






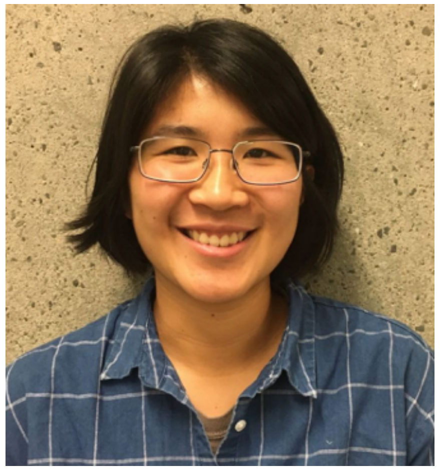




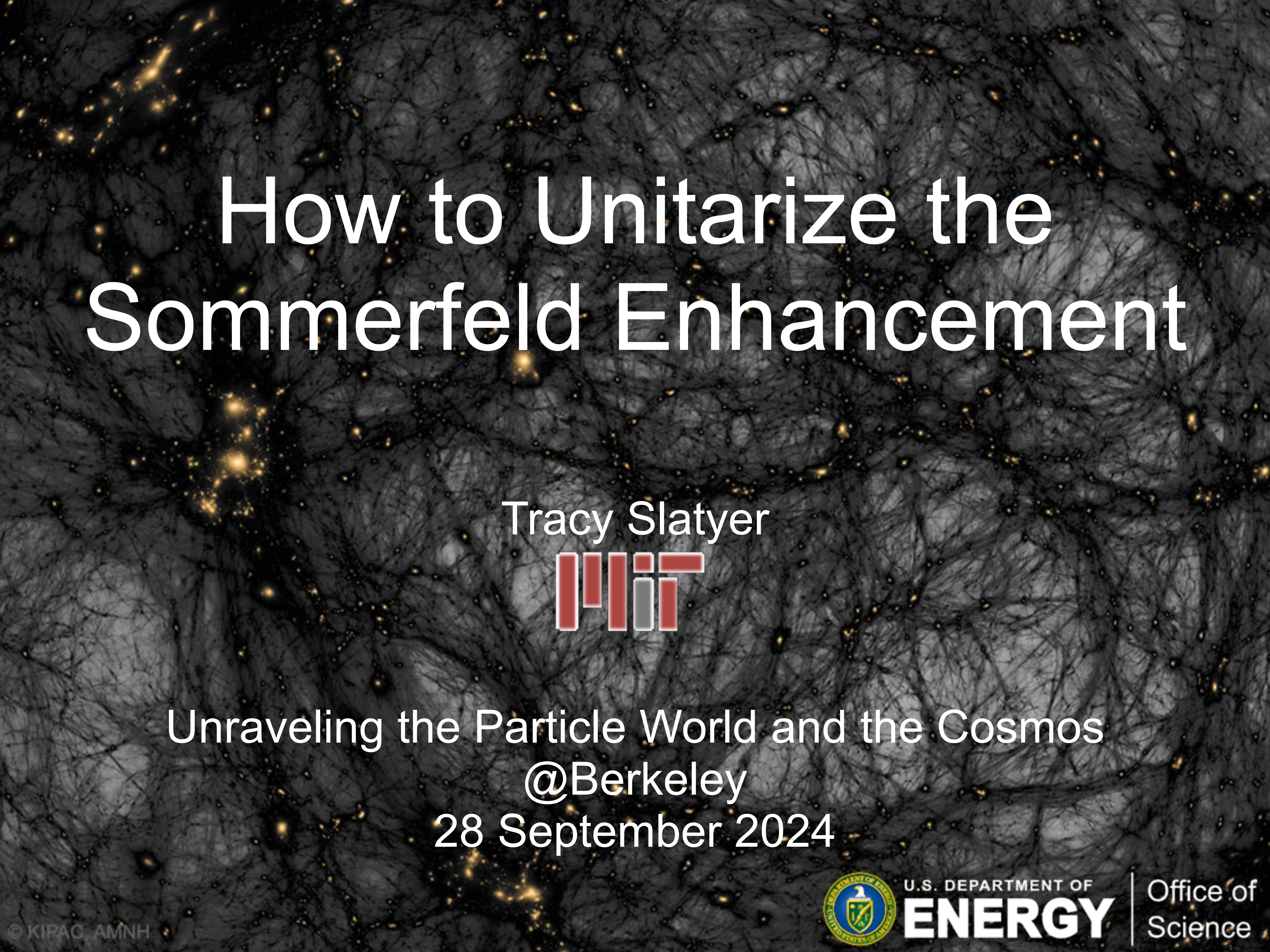
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How to Unitarize the Sommerfeld Enhancement

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The unitarity bound

- There is a mass-dependent hard upper limit on annihilation (or scattering) rates, based on unitary evolution / probability conservation - can be derived from optical theorem

- Performing a partial-wave decomposition, the lth partial wave contributes a maximum total scattering cross section (for distinguishable particles):

$$\sigma_l \leq (2l+1)\frac{4\pi}{k^2}, \quad k = M_\chi v_{\rm rel}/2$$

- Familiar from quantum mechanics:

$$\sigma = \sum_{l=0}^{\infty} \sigma_l, \quad \sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \leq (2l+1)\frac{4\pi}{k^2}$$

- For inelastic processes the bound is a factor of 4 lower

- Often framed as a limit on the dark matter mass (provided we can limit # of partial waves that contribute) - for sufficiently heavy DM, unitarity upper bound is too small to allow efficient depletion through annihilation in the early universe [see e.g. Smirnov & Beacom '19], so thermal freezeout scenarios overpredict DM density

The Sommerfeld enhancement

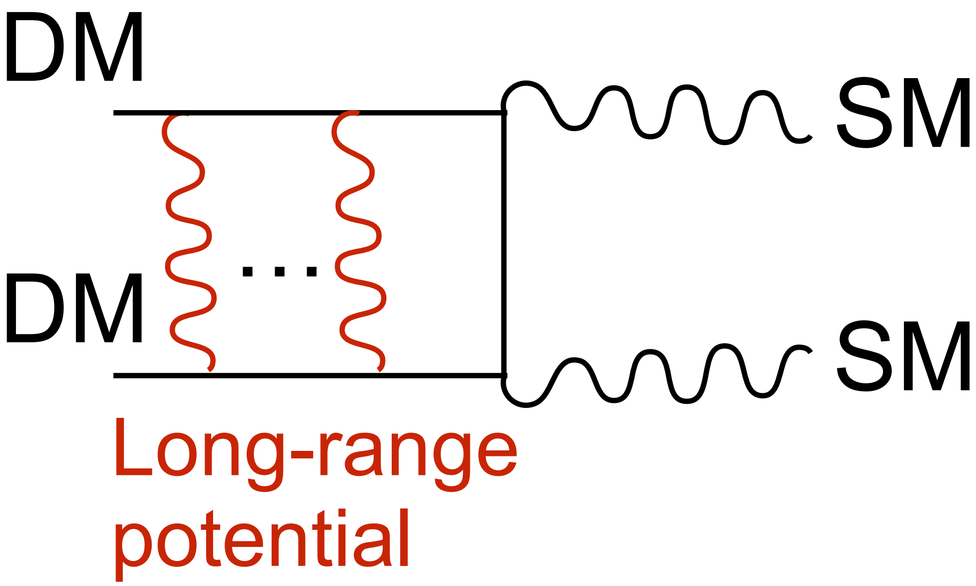
- Basic idea: long-range attractive interactions [studied for Coulomb potential by Sommerfeld 1931] enhance short-range interactions

