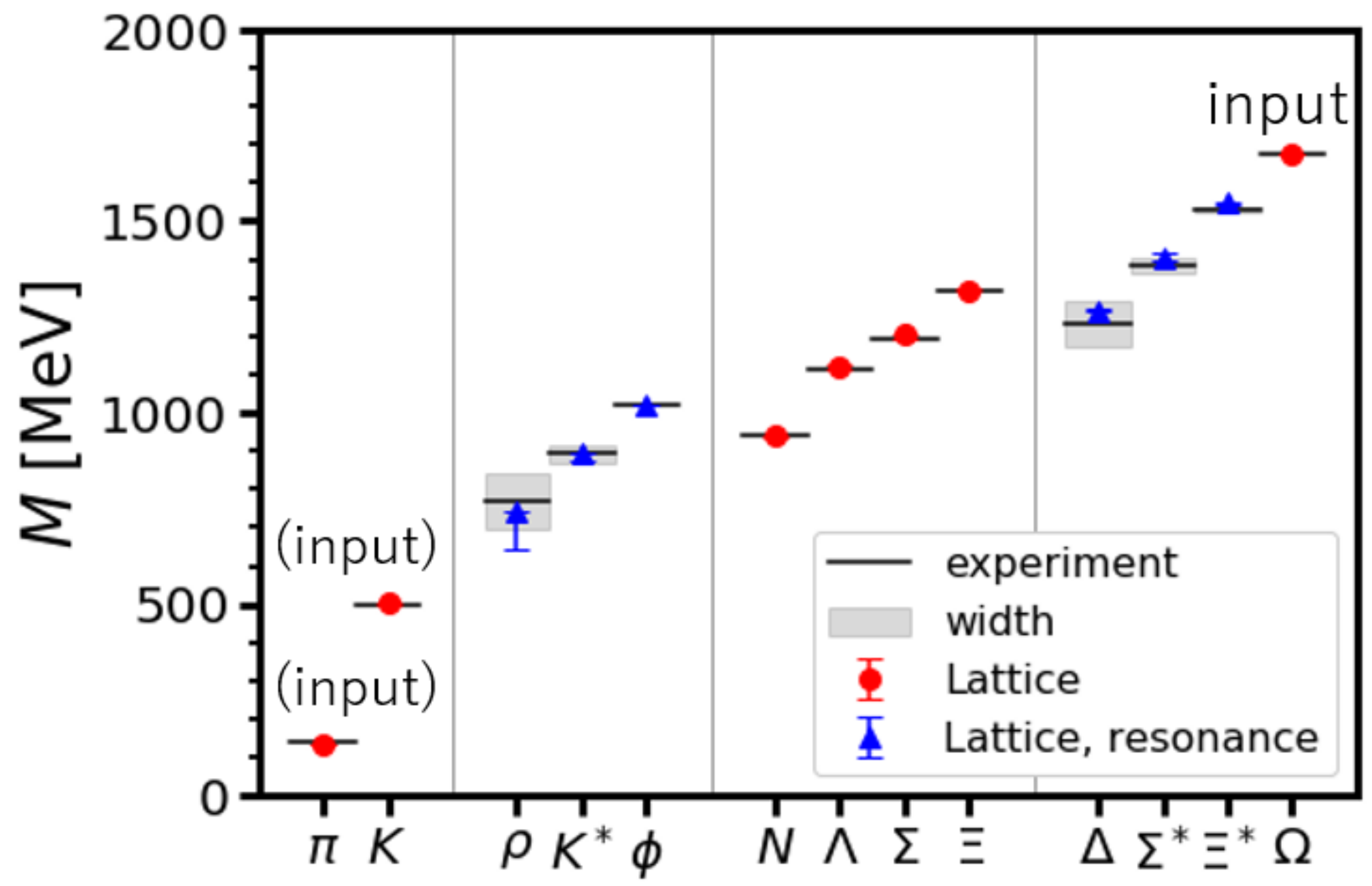
- obtain hadron masses from imaginary-time correlation functions

$$C(\tau) = \sum_x \langle \mathcal{O}(0,0)\, \mathcal{O}(x,\tau) \rangle \sim e^{-M\tau} \quad \text{—>} \quad \text{effective mass: } m_{\text{eff}}(\tau) := -\frac{d}{d\tau} \log C(\tau) \simeq M$$

effective mass



hadron spectrum



[HAL QCD collab. (2024)]

Hamiltonian formalism

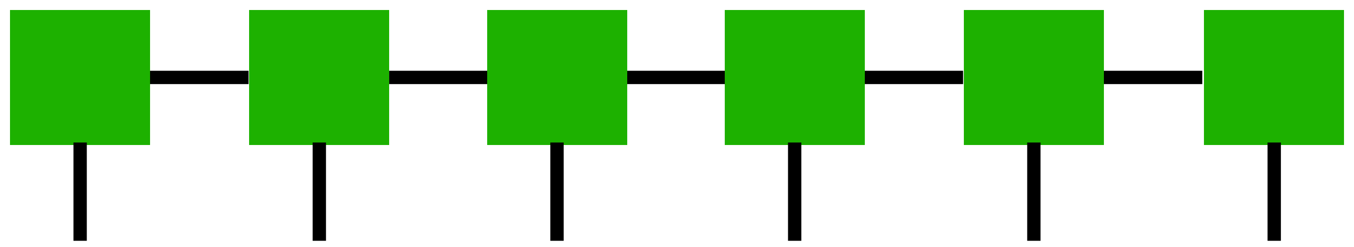
😣 Monte Carlo method cannot be applied to models with complex actions

—> sign problem (finite density QCD, topological term, real-time evolution, ⋯)

💡 Tensor network and quantum computing in Hamiltonian formalism can be complementary approaches!

👍 free from the sign problem

👍 analyze wave functions directly



aim of this work:
compute the hadron mass spectrum in Hamiltonian formalism that is applicable even when the sign problem arises

©IBM

Short summary

- compute the mass spectrum of the 2-flavor Schwinger model by distinct methods

  (1) correlation-function scheme

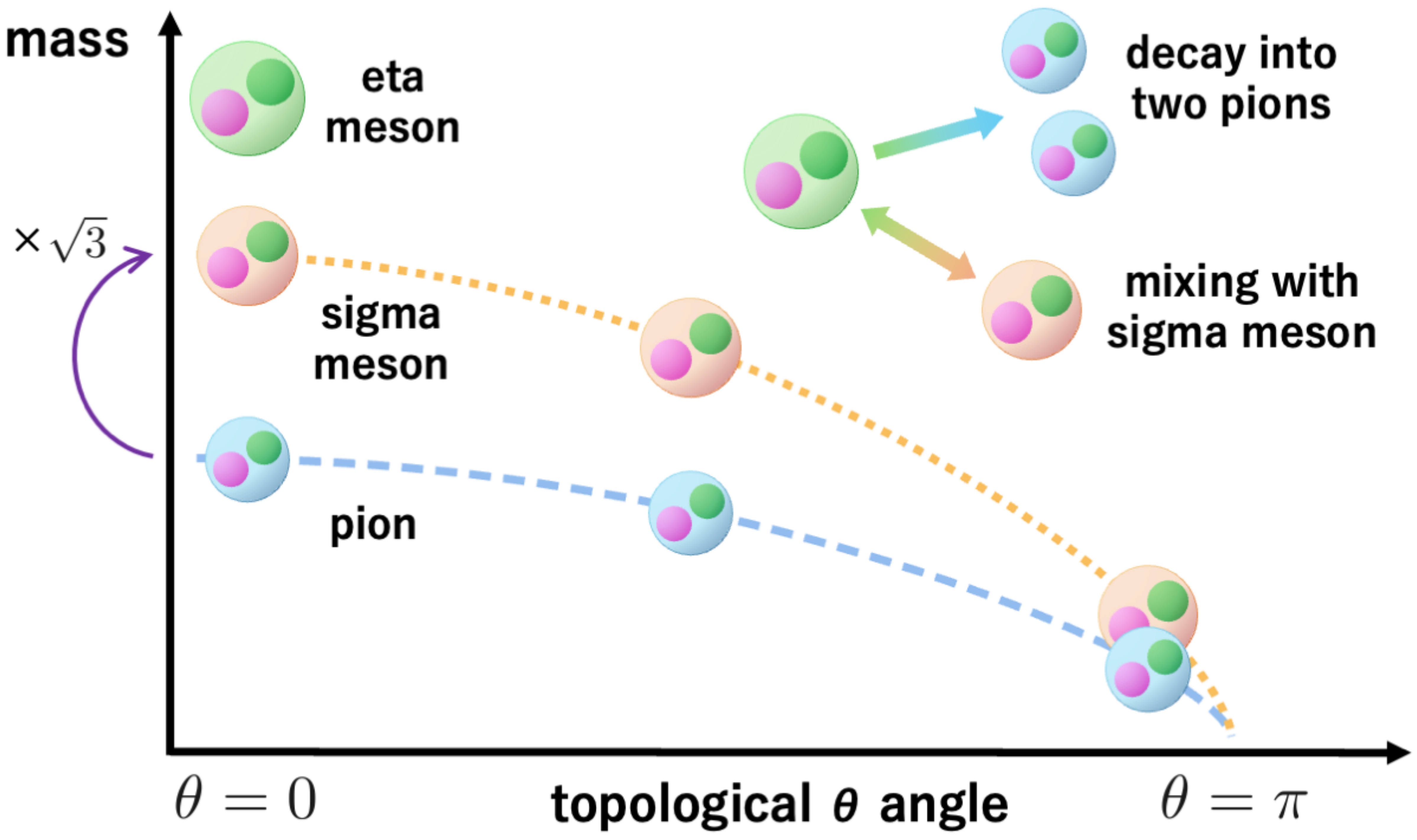
  [JHEP11 (2023) 231] [JHEP09 (2024) 155]

  (2) one-point-function scheme

  (3) dispersion-relation scheme

- θ-dependent spectra by these methods are consistent with each other and with analytic prediction

  cf.) bosonization analysis
  [Coleman (1976)] [Dashen et al. (1975)]



Outline

1. 2-flavor Schwinger model and calculation strategy

2. One-point-function scheme

3. Dispersion-relation scheme

4. Summary

Outline

1. **2-flavor Schwinger model and calculation strategy**
2. One-point-function scheme
3. Dispersion-relation scheme
4. Summary

Schwinger model with two fermions

Schwinger model = Quantum ElectroDynamics in 1+1d

- simplest nontrivial gauge theory sharing some features with QCD

$$\mathscr{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f}\left[i\bar{\psi}_f\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_f - m\bar{\psi}_f\psi_f\right]$$

sign problem if $\theta \neq 0$

- quantum numbers:

  isospin J, parity P, G-parity $G = Ce^{i\pi J_y}$

- P and G are broken at $\theta \neq 0$

  —> η becomes unstable

  due to $\eta \to \pi\,\pi$ decay and η-σ mixing

$N_f = 2$ —> three "mesons"

$\pi_a = -i\bar{\psi}\gamma^5\tau_a\psi$: $J^{PG} = 1^{-+}$

$\sigma = \bar{\psi}\psi$: $J^{PG} = 0^{++}$

$\eta = -i\bar{\psi}\gamma^5\psi$: $J^{PG} = 0^{--}$

Calculation strategy

[Kogut & Susskind (1975)]
[Dempsey et al. (2022)]

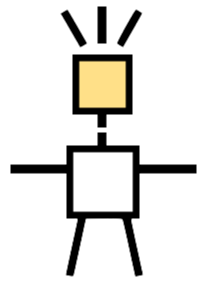
- setup: staggered fermion with open boundary

$$H=\frac{g^2a}{2}\sum_{n=0}^{N-2}\left(L_n+\frac{\theta}{2\pi}\right)^2+\sum_{f=1}^{N_f}\left[\frac{-i}{2a}\sum_{n=0}^{N-2}\left(\chi_{f,n}^\dagger U_n\chi_{f,n+1}-\chi_{f,n+1}^\dagger U_n^\dagger\chi_{f,n}\right)+m_{\rm lat}\sum_{n=0}^{N-1}(-1)^n\chi_{f,n}^\dagger\chi_{f,n}\right]$$

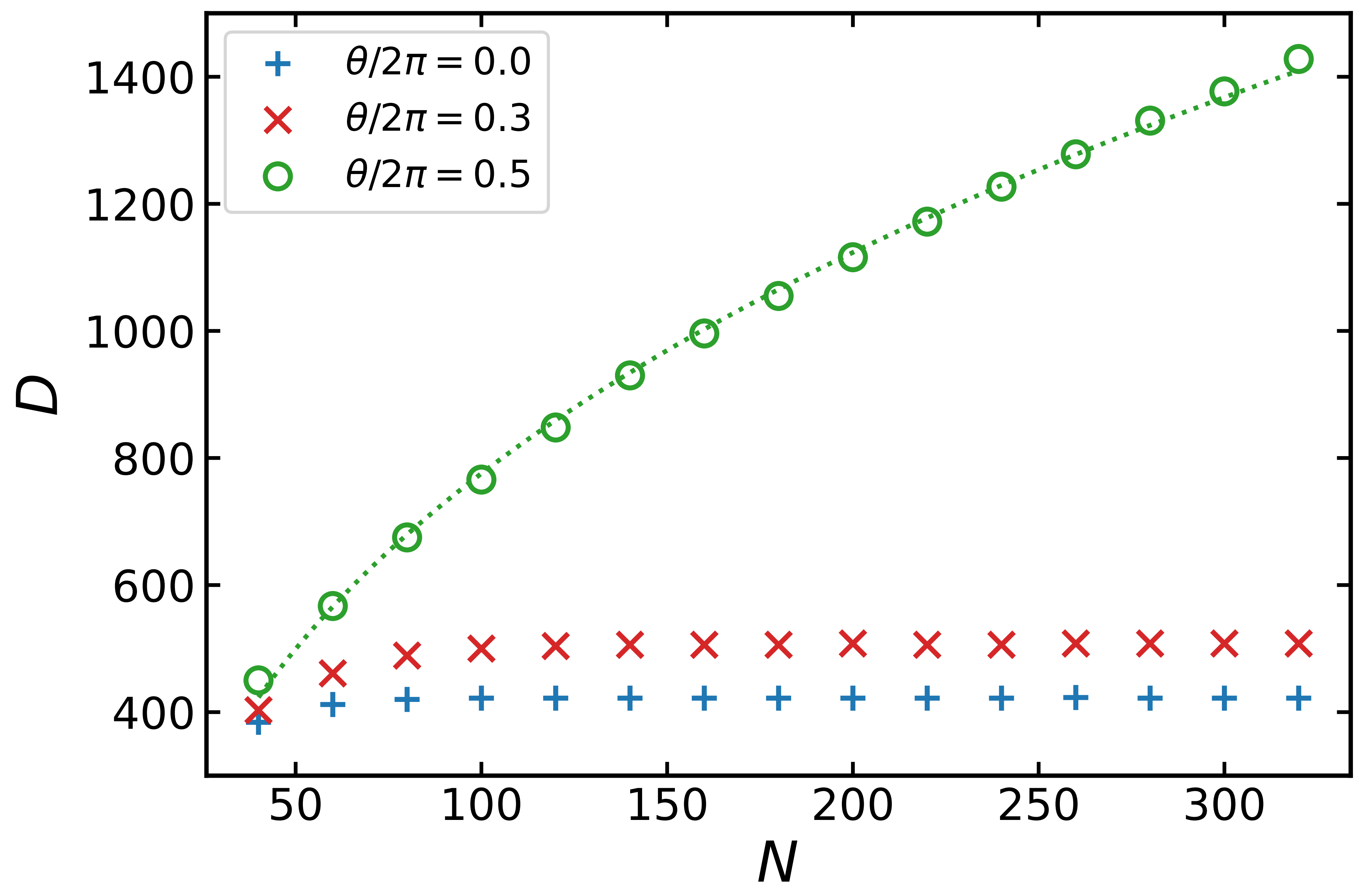
- rewrite to the spin Hamiltonian by Jordan-Wigner transformation after solving Gauss law and gauge fixing