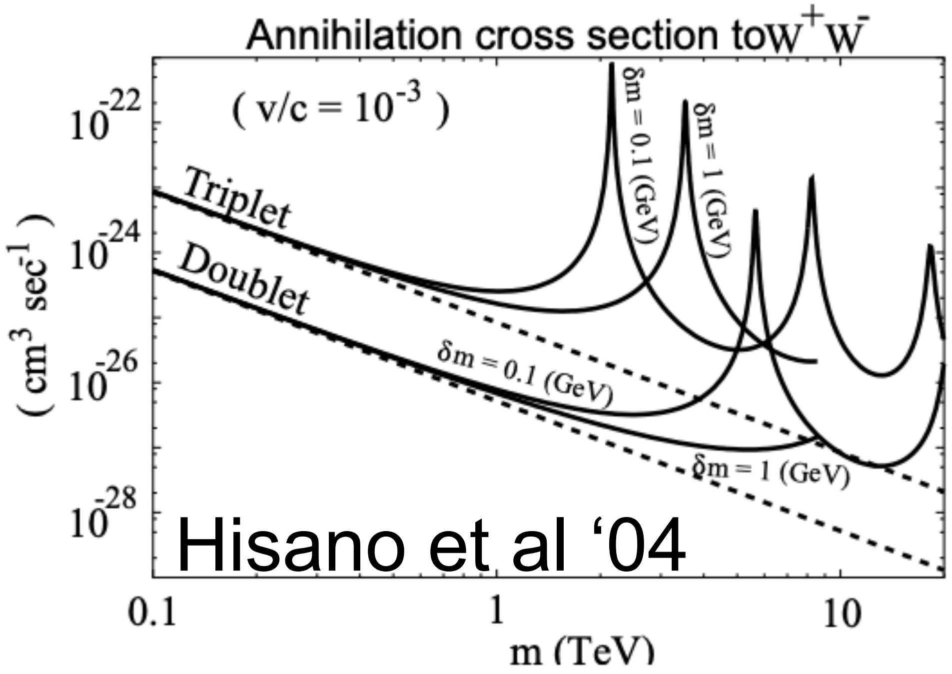
- Implications for heavy weakly interacting massive particle (WIMP) dark matter pointed out by Hisano et al hep-ph/0307216, hep-ph/0412403

- DM annihilation could be significantly enhanced by a long-range attractive interaction mediated by lighter particles, either new mediators or W/Z bosons (for heavy WIMPs)

- This is a low-velocity enhancement: classically, requires potential energy $\gtrsim$ kinetic energy

- Akin to classical gravitational focusing effect

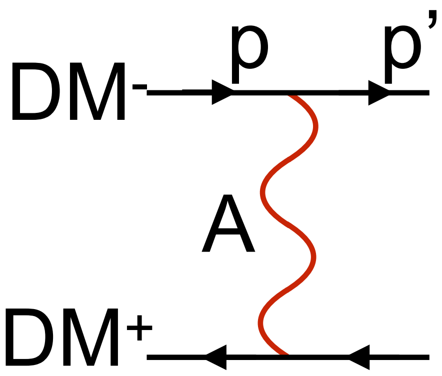




Hisano et al ‘04

Standard calculation of the Sommerfeld enhancement

- Evaluate the potential from non-relativistic limit of perturbative QFT, matched to the Born approximation
- Solve Schrödinger equation for two-particle state of incoming DM particles, obtain wavefunction $\Psi(r)$ (r=distance between particles)
- Annihilation process is high-scale i.e. short-range, so enhancement to amplitude can be estimated from $\lim_{r\to 0}\Psi(r)$
- s-wave case: $\sigma v = |\Psi(0)|^2 (\sigma v)_0$



Simplest case: Yukawa potential

$$i\mathcal{M} = \frac{ig^2}{|\mathbf{p}-\mathbf{p}'|^2 + m_A^2} 2M_{\rm DM}\delta^{ss'} 2M_{\rm DM}\delta^{rr'}$$

$$\tilde{V}(\mathbf{q}) = \frac{-g^2}{|\mathbf{q}|^2 + m_A^2}$$

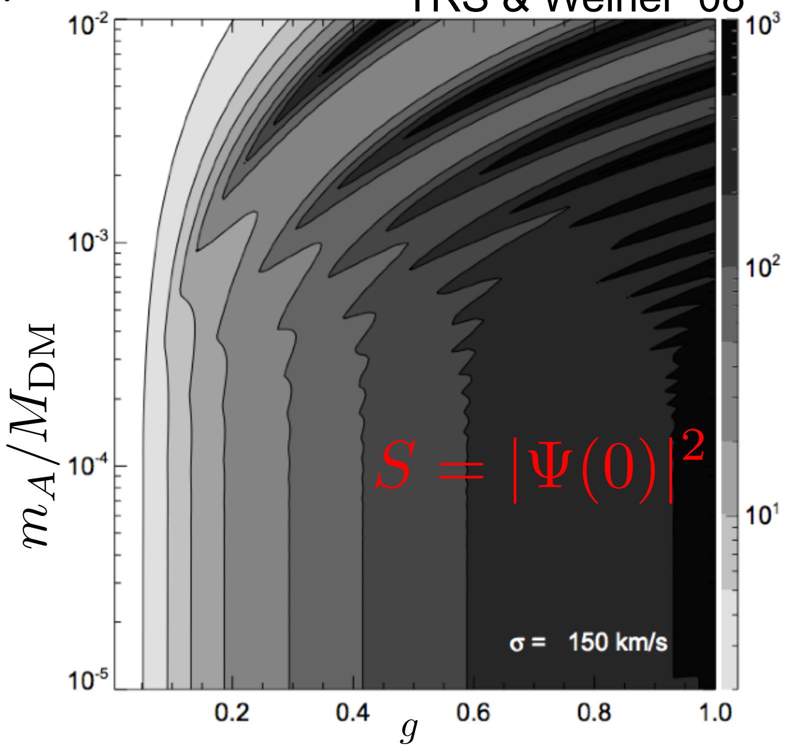
q = p - p'
= momentum transfer

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-m_A r}}{r}$$

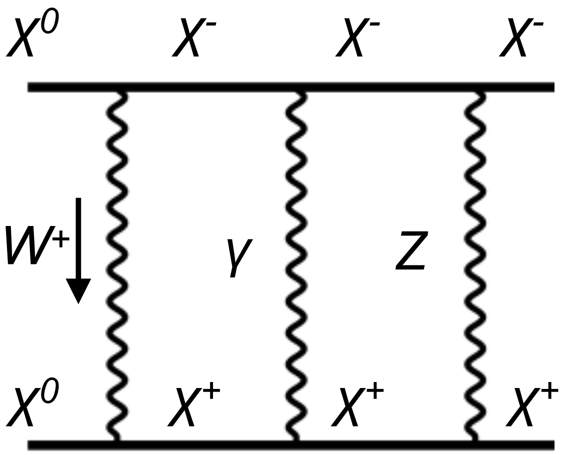
Arkani-Hamed, Finkbeiner, TRS & Weiner '08

Example of Sommerfeld enhancement at fixed DM velocity

As $m_A \to 0$,
$S \to \pi\, \alpha / v_{\rm rel}$,
$\alpha = g^2/4\pi$



Multi-state Sommerfeld enhancement



- There may be multiple two-particle states that are coupled by the potential

$$V(r) = \begin{pmatrix} 0 & -\sqrt{2}\alpha_W \frac{e^{-m_W r}}{r} \\ -\sqrt{2}\alpha_W \frac{e^{-m_W r}}{r} & 2\delta M - \frac{\alpha}{r} - \alpha_W c_W^2 \frac{e^{-m_Z r}}{r} \end{pmatrix}$$

- Generally requires mass difference vs DM-DM state to be smaller than scale of potential energy
- Occurs naturally for the case of few-TeV wino dark matter: mass splitting $\mathcal{O}(\alpha m_Z)$, potential energy scale $\mathcal{O}(\alpha_W^2 m_{\mathrm{DM}})$

$$\left(-\frac{\nabla^2}{2\mu} + V(r) - E\mathbf{1}\right)\vec{\Psi}(r) = 0$$

- Treat with non-relativistic QM: wavefunction becomes vector, potential represented by matrix

wavefunction for two-body state i acts as weighting function

"annihilation matrix" determined by phase space integral over products of matrix elements

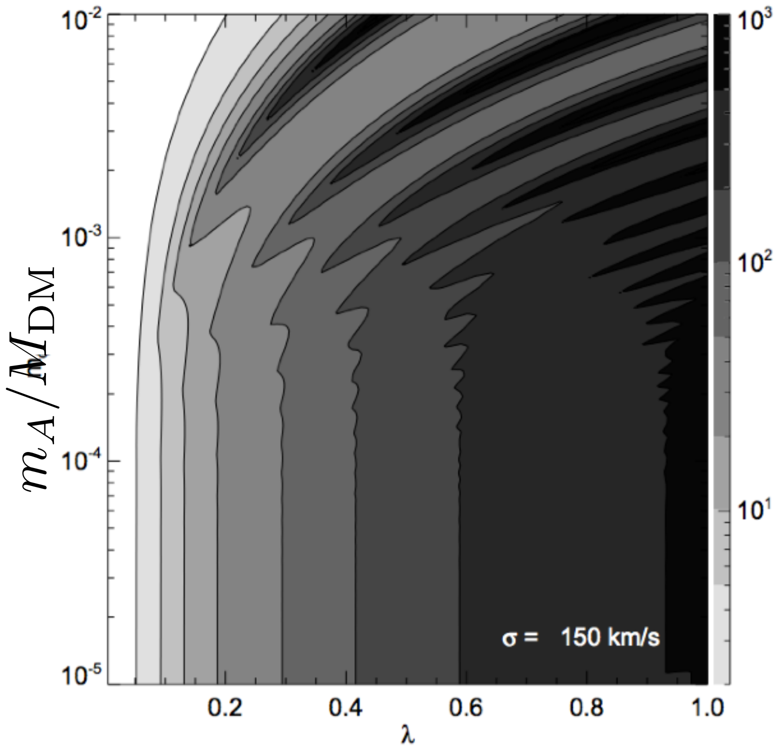
$$\sigma v = \frac{1}{(2M_{\mathrm{DM}})^2}\int d\Pi_n \left|\sum_i c_i \Psi_i(0) \mathcal{M}(i \to f)\right|^2 = c\vec{\Psi}^\dagger(0)\cdot\mathbf{\Gamma}\cdot\vec{\Psi}(0)$$

matrix elements for annihilation of two-body state i to final state f

phase space integral

constants needed to properly normalize two-body states with identical/non-identical particles

Resonances



- Attractive Coulomb potential (s-wave):

$$\frac{\pi\alpha}{v}\frac{1}{1-e^{-\pi\alpha/v}}$$

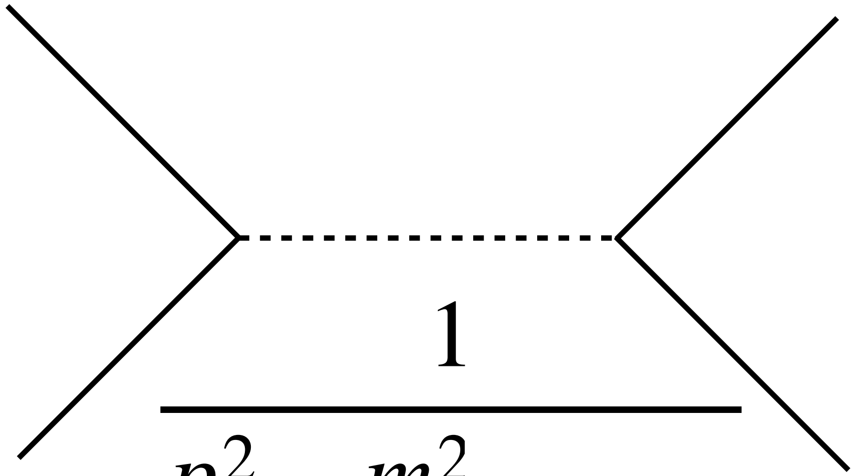
- Attractive Yukawa potential - no exact solution, but can be approximated analytically (s-wave):

$$\frac{\pi\alpha}{v}\frac{\sinh(2\pi v m_{\text{DM}}/m_\phi^*)}{\cosh(2\pi v m_{\text{DM}}/m_\phi^*)-\cos\left(2\pi\sqrt{\frac{\alpha m_{\text{DM}}}{m_\phi^*}-\frac{v^2 m_{\text{DM}}^2}{m_\phi^{*2}}}\right)}$$

- On resonance, enhancement $\propto 1/v^2$, vs 1/v off-resonance

- Physical origin of resonances = parameters where new bound states enter the spectrum / a zero-energy “bound” state exists

Unitarity violations?



$$\frac{1}{p^2 - m^2}$$

- Problem (s-wave): $\sigma v \propto 1/v^2$ on resonance, but unitarity puts an upper bound on σ that scales as $1/v^2$
- If unitarity appears to be violated then it usually signals a problem in the calculation / breakdown of an approximation
  - e.g. annihilation through a s-channel resonance appears to diverge as propagator goes on-shell $\Rightarrow$ need to take into account width of mediator, which regulates propagator
- Q: What approximation breaks down in the standard Sommerfeld calculation?
- A: Ignoring the annihilation when we solve for the wavefunction

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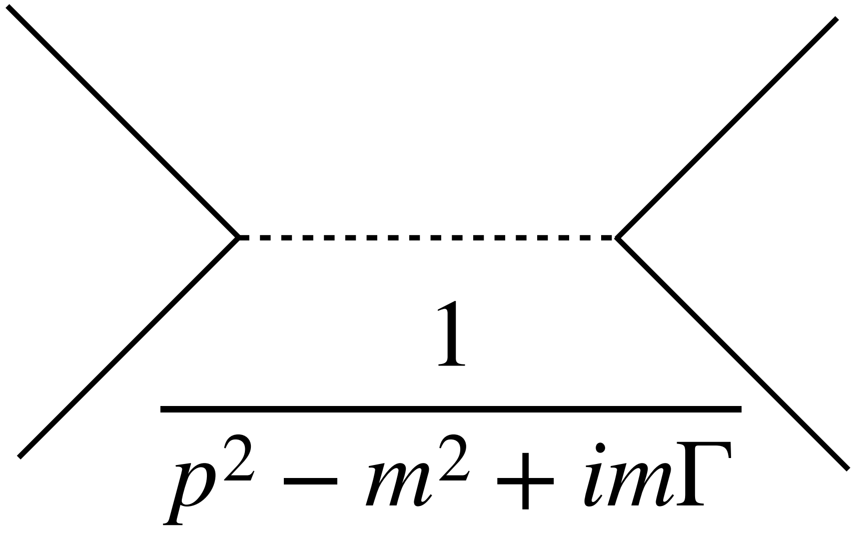
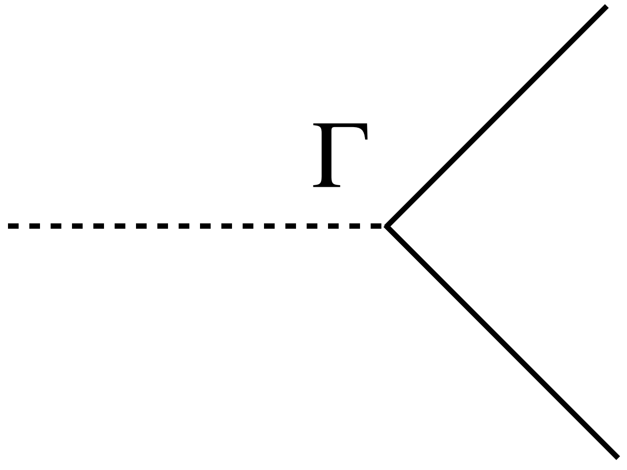


$$\frac{1}{p^2 - m^2 + im\Gamma}$$



- Q: What approximation breaks down in the standard Sommerfeld calculation?
- A: Ignoring the annihilation when we solve for the wavefunction

Correcting the Sommerfeld enhancement (2016)