- obtain the ground state as MPS by density-matrix renormalization group (DMRG)

C++ library of ITensor is used
[Fishman et al. (2022)]



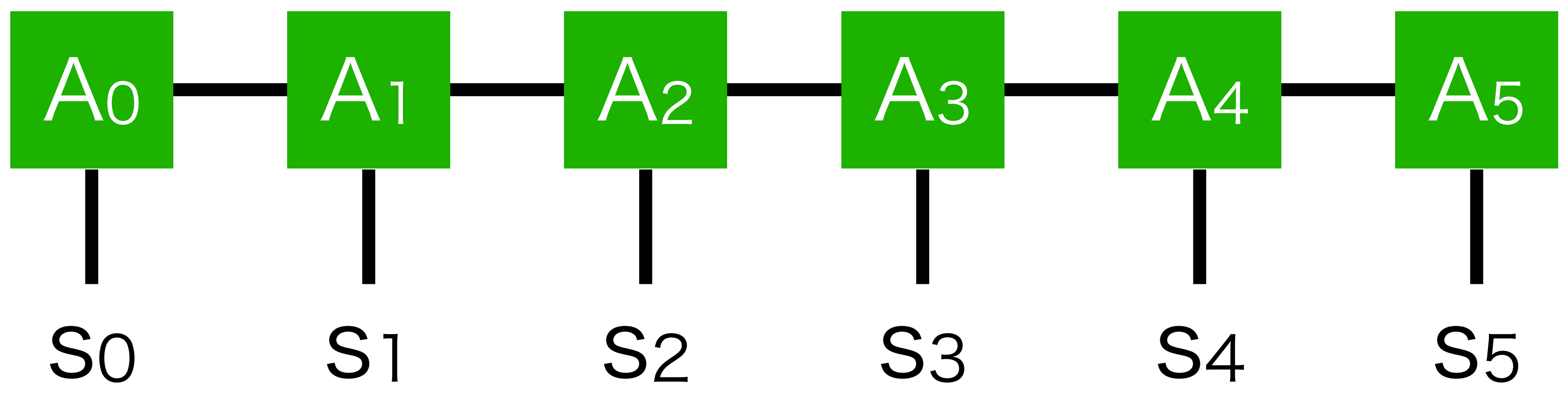
bond dim. for fixed truncation error



Approximation of states by MPS

Matrix Product State (MPS)

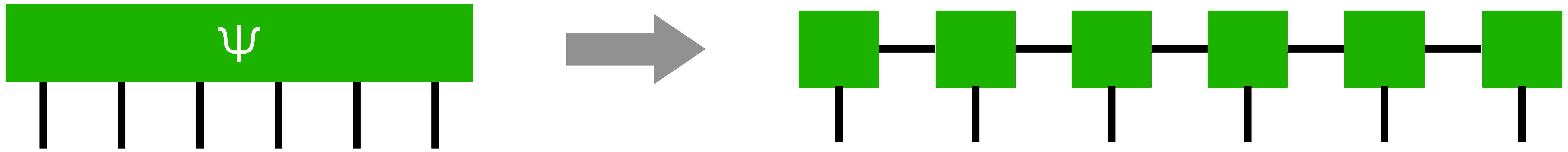
ex.) spin-1/2, N=6 sites



$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\, A_1(s_1)\, \cdots\right] |\, s_0\, s_1\, \cdots\rangle$$

- $A_i(s_i)$: $D_{i-1} \times D_i$ matrix with a spin index $s_i \in \{\uparrow, \downarrow\}$ (D_i : bond dimension)
- Any state can be written as MPS by repeating SVD, but $D_i = O(2^{N/2})$ in general.

$$|\Psi\rangle = \sum_{\{s_i\}} \Psi(s_0, s_1, \cdots)\, |\, s_0\, s_1\, \cdots\rangle$$



- Even with a cutoff $D_i \leq \mathrm{const}$, MPS efficiently approximates low-energy states of 1+1d gapped systems of any size N. → numerical cost = $O(ND^3)$

Outline

1. 2-flavor Schwinger model and calculation strategy
2. **One-point-function scheme**
3. Dispersion-relation scheme
4. Summary

Two-point correlation function?

- In lattice QCD, correlation functions are measured with spatial integral
  —> zero-momentum projection: $\sum_x \langle \mathcal{O}(0,0)\mathcal{O}(x,\tau)\rangle \sim e^{-M\tau}$

- Equal-time correlator in Hamiltonian formalism:

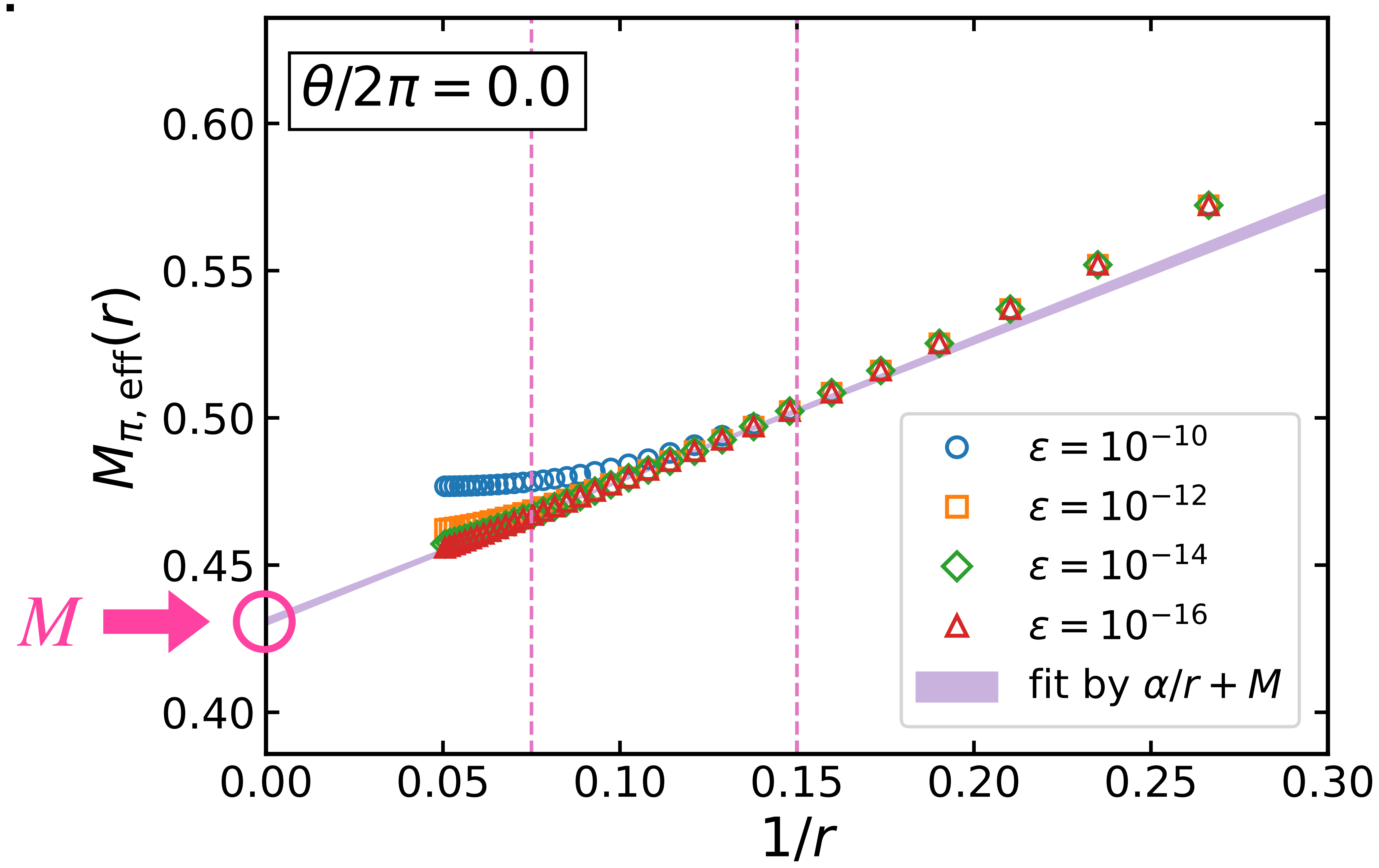
  $$C(r) = \langle \mathcal{O}(0,0)\, \mathcal{O}(r,0)\rangle \sim \frac{1}{r^\alpha} e^{-Mr}$$

  —> effective mass: $M_{\mathrm{eff}}(r) = -\frac{d}{dr}\log C(r) \sim \frac{\alpha}{r} + M$

- Bond dim. must be large enough to see $1/r$ behavior

⚠️ significant truncation effect

effective mass of π meson



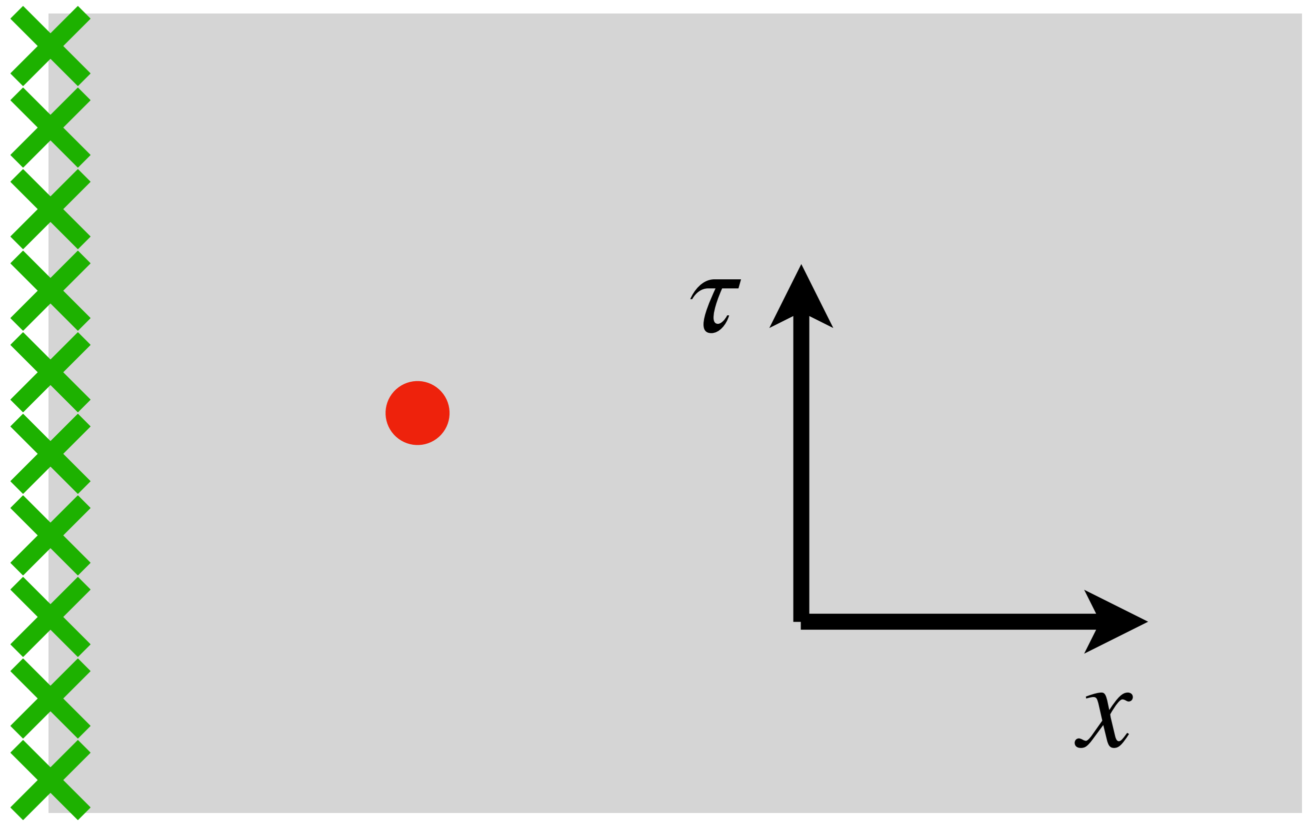
One-point-function scheme

<u>**Regarding the boundary (defect) as the source of mesons,**</u>
<u>**obtain the masses from the one-point functions**</u>

- 1pt. function $\langle \mathcal{O}(x) \rangle_{\mathrm{obc}}$ measures the correlation with the boundary state $|\mathrm{bdry}\rangle$

- $|\mathrm{bdry}\rangle$ has translational invariance in time direction
  —> zero-momentum projection —> exponential decay

$p_\tau |\mathrm{bdry}\rangle = 0$



$$\langle \mathcal{O}(x) \rangle_{\mathrm{obc}} \sim \langle \mathrm{bdry} | e^{-Hx} \mathcal{O} | 0 \rangle_{\mathrm{bulk}} \sim e^{-Mx}$$