- Blum, Sato & TRS 1603.01383: model physics responsible for annihilation as complex delta-function potential
- This is also the approach taken in the initial work by Hisano et al hep-ph/0412403, valid for l=0 (they then expanded to lowest order in the annihilation rate)
- Re-solve Schrödinger equation with additional complex potential term; in general need to renormalize delta-function contribution, calibrate to cross sections measured at high momentum

$$\sigma v \simeq \frac{\sigma v_0 \, S(v)}{\left|1+\left(\eta\sqrt{\frac{\mu^2\sigma_{sc,0}}{4\pi}-\left(\frac{\mu^2\sigma v_0}{4\pi}\right)^2}-i\frac{\mu^2\sigma v_0}{4\pi}\right)\left(T(v)+iS(v)\right)v\right|^2},$$

associated with short-range physics

derive from long-range potential

Correcting the Sommerfeld enhancement (2024a)

- Flores & Petraki 2405.02222 take a different approach, still modeling all physics in terms of a complex potential but using the optical theorem
- Resum real part of potential but treat imaginary part perturbatively
- Calculation done for single-state case (allows for multiple annihilation channels)
- Works for all partial waves, no restriction to contact interactions
- Simplest version of result for regulated amplitude is:

$$M_{\ell,\mathrm{reg}}^{\mathrm{inel},i} = \frac{M_{\ell,\mathrm{unreg}}^{\mathrm{inel},i}}{1 + \sigma_{\ell,\mathrm{unreg}}^{\mathrm{inel}} / \sigma_{\ell}^{U}},$$

but doesn't apply to contact interactions / amplitudes that fall off too slowly at large momentum

Correcting the Sommerfeld enhancement (2024b)

- New work by Parikh, Sato & TRS (2410.XXXXX, to appear)

- Stick to non-relativistic quantum mechanics framework but avoid non-Hermitian potential

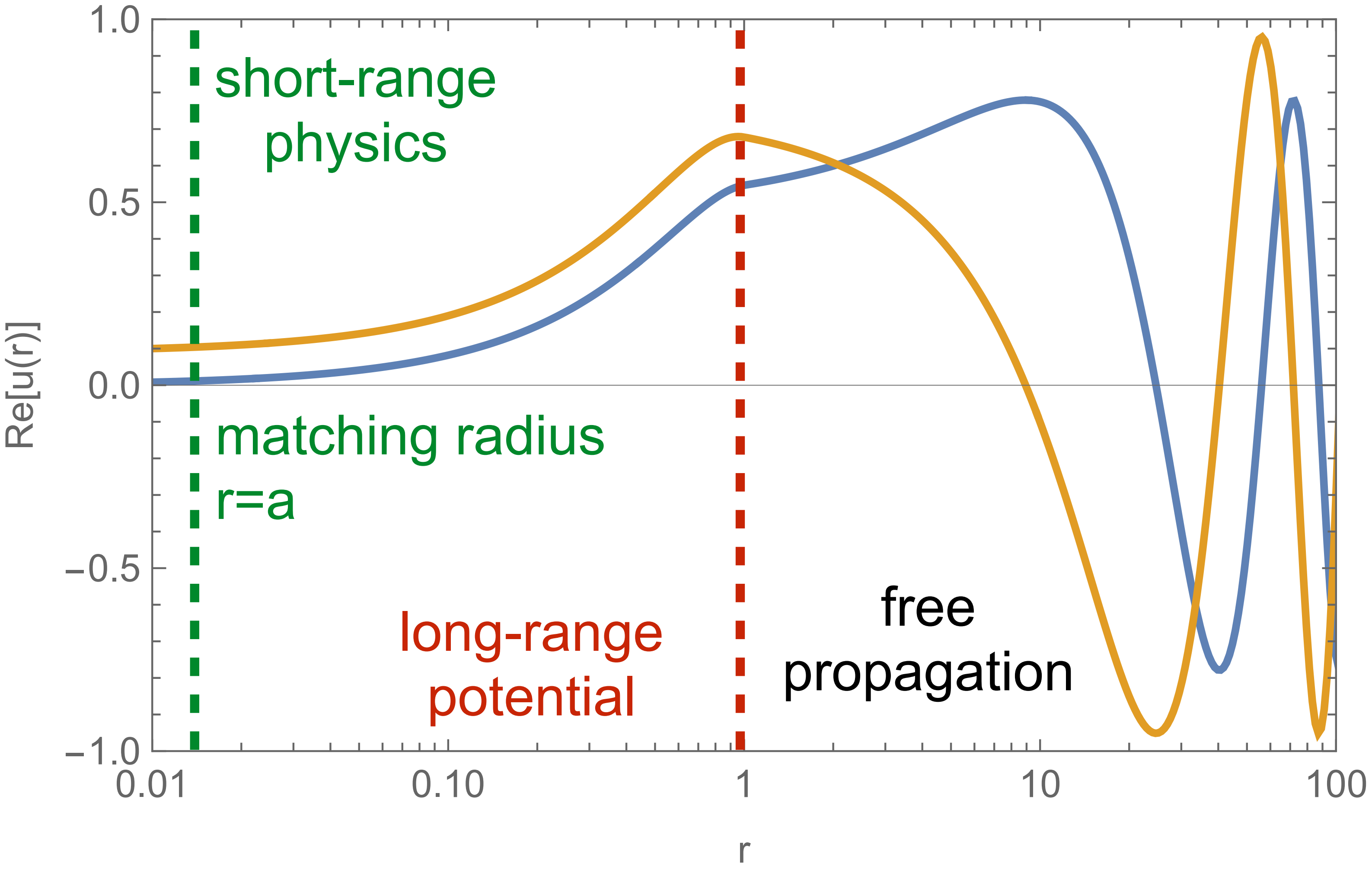
- Works for all partial waves and for multi-state systems, provided we can assume that annihilation/absorptive physics is localized to r < a for some matching radius a

- Position-space matching approach initially motivated by Agrawal et al 2003.00021 (which used a similar approach to study scattering in apparently-singular potentials)

- General idea: use non-relativistic QM for r > a, for r < a match onto appropriate QFT S-matrix (calculated perturbatively)

High-scale physics = modified boundary conditions

Square well example



solution with standard BC at r=0

pure-outgoing solution with modified BC at r=0 (sourced by short-range physics)

- Standard wavefunction calculation imposes regularity at r=0
- We instead impose BCs at a matching radius r=a, set by energy scale of short-range physics (e.g. for annihilation $a \sim 1/M_{\rm DM}$); BCs determined by short-range S-matrix
- Nonperturbative evolution from r=a to r=∞ resums effects of long-range potential

Solving for the wavefunction with modified BCs

- Solve standard Schrödinger equation from $r = a \rightarrow \infty$
- Standard solution $w(r)$ = incoming plane wave with standard normalization, regular as $r \rightarrow 0$
- 2nd order linear ODE = 2 independent solutions, pick one other to form a basis
- We use irregular solution that is purely outgoing ($\propto e^{ipr}$) as $r \rightarrow \infty$, with simple normalization at r=0, denote $\tilde{w}(r)$
- Full solution: $u(r) = w(r) + R\tilde{w}(r)$ (has correct normalization for incoming wave)
  Adjust factor R to match desired boundary conditions at r=a
- Can read off the outgoing wave and hence S-matrix in terms of the baseline S-matrix (what we would get with R=0) and R

$$u(r) \rightarrow \frac{1}{2i}\left((-i)^\ell S_\ell e^{ipr} - i^\ell e^{-ipr}\right), r \rightarrow \infty$$

$$\overset{\text{full S-matrix}}{\boxed{S}} = \underset{\text{baseline S-matrix}}{\boxed{S_0}}(1 + 2ip^\ell \overset{\text{encodes short-distance physics}}{\boxed{R}}\underset{\text{Sommerfeld factor}}{\boxed{\Sigma_0^*}})$$

Matching modified BCs to the short-range S-matrix

- Short-range S-matrix / scattering amplitude encodes what happens if you send in a plane wave from r=a to r=0 and measure outgoing wave at r=a