👍 TN truncation effect is much smaller

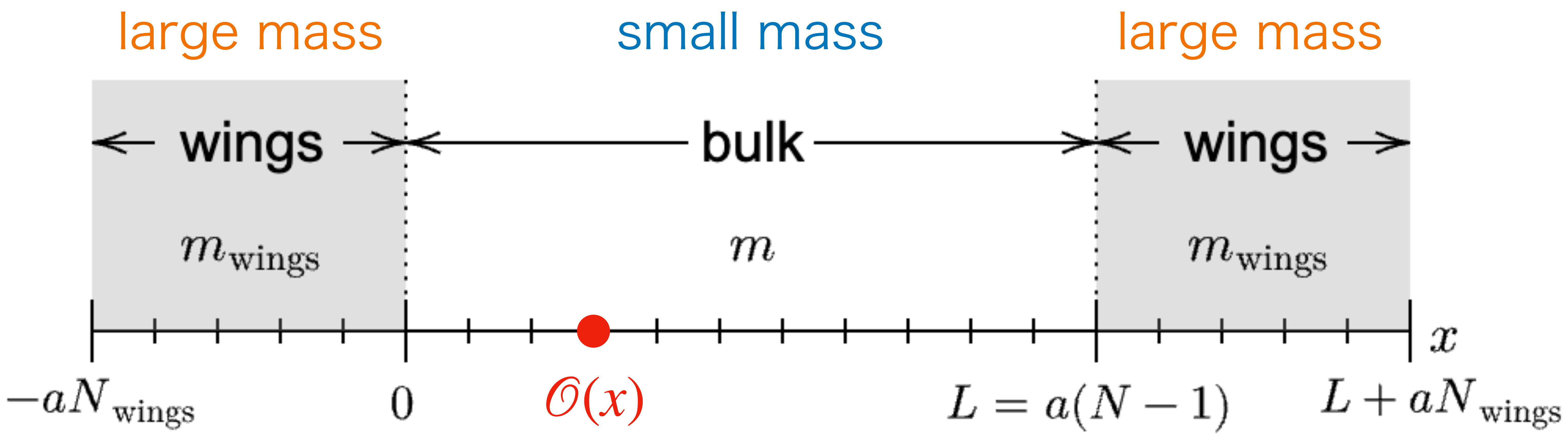
cf.) wall source method

Some technical improvement

- The boundary state $\langle \mathrm{bdry}|$ is specified by "the wings" attached to the lattice, which have the same quantum number as the target meson

  e.g.) Dirichlet b.c. ⋯ $m_{\mathrm{wings}} \gg m$

  isospin-breaking b.c. ⋯ $m_{\mathrm{wings}} = m_0 \exp(\pm i \Delta \gamma^5)$



Spectrum by the one-point function

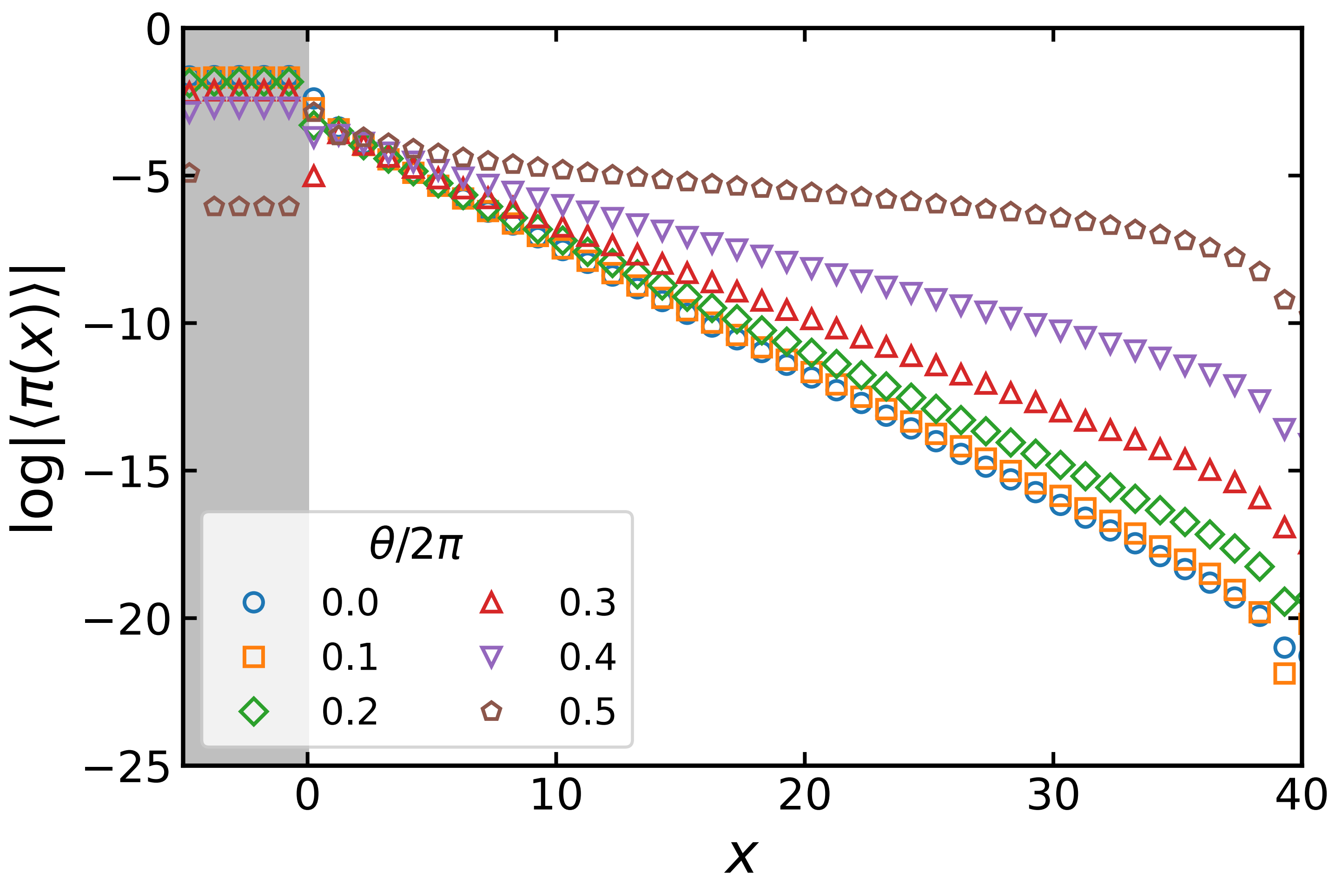
- Dirichlet b.c. for the singlets / isospin-breaking b.c. for the triplets
- Assuming $\langle \mathcal{O}(x) \rangle \sim Ae^{-Mx} + Be^{-(M+\Delta M)x}$,

  the effective mass should be $\sim M + \dfrac{\Delta M}{1 + Ce^{\Delta Mx}}$

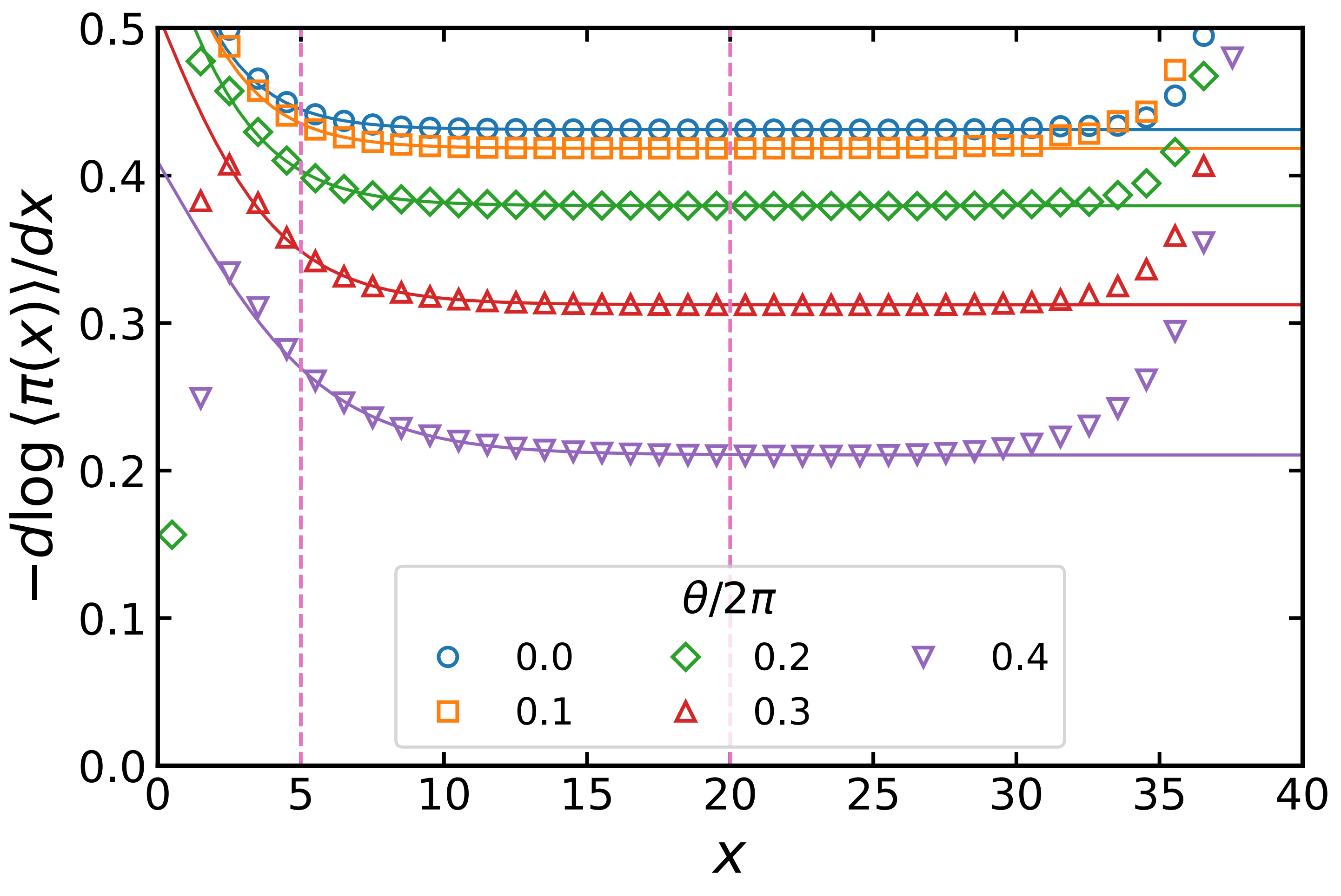
$N = 320,\ a = 0.25$

$m = 0.1,\ m_0 = 10,\ \Delta = 0.1$

pion

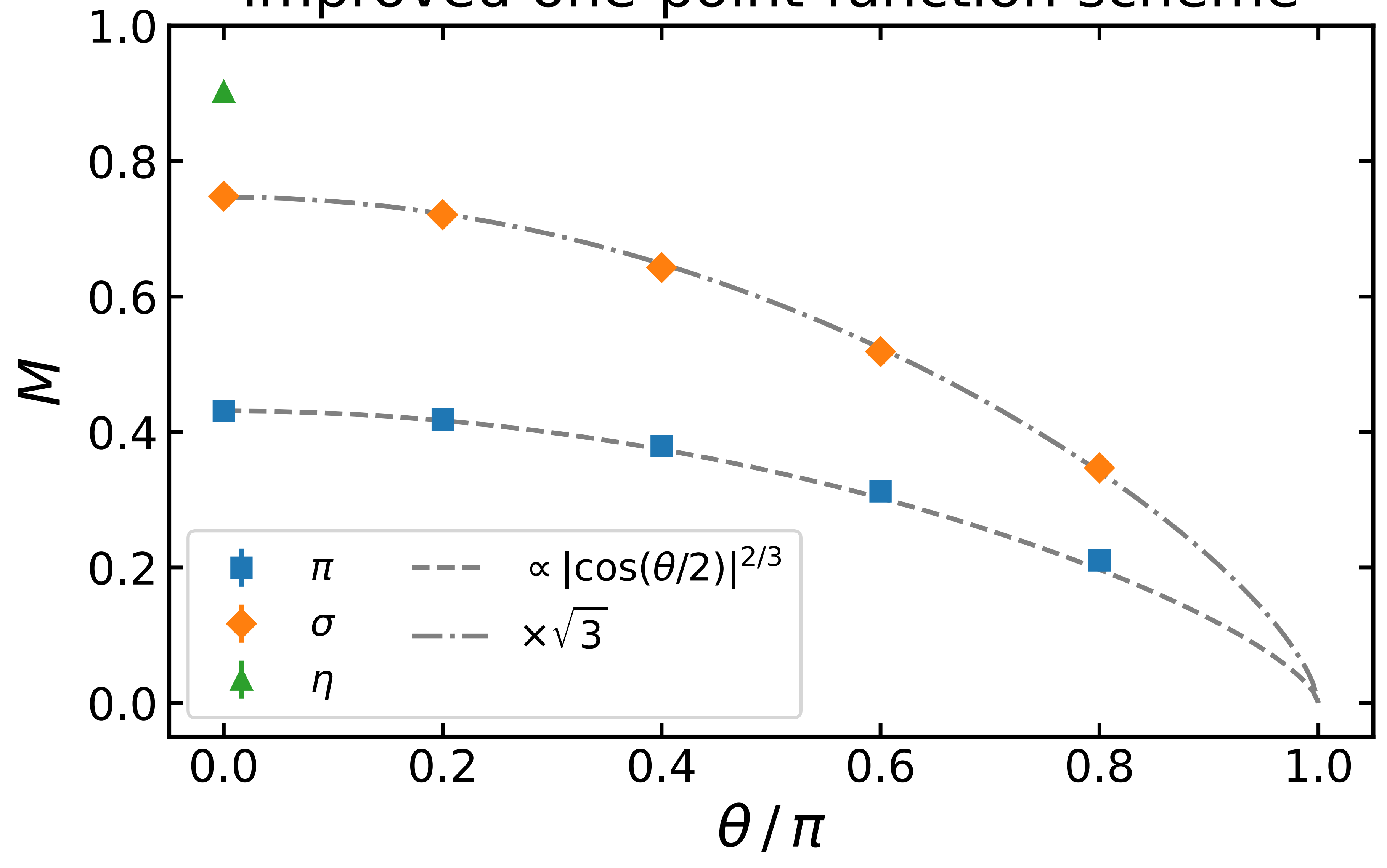


effective mass



summary

improved one-point-function scheme



Outline

1. 2-flavor Schwinger model and calculation strategy
2. One-point-function scheme
3. **Dispersion-relation scheme**
4. Summary

Dispersion-relation scheme

Obtain the dispersion relation $E = \sqrt{K^2 + M^2}$ directly from the excited states (momentum excitations of the mesons)

- ℓ-th excited state $|\Psi_\ell\rangle$

  = the lowest energy eigenstate satisfying $\langle\Psi_{\ell'}|\Psi_\ell\rangle = 0$ for $\ell' = 0, 1, \cdots, \ell - 1$

- obtained by DMRG, adding the projection term to H

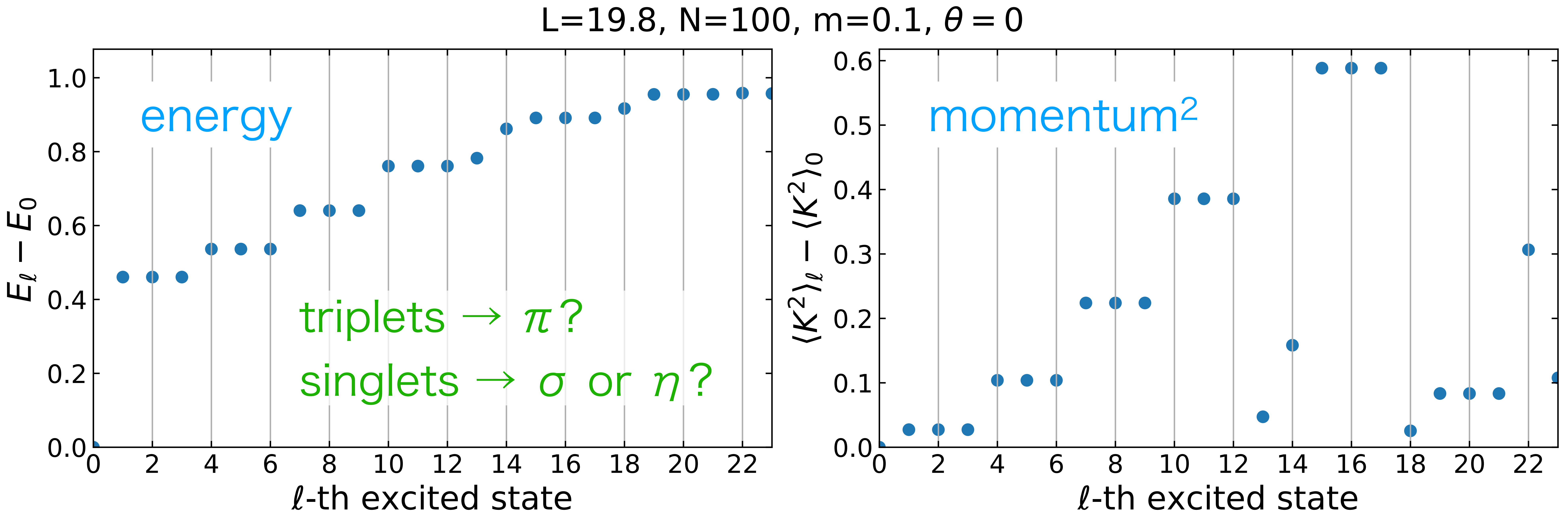
  [Stoudenmire & White (2012)]
  [Banuls et al. (2013)]

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| \quad \text{—> cost function: } \langle\Psi_\ell|H|\Psi_\ell\rangle + W \sum_{\ell'=0}^{\ell-1} \left|\langle\Psi_{\ell'}|\Psi_\ell\rangle\right|^2 \qquad W > 0$$

- measure the energy E and the total momentum $K = \sum_f \int dx\, \psi_f^\dagger (i\partial_x - A_1)\psi_f$

Energy spectrum at $\theta = 0$

- energy gap: $\Delta E_\ell = E_\ell - E_0$
- momentum square: $\Delta K_\ell^2 = \langle K^2 \rangle_\ell - \langle K^2 \rangle_0$
- identify the states by measuring quantum numbers: $\mathbf{J}^2,\ J_z,\ G = Ce^{i\pi J_y}$



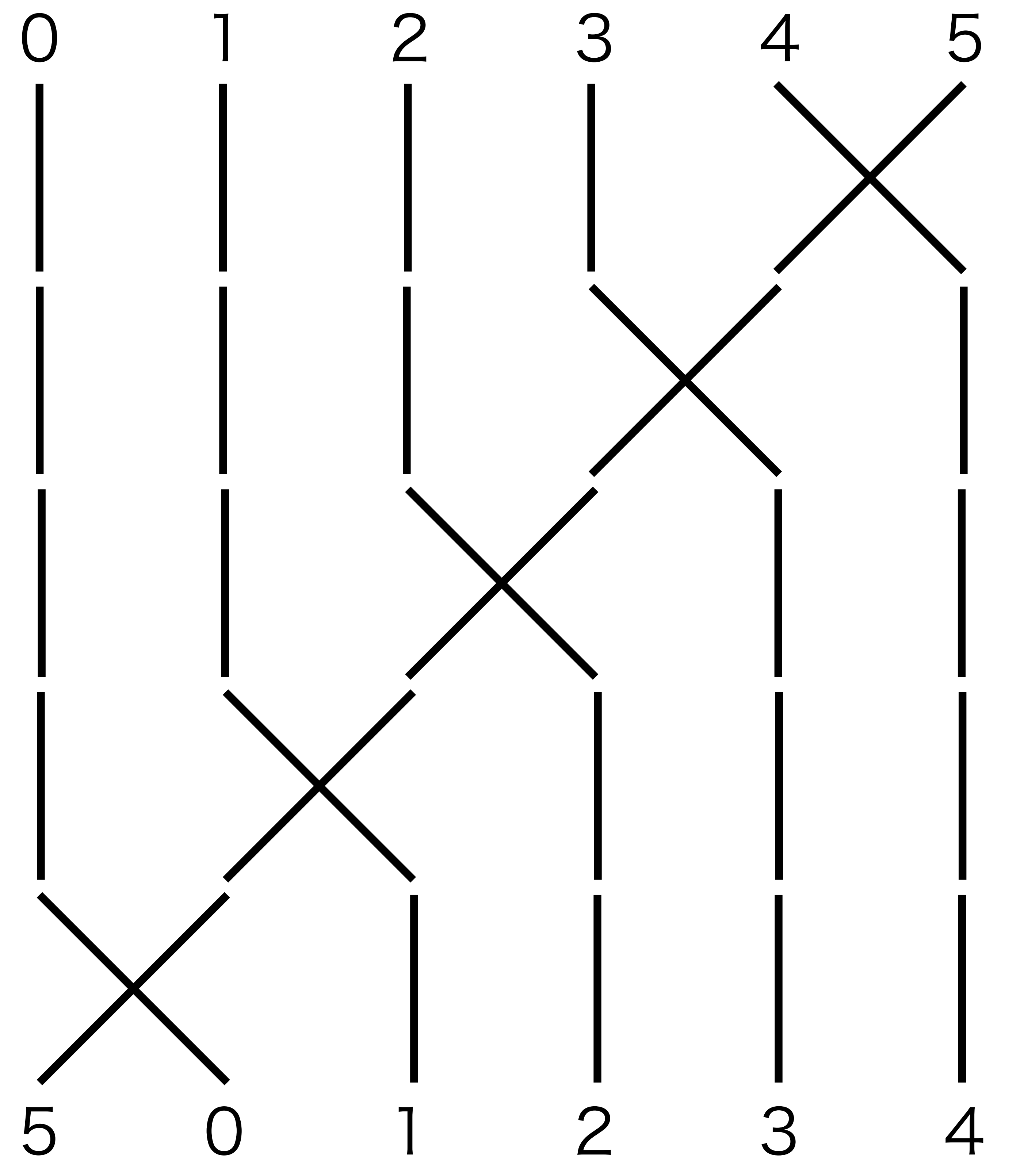
Quantum numbers

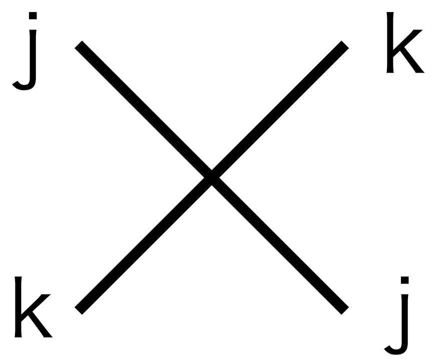
- isospin: $J_a = \frac{1}{2}\int dx\, \psi^\dagger \tau^a \psi \qquad [H, \mathbf{J}^2] = [H, J_z] = 0$

- charge conjugation:
  = exchange particles/anti-particles
  = exchange even/odd sites and flip each spin
  = 1-site translation and σ^x operator
  $$C := \prod_{f=1}^{N_f}\left(\prod_{n=0}^{N-1}\sigma^x_{f,n}\right)\left(\prod_{n=0}^{N-2}(\mathrm{SWAP})_{f;N-2-n,N-1-n}\right)$$
  $[H, C] \neq 0$ due to the boundary

- G-parity: $G = C\exp(i\pi J_y)$

1-site translation





$$= (\mathrm{SWAP})_{f;j,k} = \frac{1}{2}\left(\mathbf{1}_{f,j}\mathbf{1}_{f,k} + \sum_a \sigma^a_{f,j}\sigma^a_{f,k}\right)$$

Result of quantum numbers

results at $\theta = 0$

- triplets: $\mathbf{J}^2 = 2,\ \ J_z = (0, \pm 1),\ \ G > 0$

  —> pion ($J^{PG} = 1^{-+}$)

- singlets: $\mathbf{J}^2 = 0,\ \ J_z = 0,$

  $G > 0$ ($\ell = 13,14,22$) —> sigma meson ($J^{PG} = 0^{++}$)

  $G < 0$ ($\ell = 18,23$) —> eta meson ($J^{PG} = 0^{--}$)

triplets

ℓ	$\boldsymbol{J}^2$	J_z	G	P
1	2.00000004	0.99999997	0.27872443	-6.819×10^{-8}
2	2.00000012	-0.00000000	0.27872416	-6.819×10^{-8}
3	2.00000004	-0.99999996	0.27872443	-6.819×10^{-8}
4	2.00000007	0.99999999	0.27736066	7.850×10^{-8}
5	2.00000006	0.00000000	0.27736104	7.850×10^{-8}
6	2.00000009	-0.99999998	0.27736066	7.850×10^{-8}
7	2.00000010	1.00000000	0.27536687	-8.838×10^{-8}
8	2.00000002	0.00000000	0.27536702	-8.837×10^{-8}
9	2.00000007	-0.99999998	0.27536687	-8.838×10^{-8}
10	2.00000007	0.99999998	0.27356274	9.856×10^{-8}
11	2.00000005	0.00000001	0.27356277	9.856×10^{-8}
12	2.00000007	-0.99999999	0.27356274	9.856×10^{-8}
15	1.99999942	0.99999966	0.27173470	-1.077×10^{-7}
16	2.00000052	0.00000000	0.27173482	-1.077×10^{-7}
17	2.00000015	-1.00000003	0.27173470	-1.077×10^{-7}
19	2.00009067	1.00004377	0.27717104	-3.022×10^{-8}
20	2.00002578	-0.00000004	0.27717020	-3.023×10^{-8}
21	2.00003465	-1.00001622	0.27717104	-3.023×10^{-8}

singlets

ℓ	$\boldsymbol{J}^2$	J_z	G	P
0	0.00000003	-0.00000000	0.27984227	3.896×10^{-7}
13	0.00000003	0.00000000	0.27865844	1.273×10^{-7}
14	0.00000003	0.00000000	0.27508176	-2.765×10^{-8}
18	0.00000028	0.00000006	-0.27390909	-6.372×10^{-7}
22	0.00001537	0.00000115	0.26678987	7.990×10^{-8}
23	0.00003607	-0.00000482	-0.27664779	5.715×10^{-7}

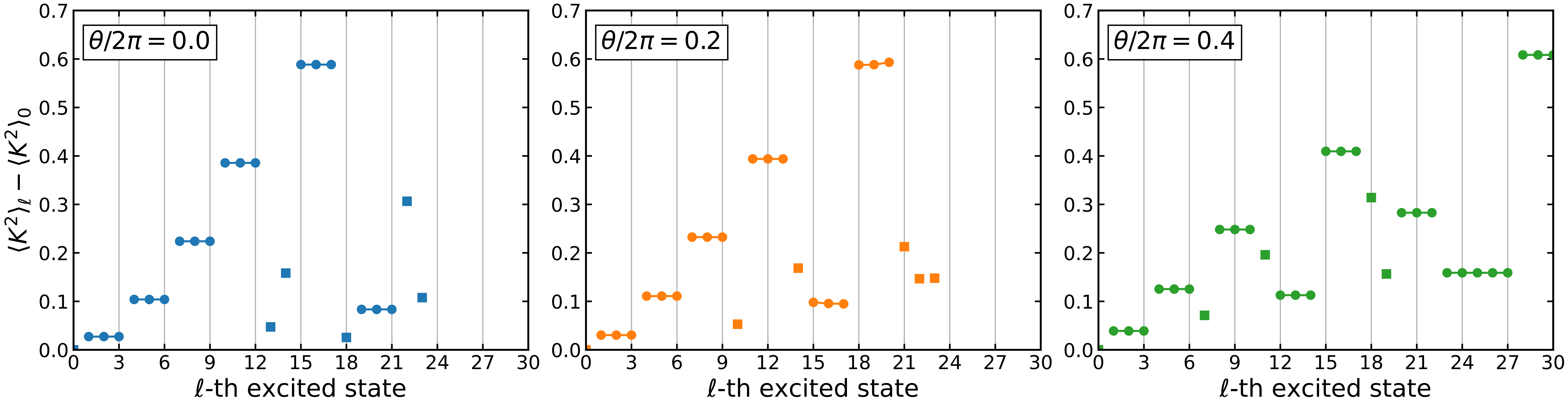
Extension to $\theta \neq 0$

- G-parity is no longer the quantum number —> η disappears
- singlet projection to obtain σ with reasonable computational cost

$$H_\ell = H + W \sum_{\ell'=0}^{\ell-1} |\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| + W_J \mathbf{J}^2$$