- Match value and first derivative of wavefunction at r=a to

$$C(s_\ell(pr) + f_s p(c_\ell(pr) + is_\ell(pr)))$$

constant

scattering amplitude

partial wave components of plane wave (cos-like and sin-like)

- Solve for R in terms of f_s, plug into expression for S-matrix, do algebra (matrix algebra in multi-state case)

Corrected S-matrix

- After the dust clears, this is what the full multi-state result for the S-matrix looks like:

$$S_\ell = S_{0,\ell}\left(1 + 2iP\Sigma_{0,\ell}^\dagger\left[\kappa_\ell^{-1} - i\Sigma_{0,\ell}P^2\Sigma_{0,\ell}^\dagger\right]^{-1}\Sigma_{0,\ell}P\right),$$

$$\kappa_\ell^{-1} \equiv \left[\tilde{P}\alpha_{b,\ell}(0)\tilde{P}^{-2}\right]\left(\hat{f}_{s,\ell}^{-1}\left[\tilde{P}\alpha_{b,\ell}(0)\tilde{P}^{-2}\right]^T\tilde{P}^{2\ell} - \tilde{P}\alpha_{\tilde{G}_\ell}(a)\right)(\tilde{P}^\dagger)^{-2\ell}.$$

- κ_ℓ can be thought of as a corrected short-range amplitude (corrected by the long-range potential)
- $\Sigma_{0,\ell}$ = standard Sommerfeld factor matrix (for amplitude, not cross section)
- $P, \tilde{P}$ = diagonal matrices encoding momentum factors
- $\hat{f}_s$ = short-range amplitude after removing any contribution already included in the “long-range potential” used to compute $w(r), \tilde{w}(r)$
- $\alpha_{b,\ell}(0), \alpha_{\tilde{G}_\ell}(a)$ = matrices that can be read off from $w(r), \tilde{w}(r)$ solutions, sensitive to matching radius but not to short-range amplitude

Corrected cross-section

- General result (i indexes the initial two-body state):

$$\sigma_{i,\text{ann}} = \underset{\uparrow}{c_i}\frac{\pi}{p_i^2}\sum_{\ell}(2\ell+1)(1 - S_\ell^\dagger \underset{\uparrow}{S_\ell})_{ii}$$

identical-particle factor   full S-matrix

- Annihilation rate is encoded in apparent non-unitarity of full S-matrix element (because we have truncated the S-matrix to the space of non-relativistic two-particle states in the QM calculation)

- If all the absorptive physics is short-distance, we can assume $S_{0,\ell}$ (S-matrix with only long-range potential) is unitary

- Then in terms of the "corrected short-distance amplitude" κ_ℓ, we get:

$$(\sigma_{i,\text{ann}} v_{\text{rel}})_\ell = c_i \frac{4\pi i}{M_{\text{DM}}}(2\ell+1)\left[\Sigma_\ell^\dagger(\kappa_\ell^\dagger - \kappa_\ell)\Sigma_\ell\right]_{ii}, \quad \Sigma_\ell = \left[1 - i\Sigma_{0,\ell}P^2\Sigma_{0,\ell}^\dagger\kappa_\ell\right]^{-1}\Sigma_{0,\ell}$$

corrected Sommerfeld enhancement

Checking unitarity

- There are long-range corrections encoded in κ_ℓ but they're not necessary to confirm the preservation of unitarity

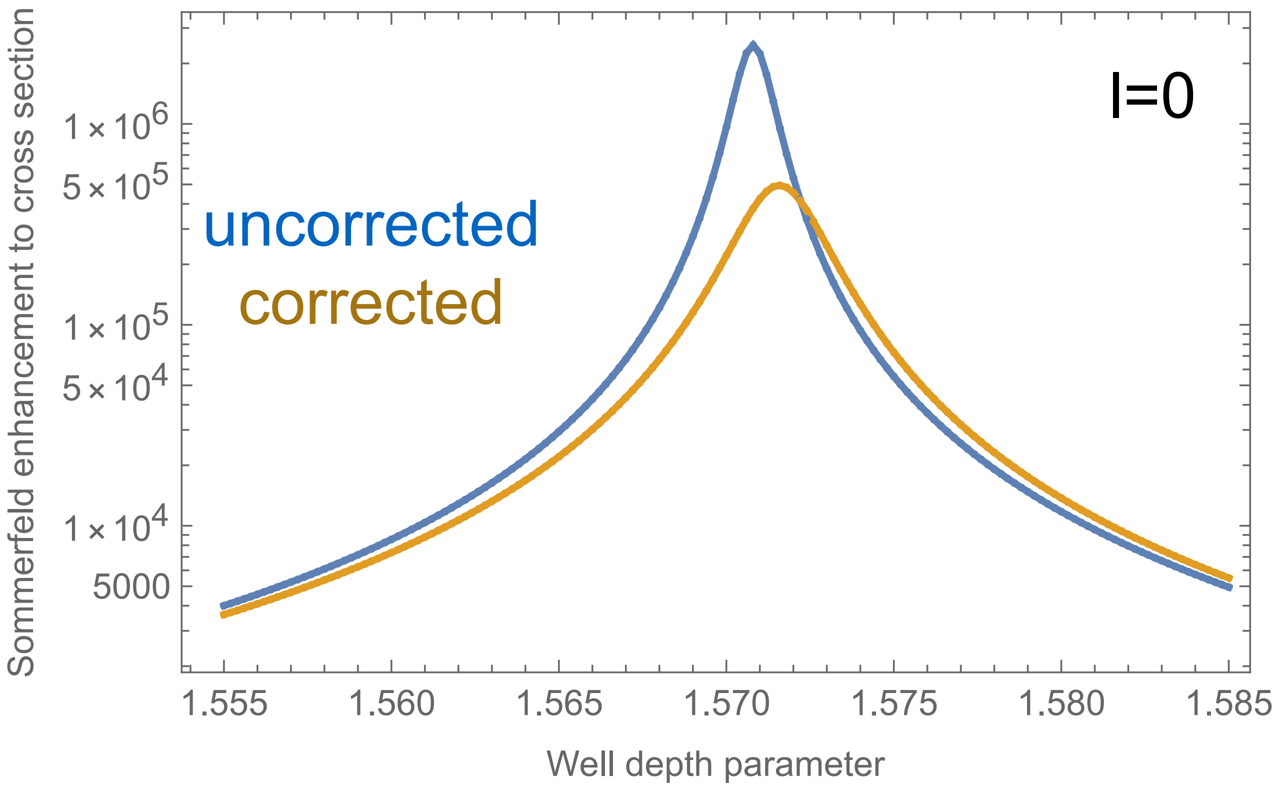
- Single-state case: $4\pi i \Sigma_\ell^\dagger (\kappa_\ell^\dagger - \kappa_\ell)\Sigma_\ell = \dfrac{8\pi}{p}\dfrac{p\text{Im}\kappa_\ell |\Sigma_{0,\ell}|^2}{|1 - ip\kappa_\ell |\Sigma_{0,\ell}|^2|^2}$

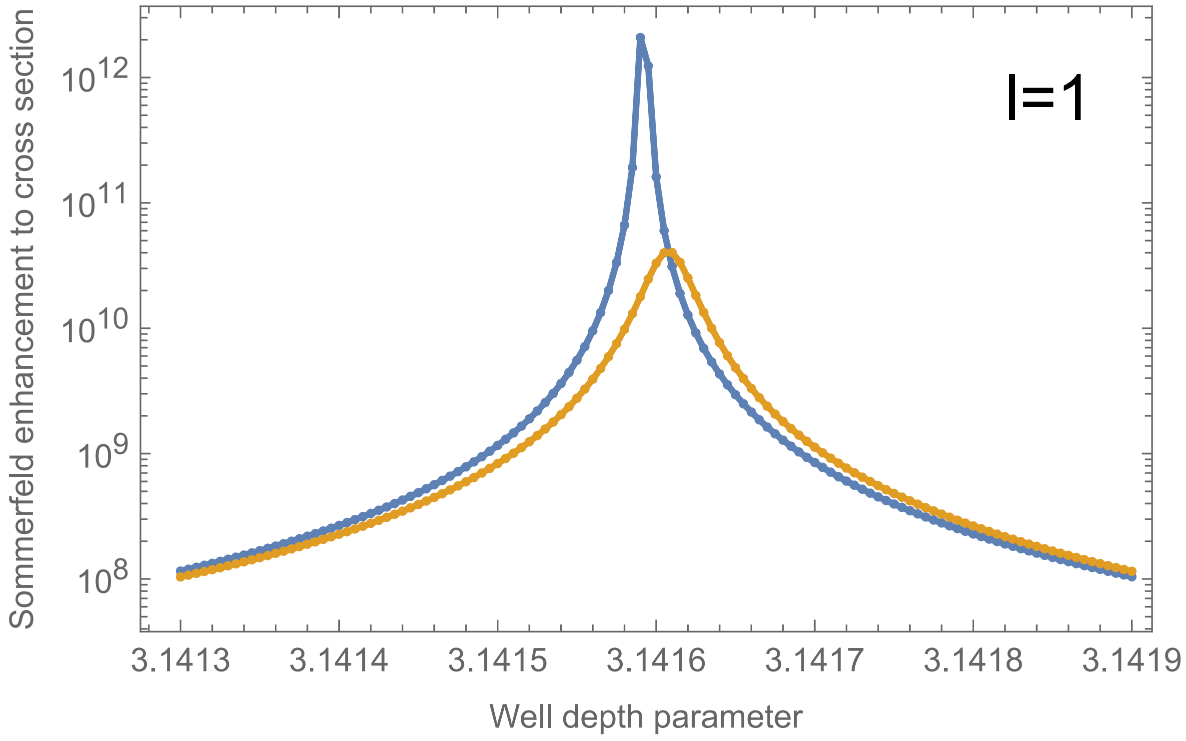
$$\leq \frac{8\pi}{p}\frac{p\text{Im}\kappa_\ell |\Sigma_{0,\ell}|^2}{(1 + p\text{Im}\kappa_\ell |\Sigma_{0,\ell}|^2)^2 + (p\text{Re}\kappa_\ell |\Sigma_{0,\ell}|^2)^2} \leq \frac{2\pi}{p}$$

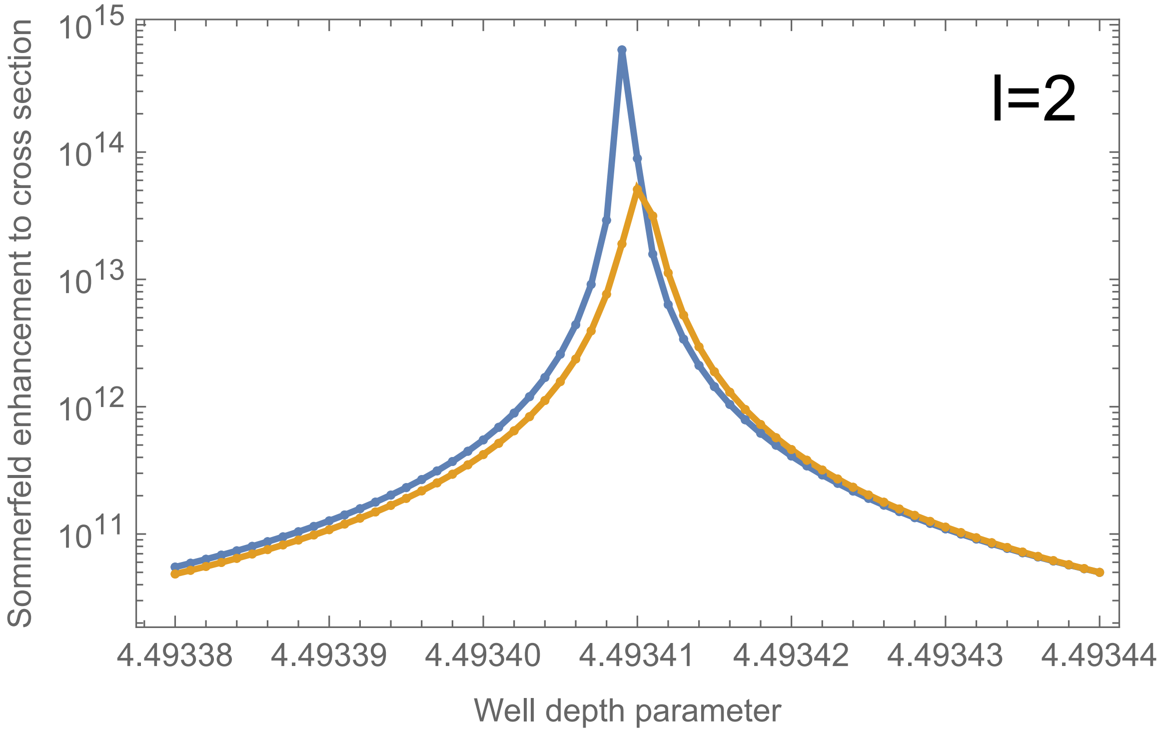
- When the Sommerfeld enhancement becomes very large, so does the denominator term, and drives the final result below (often well below) the unitarity bound

Square well example (single-state, l=0,1,2)

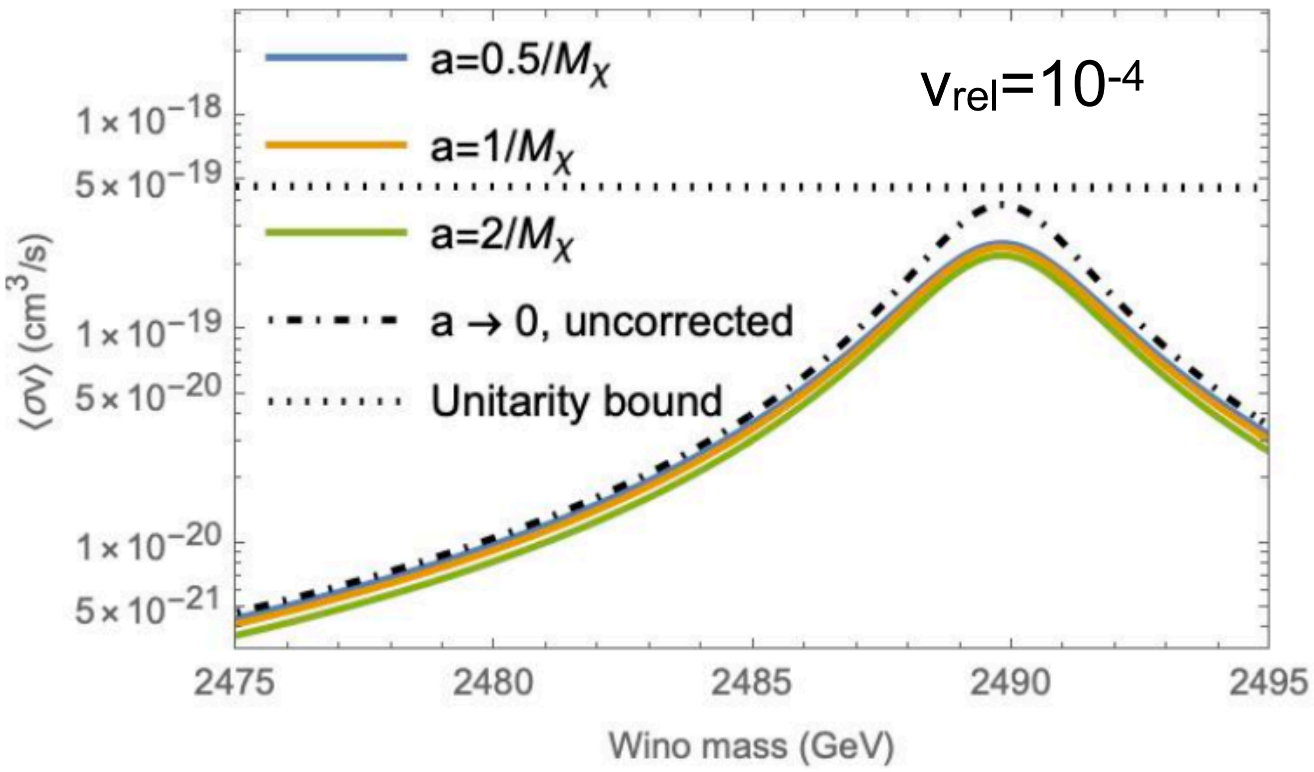
- The finite square well is analytically solvable and supports bound states. Zero-energy bound states yield resonances.
- In this case the whole calculation can be done analytically, take $a \to 0$ limit.
- Short-distance amplitude set arbitrarily here. Real part of amplitude responsible for shift in resonance position.