cf.) quantum-number preserving DMRG

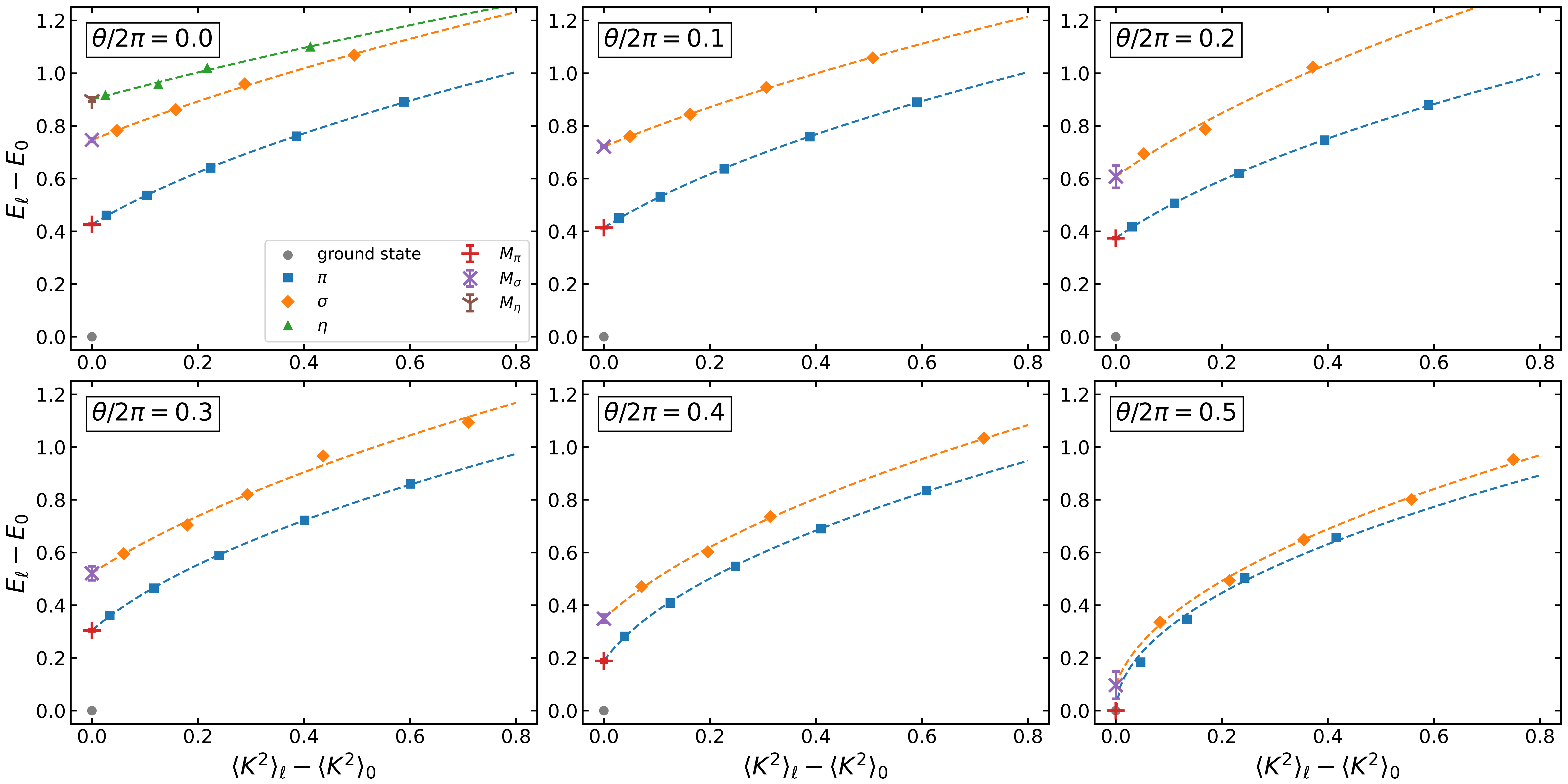
momentum^2



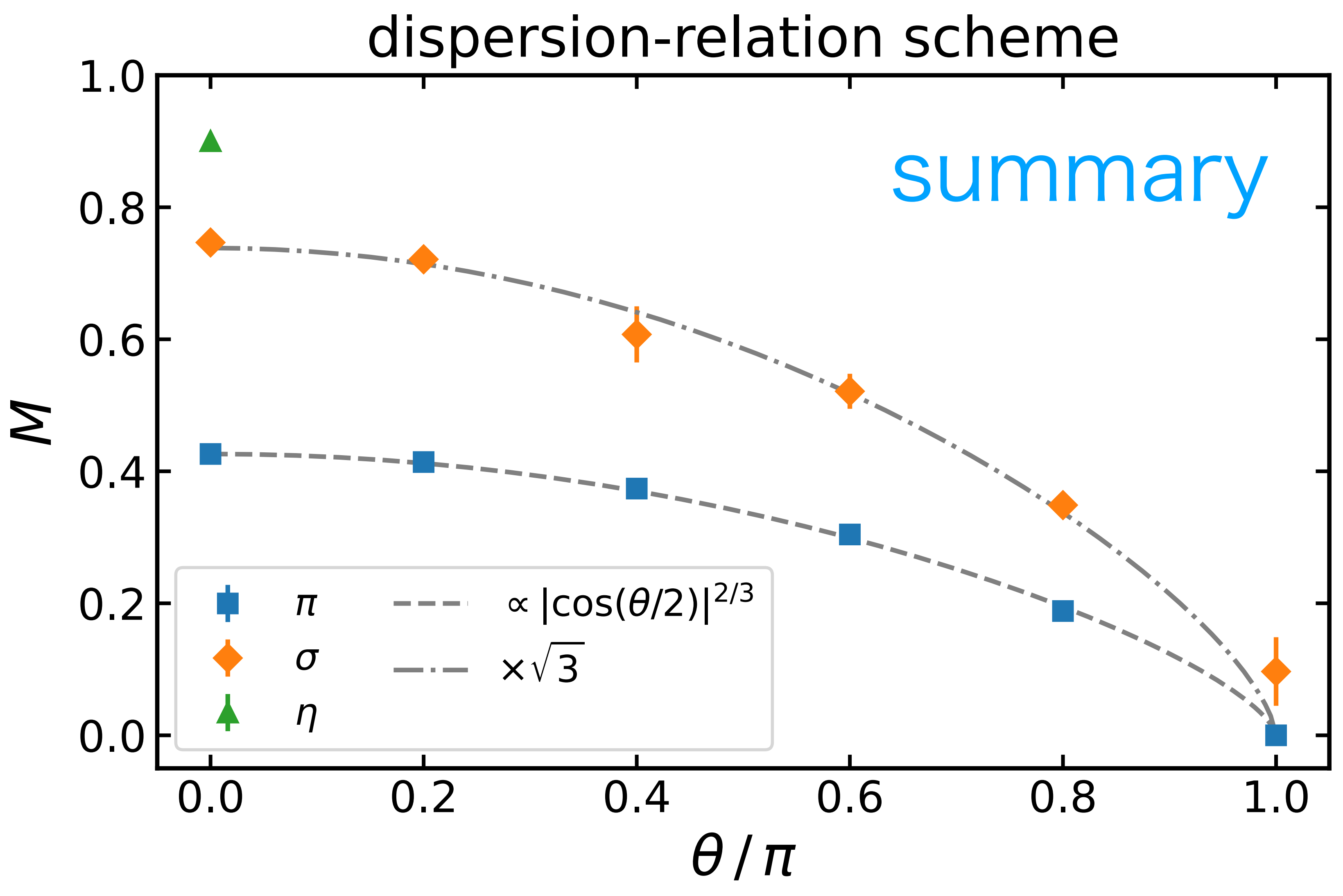
Result of dispersion relation

- plot ΔE_ℓ against ΔK_ℓ^2 and fit the data by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$ for each meson

energy vs momentum²



Around $\theta/2\pi = 0.2$, σ is contaminated by a remnant of η due to the mixing



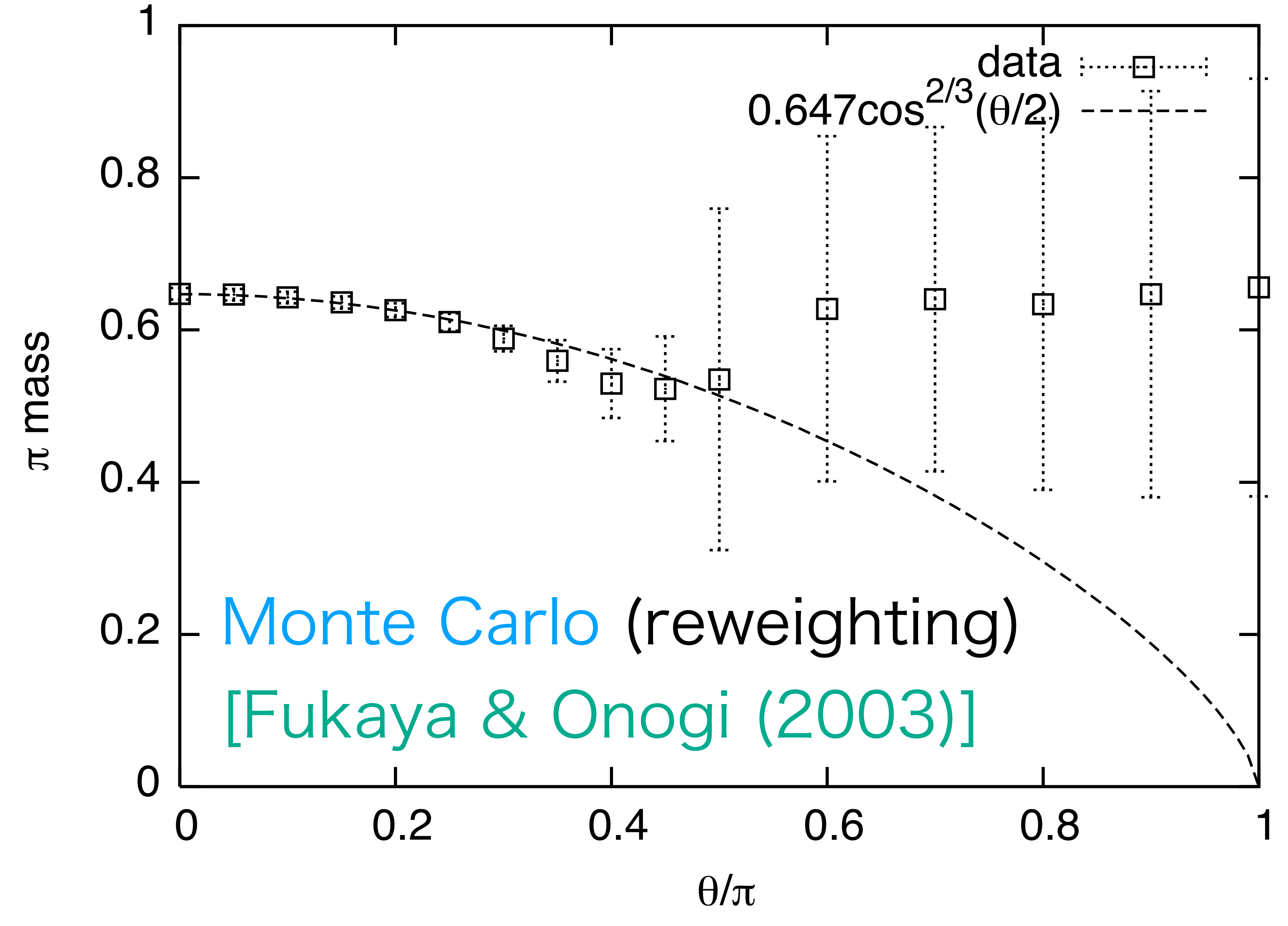
Outline

1. 2-flavor Schwinger model and calculation strategy
2. One-point-function scheme
3. Dispersion-relation scheme
4. **Summary**

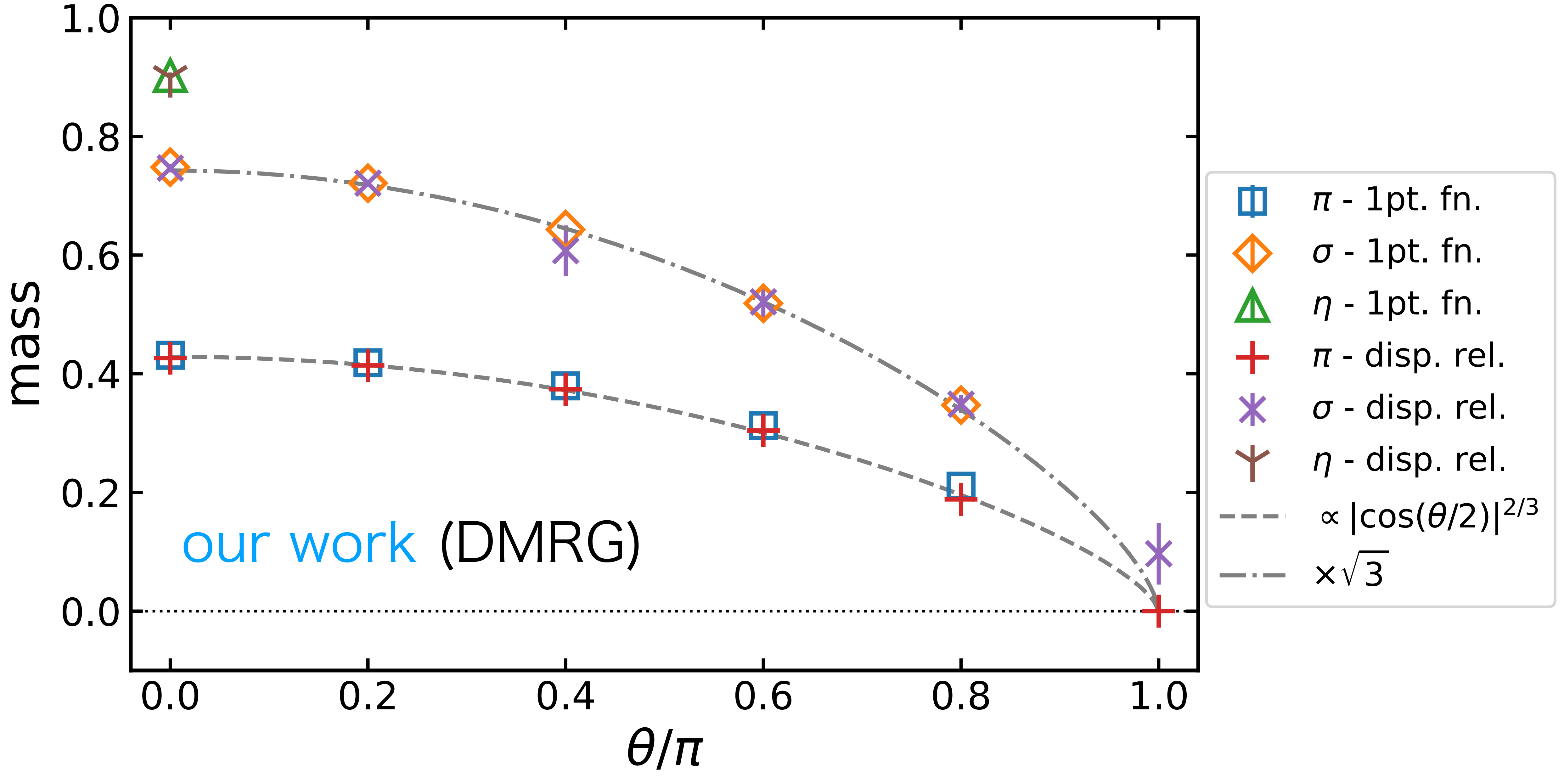
Summary

- The two schemes give consistent results and look promising
- consistent with predictions by the bosonization [Coleman (1976)] [Dashen et al. (1975)]

$$M_\pi(\theta) \propto |\cos(\theta/2)|^{2/3} \qquad M_\sigma(\theta)/M_\pi(\theta) = \sqrt{3}$$



applicable even in large θ region!

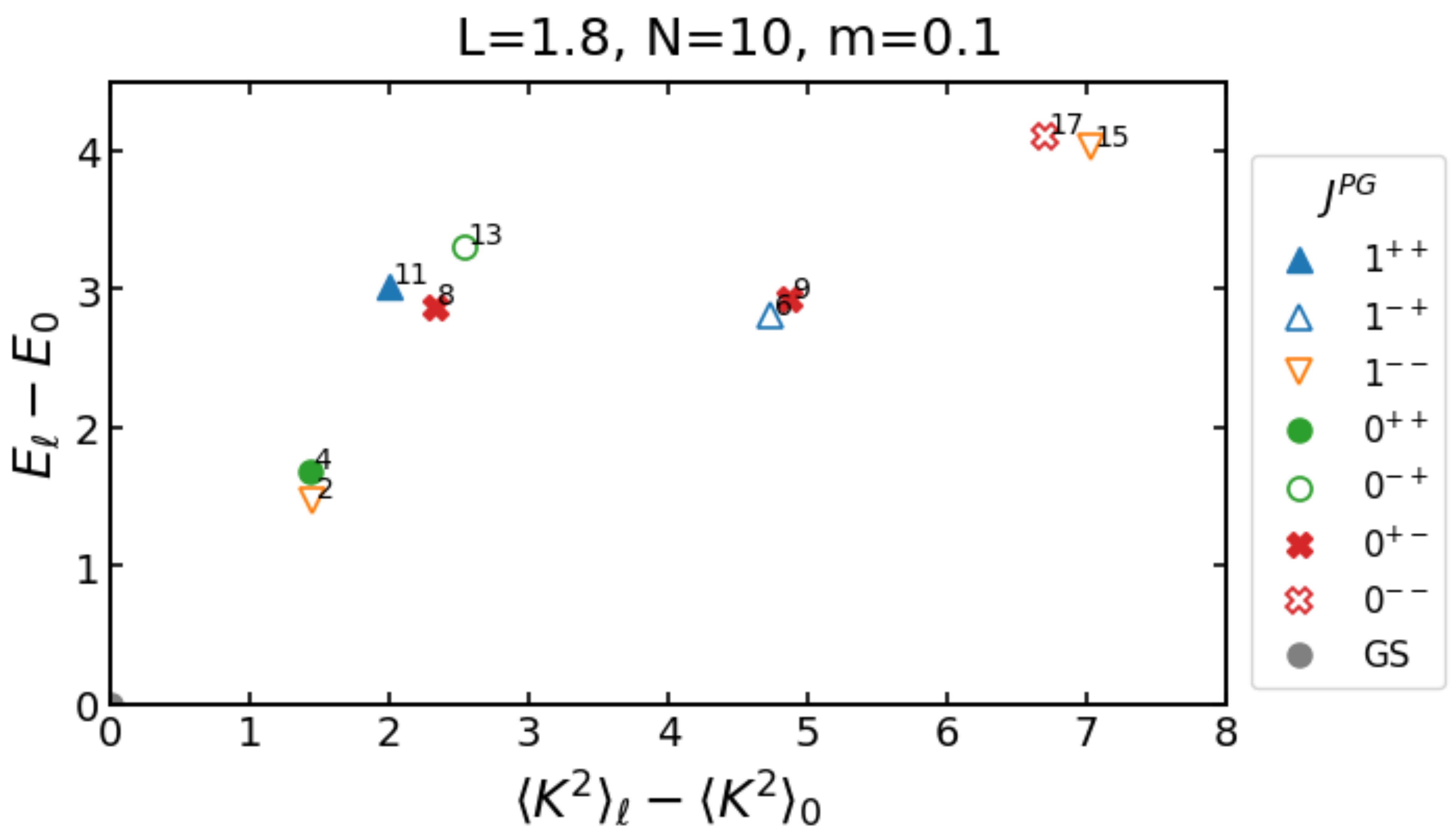


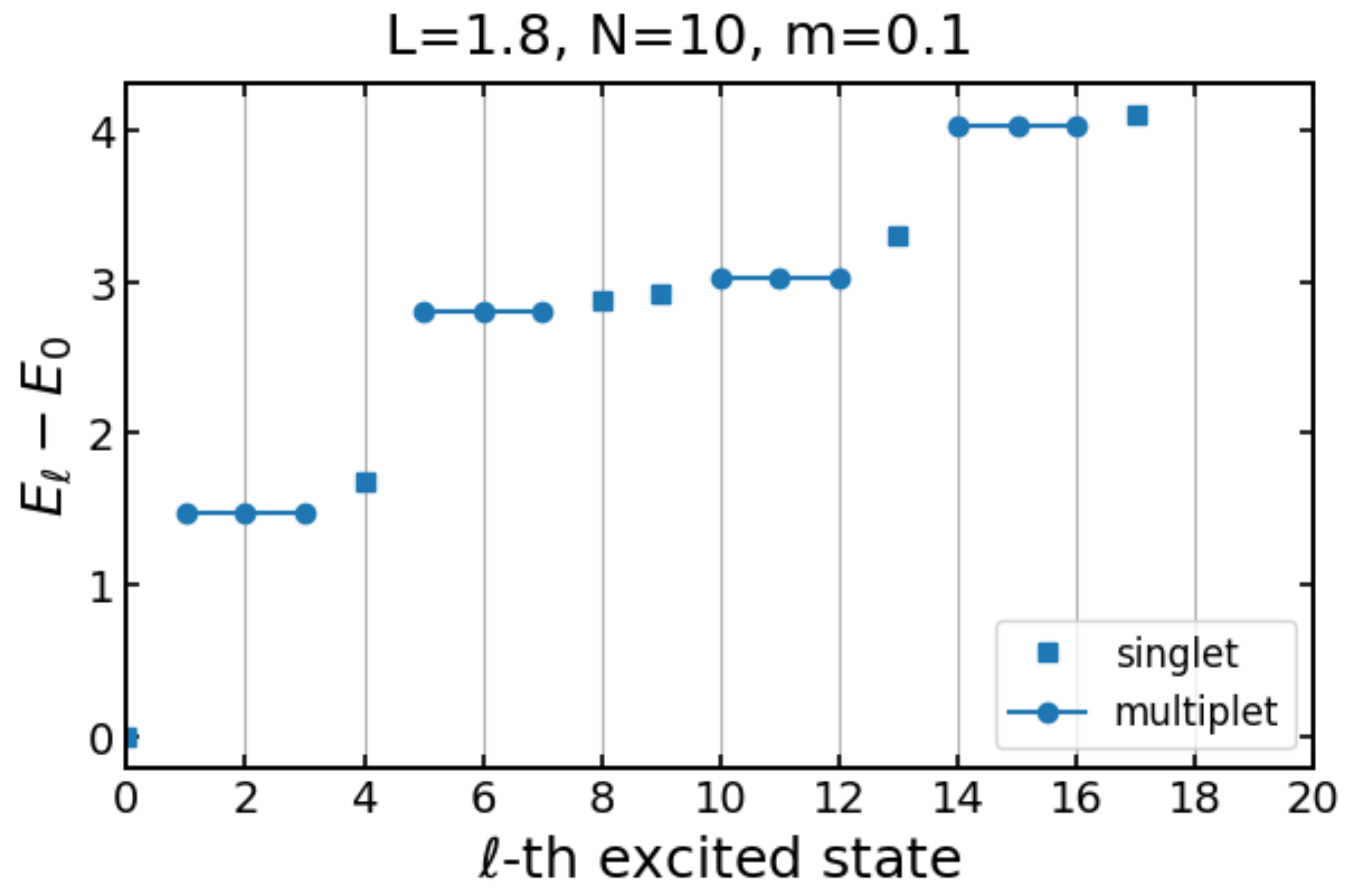
Discussion and prospect

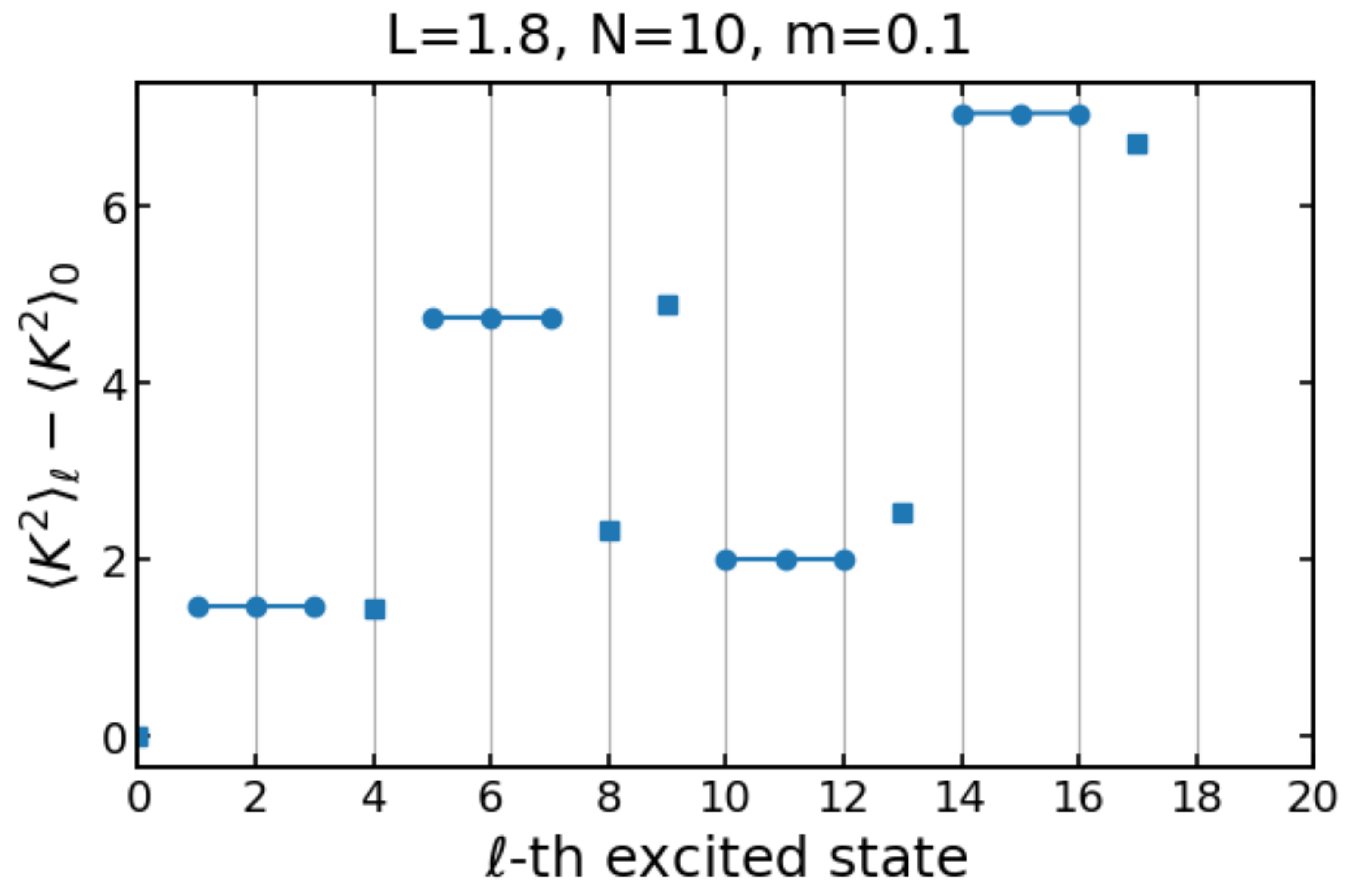
- Large volume limit and continuum limit [Schwägerl et al. (2025)]
- 2+1 dimensions
- Analyses using the wave functions for the scattering problem
- Real-time evolution to study the decay of unstable mesons
  - TN method: Time-Dependent Variational Principle? [Haegeman et al. (2011)] [Yang & White (2020)]
  - TN-QC hybrid method: encoding TN states into quantum circuits? [Shirakawa et al. (2021)]
- Full Quantum algorithm to find excited states?

Small N: N = 10, a = 0.2 (20 qubits)

- The lowest excitations are not consistent with pion triplets 1^{-+}
- ⚠️ $L = a(N-1) = 1.8$ is smaller than $1/M_\pi \sim 2.4$