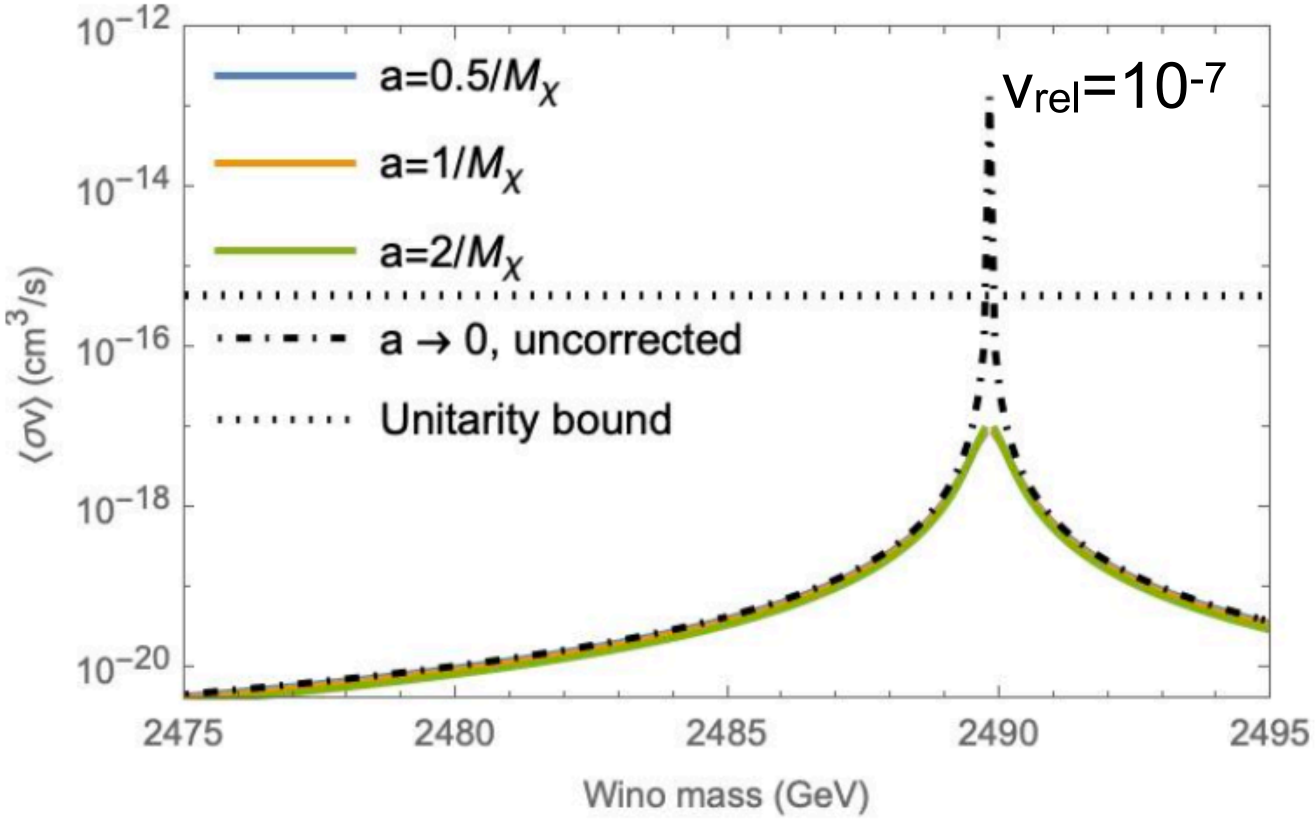






Wino example (multi-state, l=0)



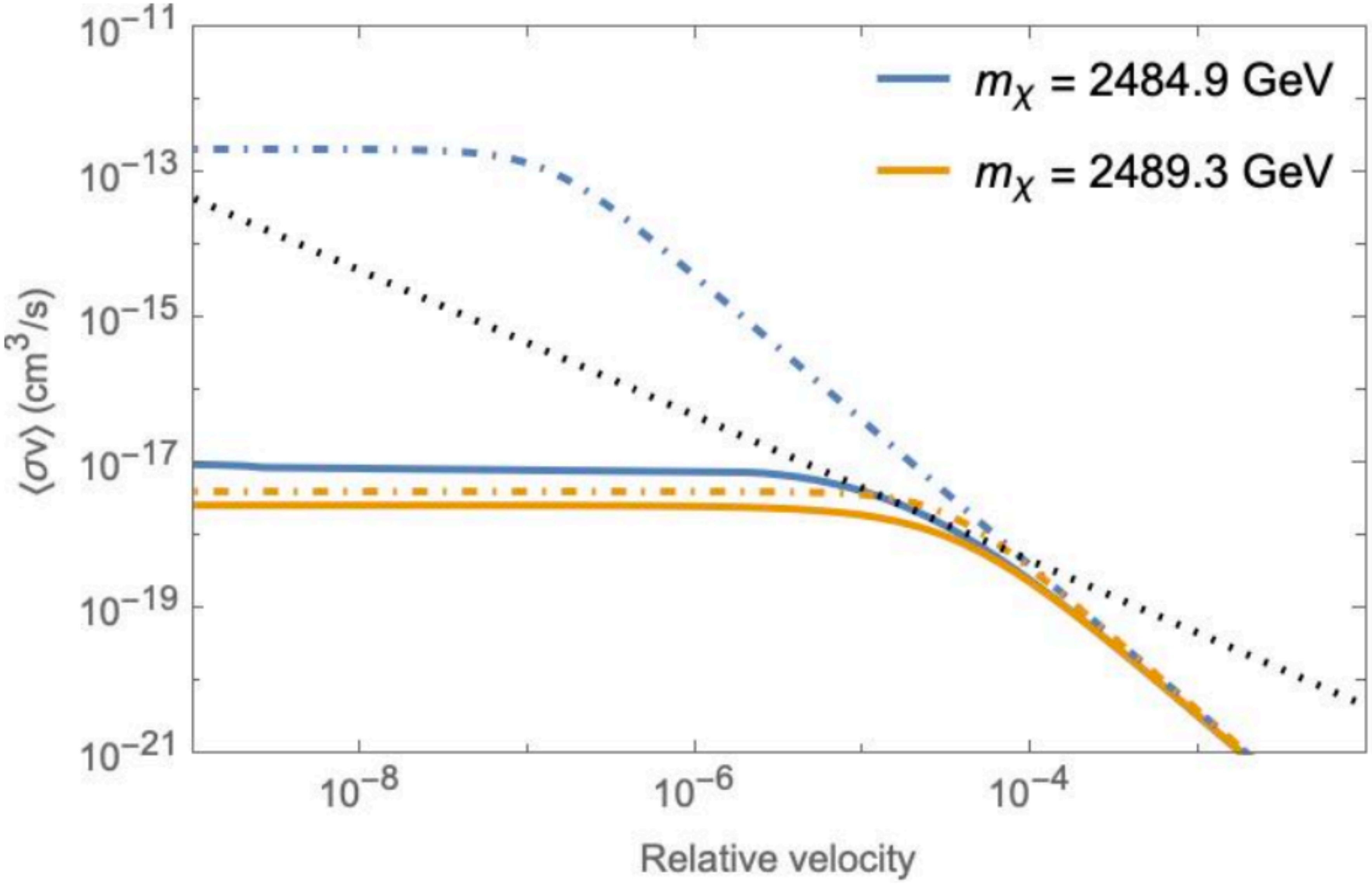


- A more physical example is the wino (fermion DM in triplet of $SU(2)_W$)
- First s-wave resonance around 2.5 TeV (using NLO potential derived in Beneke et al 1909.04584)