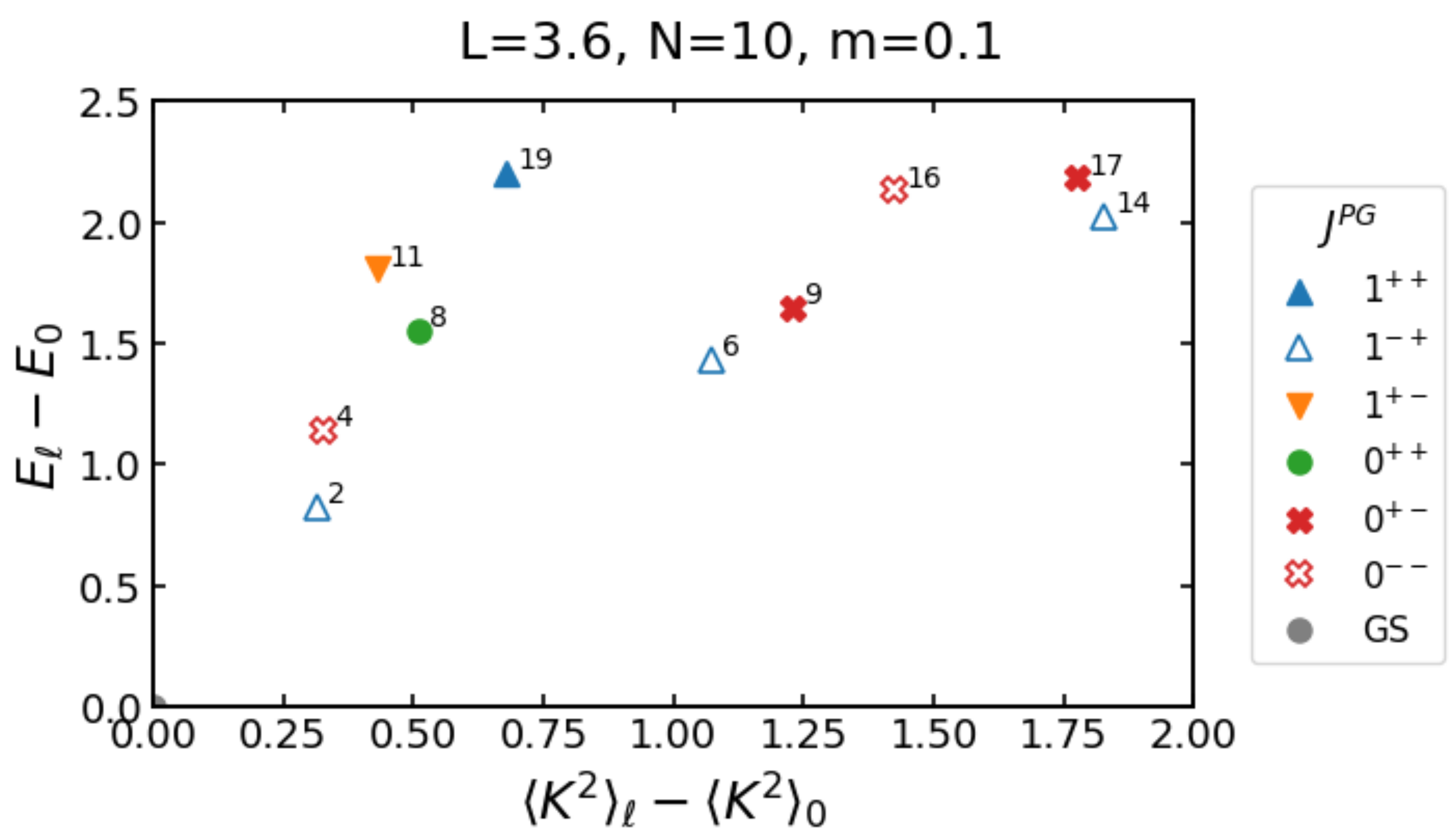


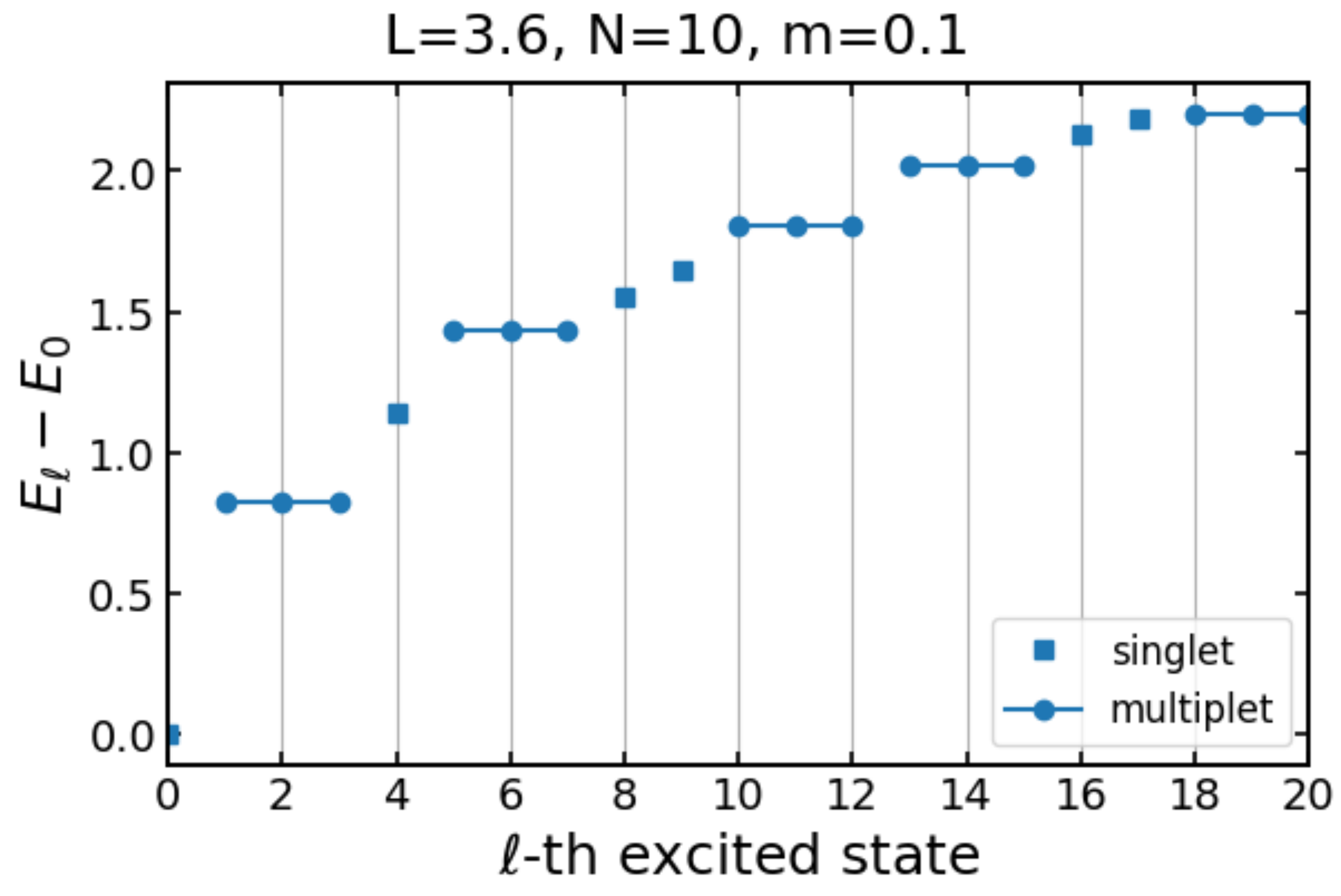


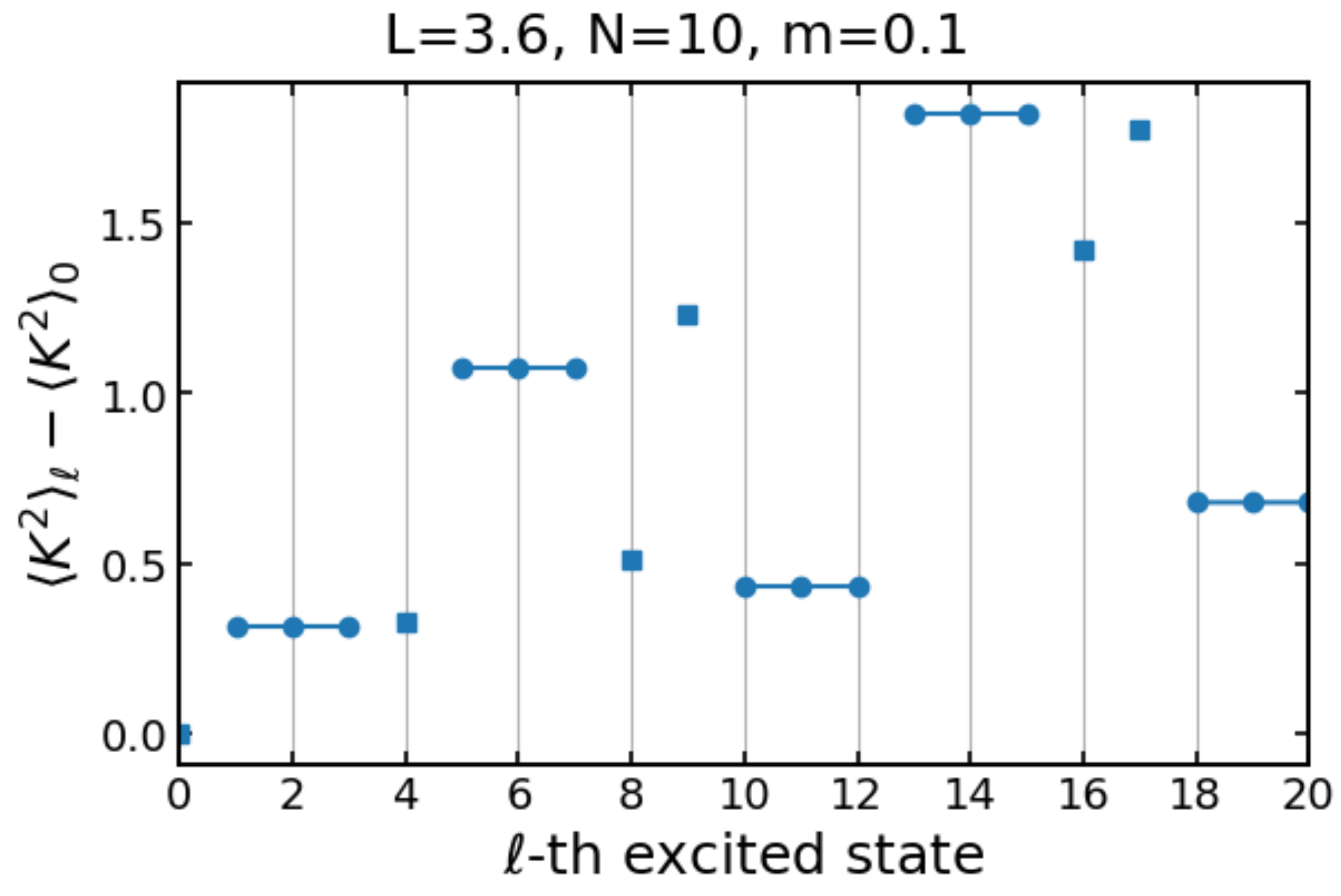


Small N: N = 10, a = 0.4 (20 qubits)

- The lowest excitations become consistent with pion triplets 1^{-+}
- $L = a(N-1) = 3.6 > 1/M_\pi \sim 2.4$





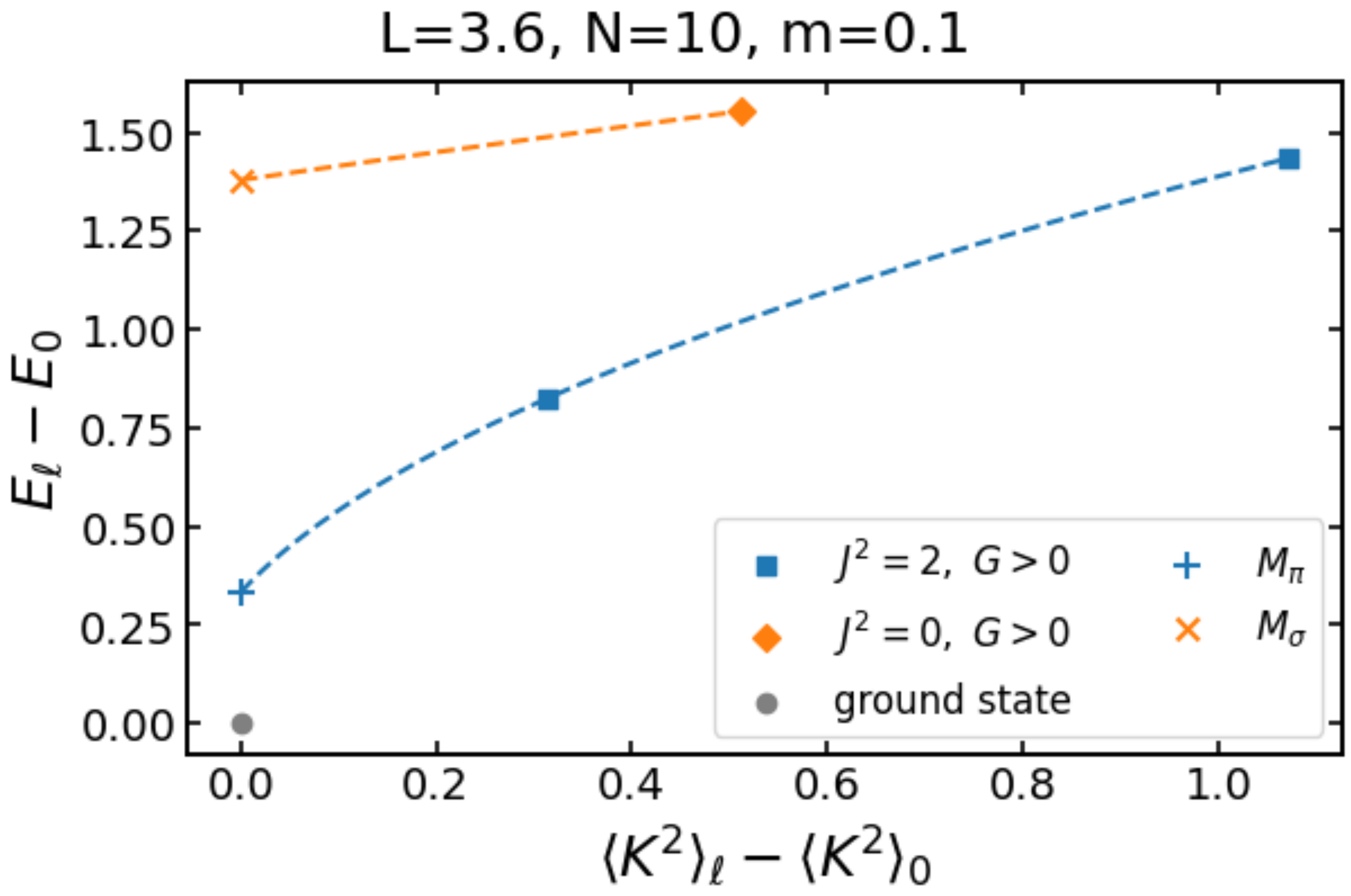


Dispersion relation for L=3.6

- $\ell = (1,2,3), (5,6,7)$ —> pions?
- $\ell = 8$ —> sigma meson?
- solutions of $E = \sqrt{b^2K^2 + M^2}$