Velocity dependence

- Plot shows velocity dependence of cross section (uncorrected=dot-dashed, corrected =solid)

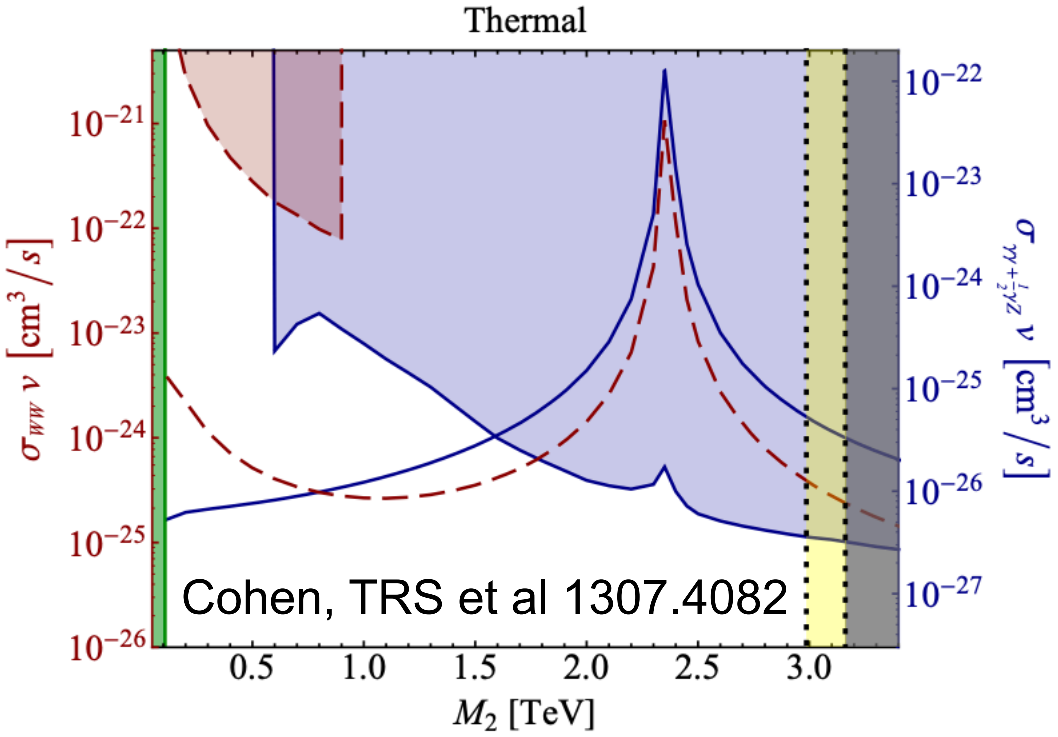
- Hitting unitarity causes enhancement to saturate (does not continue to follow unitarity bound at lower velocity)



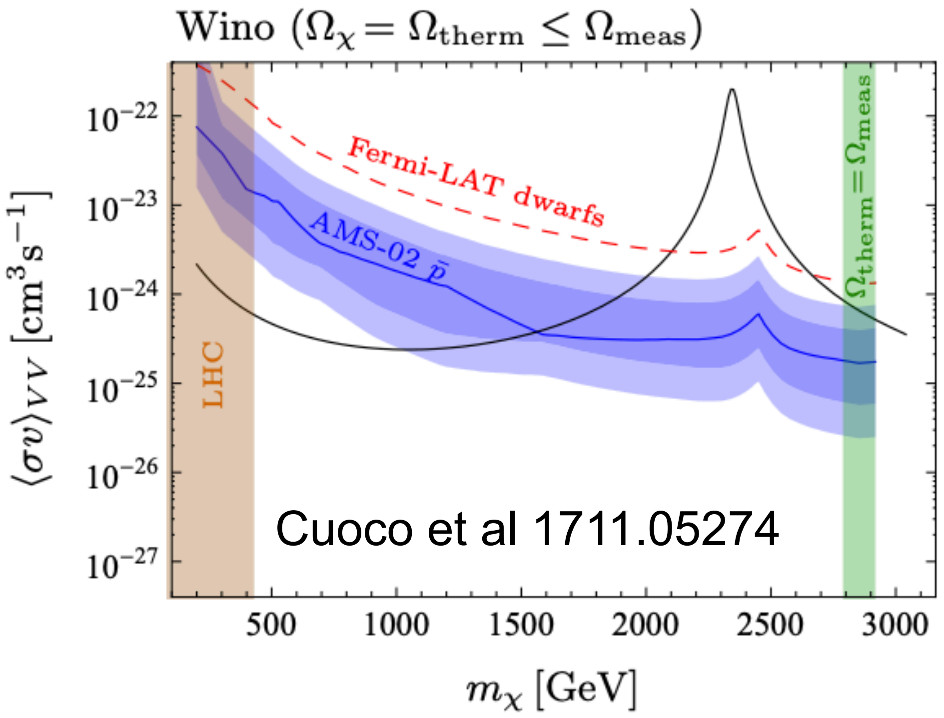
- Asymptotic low-velocity value at resonance consistent with calculation of zero-range effective field theory for wino (ZREFT) from Braaten et al 1712.07142

Phenomenological implications?

- Not relevant for indirect detection of wino - for parameter space where this correction matters, strongly excluded already (by gamma-rays/antiprotons)
- Possibly an issue for higher-mass DM, where cross sections approaching the unitarity bound at freezeout are needed to yield correct relic density
- Also potentially relevant for scenarios with larger coupling - cross section closer to unitarity bound without requiring strong resonant enhancement



Cohen, TRS et al 1307.4082



Cuoco et al 1711.05274

Summary

- The standard calculation of the Sommerfeld enhancement leads to apparent violation of unitarity on resonance peaks

- This is a consequence of solving for the wavefunction deformation by the long-range potential without accounting for probability loss to annihilation

- Multiple methods to unitarize the calculation; we have shown how to modify the non-relativistic QM calculation to account for a hard/short-range annihilation process, viable for multi-state systems and arbitrary partial waves

Backup slides

The meaning of the α factors

- Earlier: "$\alpha_{b,\ell}(0), \alpha_{\tilde{G}_\ell}(a)$ = matrices that can be read off from $w(r), \tilde{w}(r)$ solutions, sensitive to matching radius but not to short-range amplitude"
- What is the intuition for these objects?
- Suppose at each r we separate the wavefunction into a "incoming plane wave" component and a "purely outgoing wave" component (by matching values+1st derivatives)
- $\alpha_{b,\ell}(0)$ tells us about how much the 'incoming plane wave' component grows from r=a to r=0, in the regular solution for the long-range potential (goes to $\sqrt{p}$ when there is no evolution between r=0 and r=a)
- $\alpha_{\tilde{G}_\ell}(a)$ tells us about the size of the 'incoming plane wave' component at r=a in the solution for the long-range potential that is purely outgoing at infinity (goes to zero when long-range potential is negligible)

Bound states and final-state Sommerfeld effects

- This method is a good fit for scenarios where the DM “annihilates” into only a slightly lighter state and so the final state also experiences Sommerfeld enhancement - just use multi-state formalism

- Formation of bound states via radiation of light mediator is also generically present and should contribute to possible final states / inclusive cross section - will suppress annihilation rate further when rates approach unitarity

- Work by [] finds cases where this process appears to violate unitarity - can happen because bound-state formation is treated perturbatively, not included in wavefunction calculation

- Not an obvious fit for this approach because there is not a clear separation of scales (bound-state formation process is not short-range/high-energy); Petraki & Flores ’24 approach using optical theorem may be a better fit