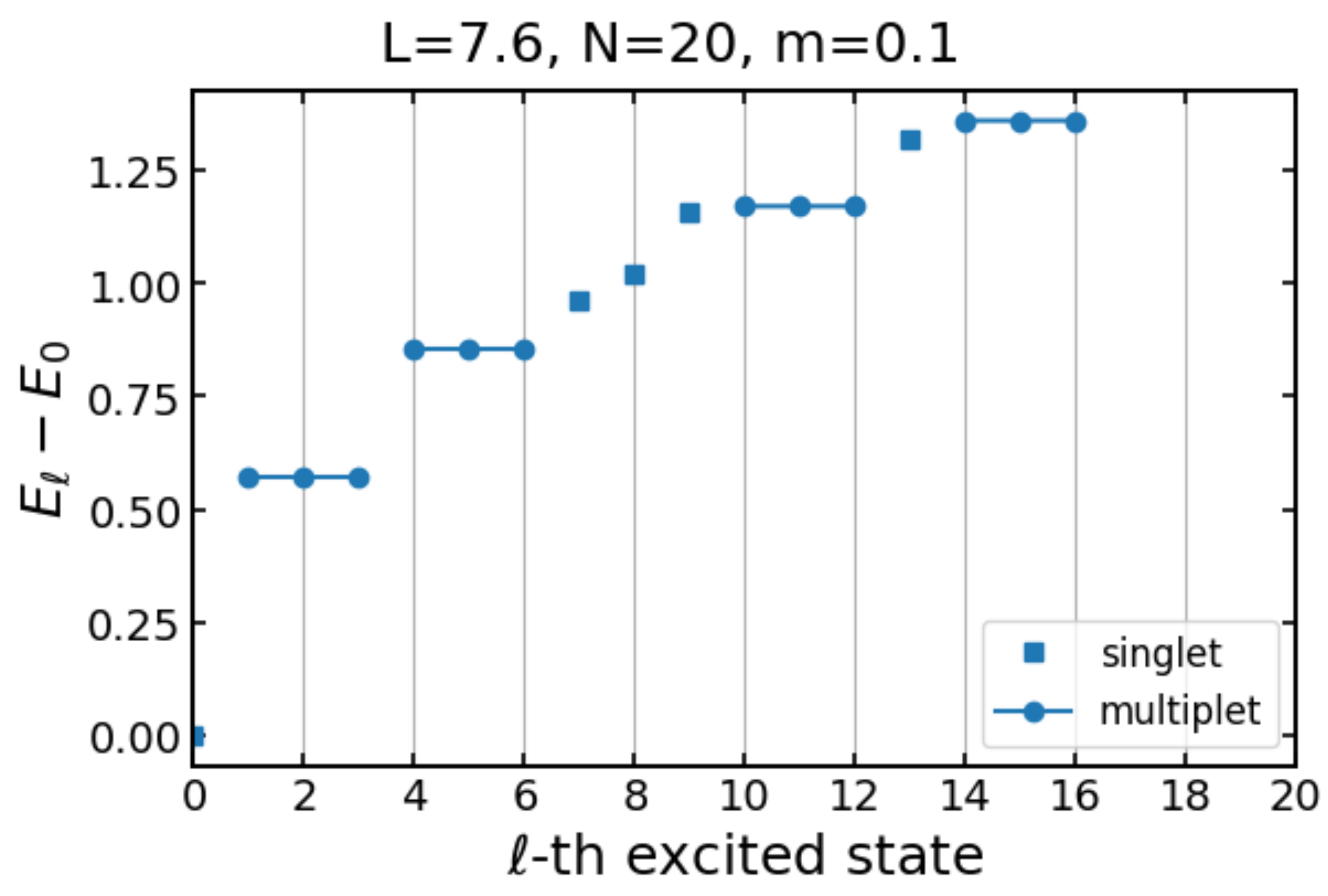
$M_\pi = 0.332,\ b = 1.35$

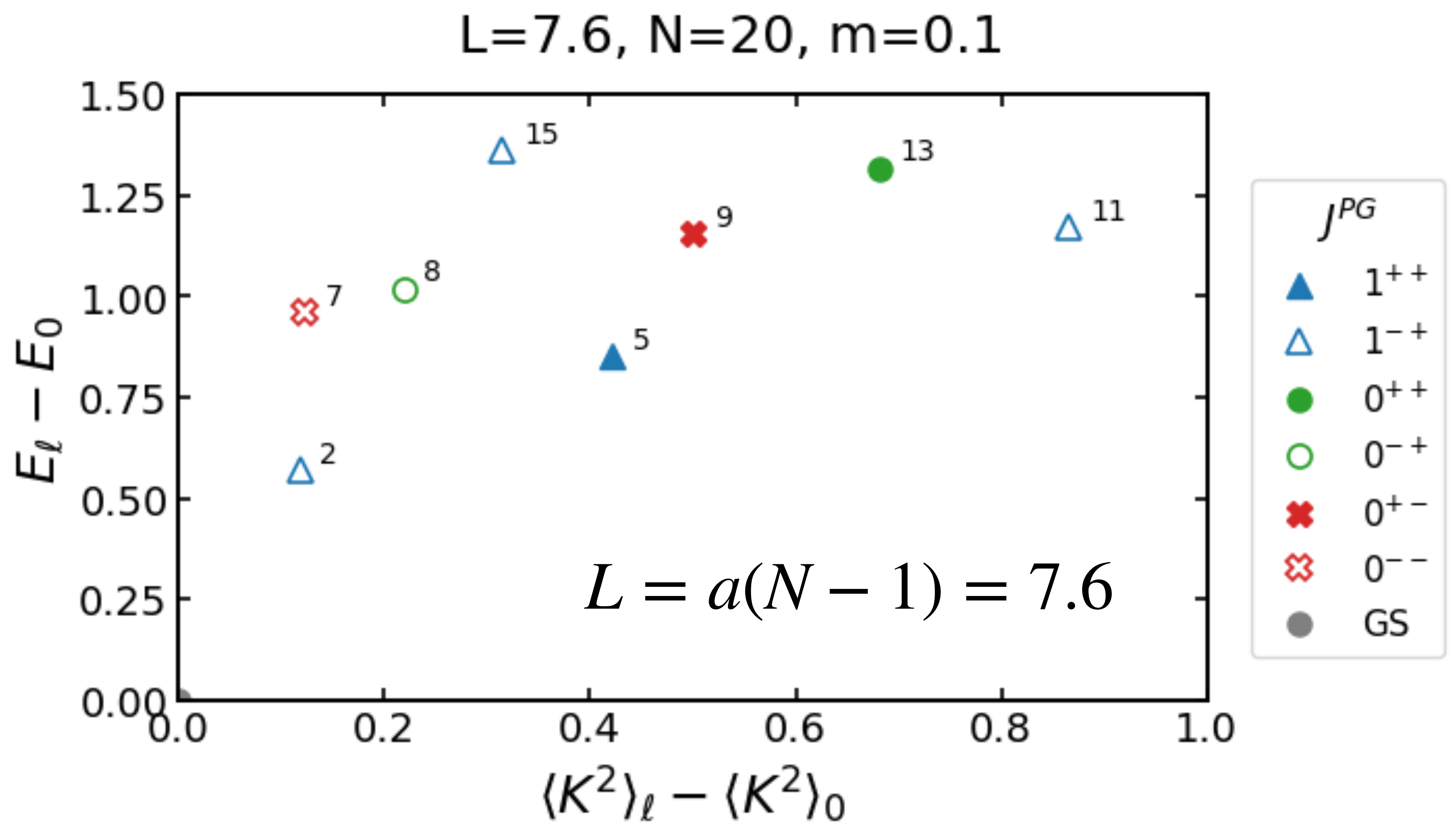
$M_\sigma = 1.38,\ b := 1$

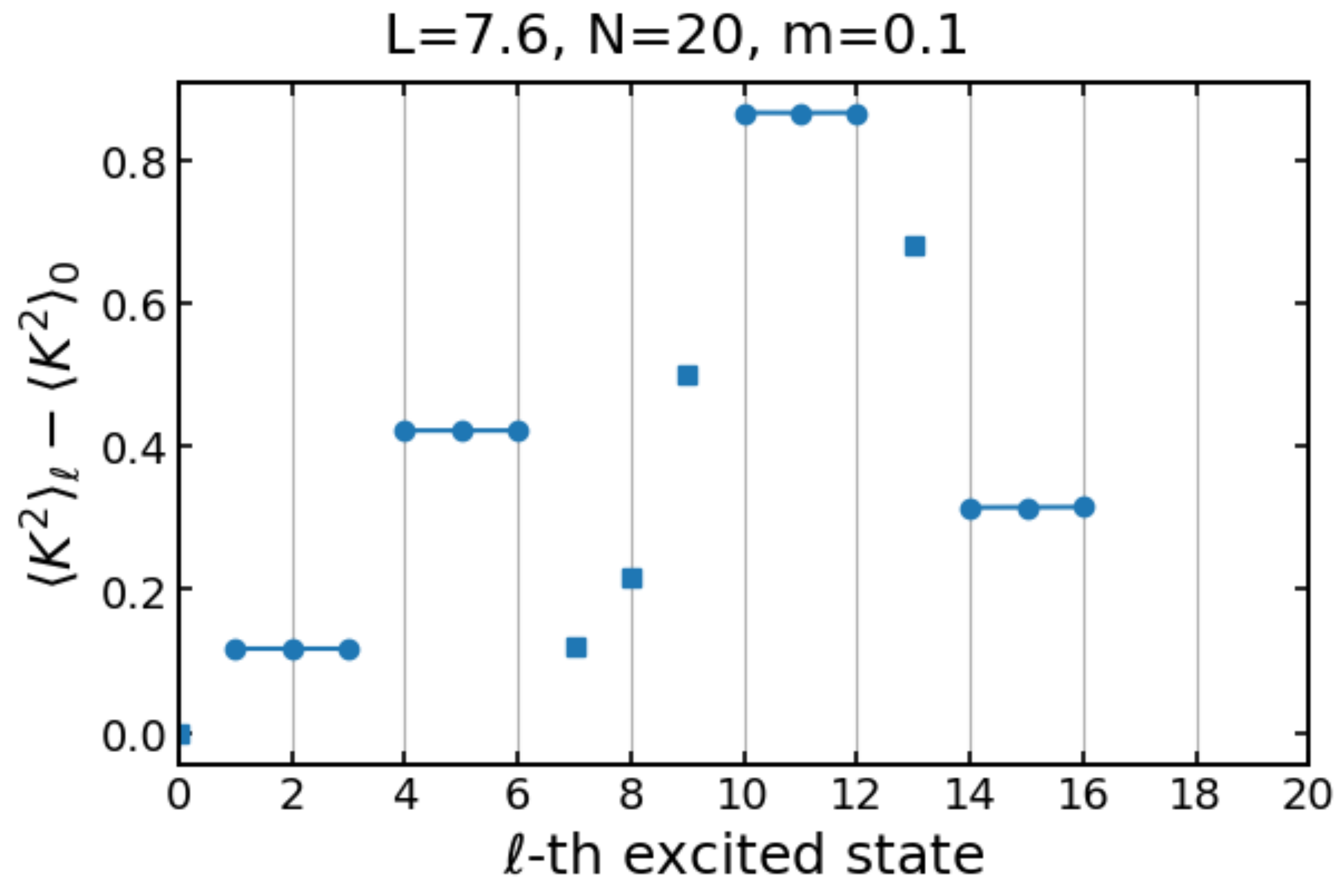


Middle N: N = 20, a = 0.4 (40 qubits)

- The behavior of the pions
  approaches the case of $N = 100$!
- sigma and eta mesons start to appear







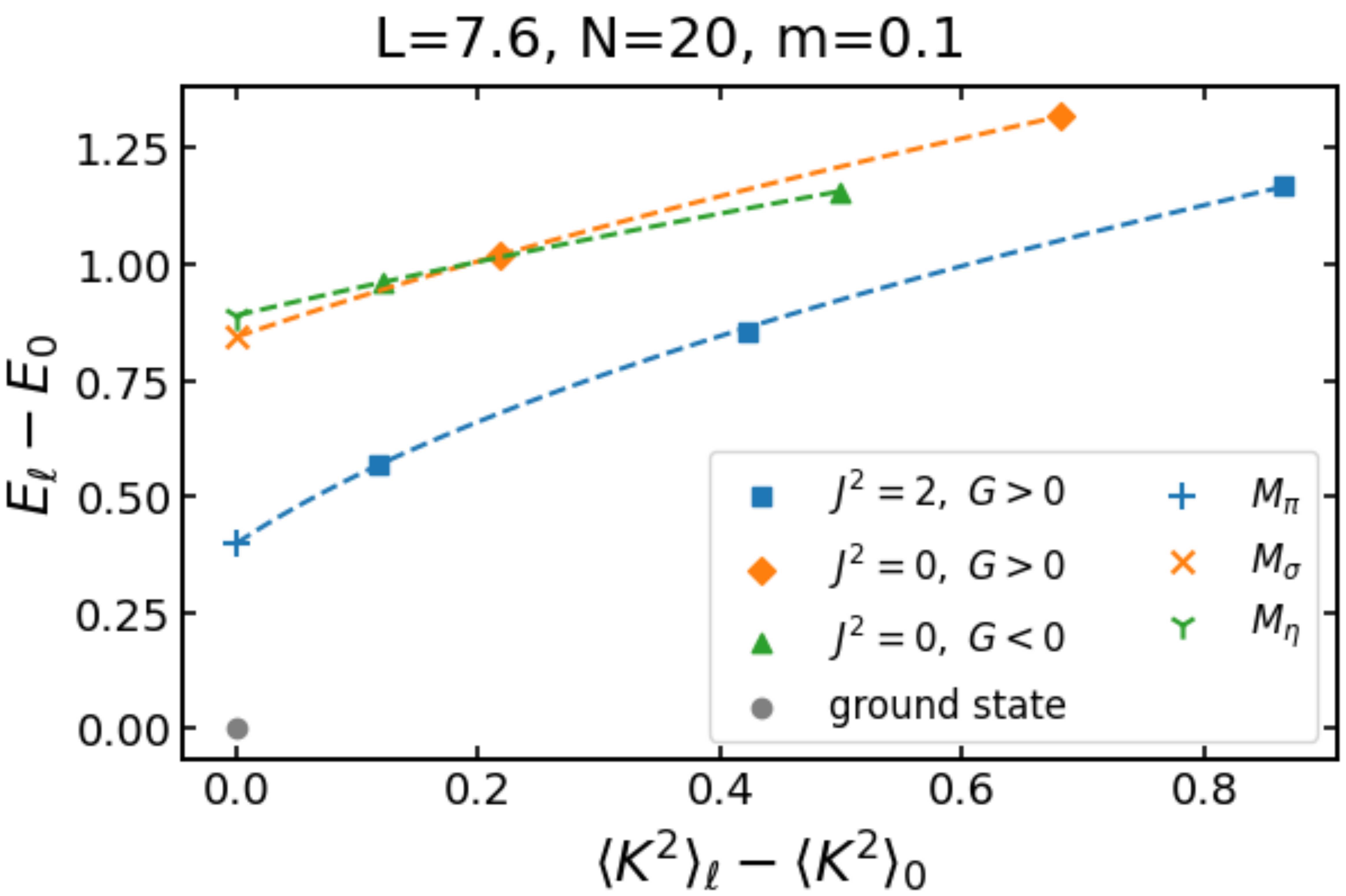
Dispersion relation for L=7.6

- $\ell = (1,2,3), (4,5,6), (10,11,12)$ —> pions?
- $\ell = 8,13$ —> sigma meson?
- $\ell = 7,9$ —> eta meson?
- fitting/solutions of $E = \sqrt{b^2K^2 + M^2}$

$M_\pi = 0.40(2),\ b = 1.18(2)$

$M_\sigma = 0.843,\ b = 1.23$

$M_\eta = 0.889,\ b = 1.05$



consistent with $M_\pi = 0.426(2)$ for N=100

Thank you for listening.