Survey of Lattice QCD

UC Berkeley Phys 290e

André Walker-Loud

Survey of Lattice QCD

- *Why do we use/need lattice QCD?*
- *What is lattice QCD?*
- *Some details of Lattice QFT*
- *Some select results*

some slide material borrowed
from Andrea Shindler, MSU

Survey of Lattice QCD

Why do I use lattice QCD?

- *Understanding Nuclear Physics from QCD*
- *Testing the Standard Model at low-energy in nuclear environments*

Nuclear Physics from QCD

- QCD is The fundamental theory of the strong interactions

$$E^{(i)}_{N,Z,S} = \Lambda_{QCD} \times f^{(i)}_{N,Z,S}\left(\frac{m_u}{\Lambda_{QCD}}, \frac{m_d}{\Lambda_{QCD}}, \frac{m_s}{\Lambda_{QCD}}, \alpha_{f.s.}, \Lambda^2_{QCD} G_F\right)$$

these energy levels range from a few KeV to MeV to many GeV

We would like to understand the spectrum and transitions/reaction rates in nuclear physics directly from QCD

Nuclear Physics from QCD

- *There are well known fine-tunings in nature that have a significant impact on our existence*

$$M_n - M_p, \quad B_d, \quad \text{triple alpha process and } {}^{12}C, \quad \cdots$$

How sensitive are these fine-tunings to variations of fundamental parameters in the Standard Model?

How sensitive is the Universe as we know it to variations in these fundamental parameters?

→ *need a solution to QCD*

Nuclear Physics from QCD

- *What is the weak fusion rate*

  $$p + p \rightarrow d + \nu_e + e^+$$

  as a function of parameters in the Standard Model?

- *What is the composition and equation of state of dense nuclear matter in neutron stars?*

- ...

Nuclear Physics from QCD

- These are examples of understanding QCD to connect interesting nuclear physics to the fundamental theory
- There is another very compelling reason - depending on your taste - you will find more or less compelling (or the same, like me)

Testing the Standard Model at low-energy

- *With the discovery of the Higgs boson, the Standard Model (SM) is now complete*
- *However, the LHC has turned up no hints of any physics beyond the Standard Model (BSM)*
- *Further, there is almost NO terrestrial experimental hints for any physics BSM*

the exceptions: muon anomalous magnetic moment
proton radius puzzle

Testing the Standard Model at low-energy

- *muon anomalous magnetic moment*

  the numerical size of the discrepancy between theory and experiment is the size of a one-loop SM correction

  This makes it difficult to understand this coming from high-energy BSM physics - as there is no room in any other SM comparison for a correction the size of one-loop electro-weak

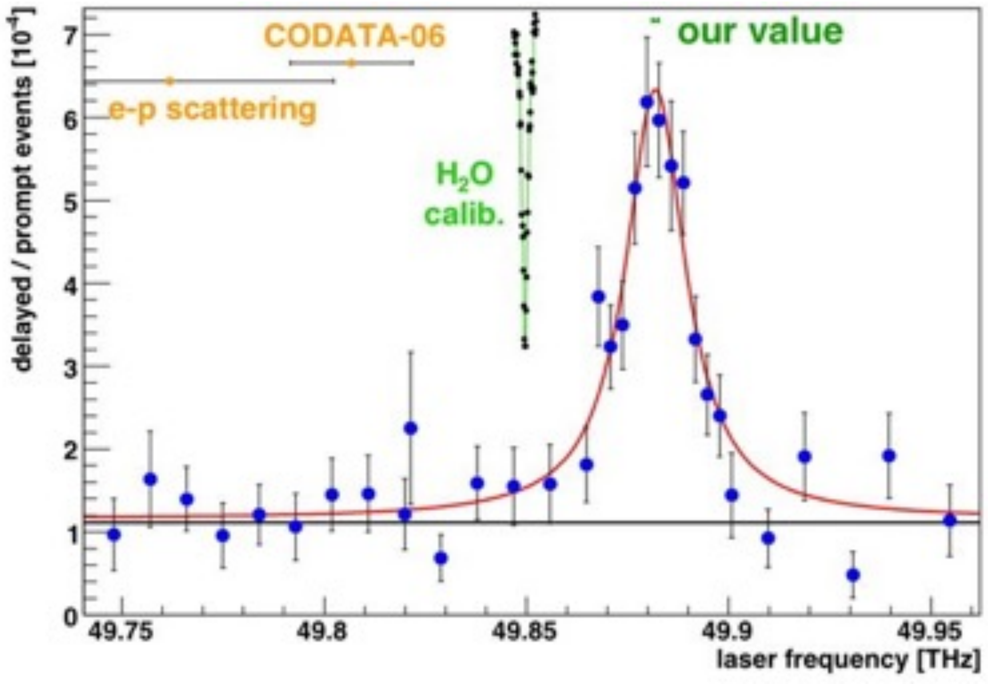
could the BSM physics come from weakly coupled light degrees of freedom?

Testing the Standard Model at low-energy

- *proton radius puzzle*

the discrepancy between the quoted value of the proton charge radius $\langle r_E^2 \rangle \equiv -6 \frac{\partial G_E(Q^2)}{\partial Q^2}\Big|_{Q^2=0}$

measured in muonic-hydrogen and e-p scattering is ~7 sigma!



The determinations of this quantity have been put under extreme scrutiny - while the resolution is still a mystery - it is fair to say many people working on this subject suspect the systematics in e-p are underestimated

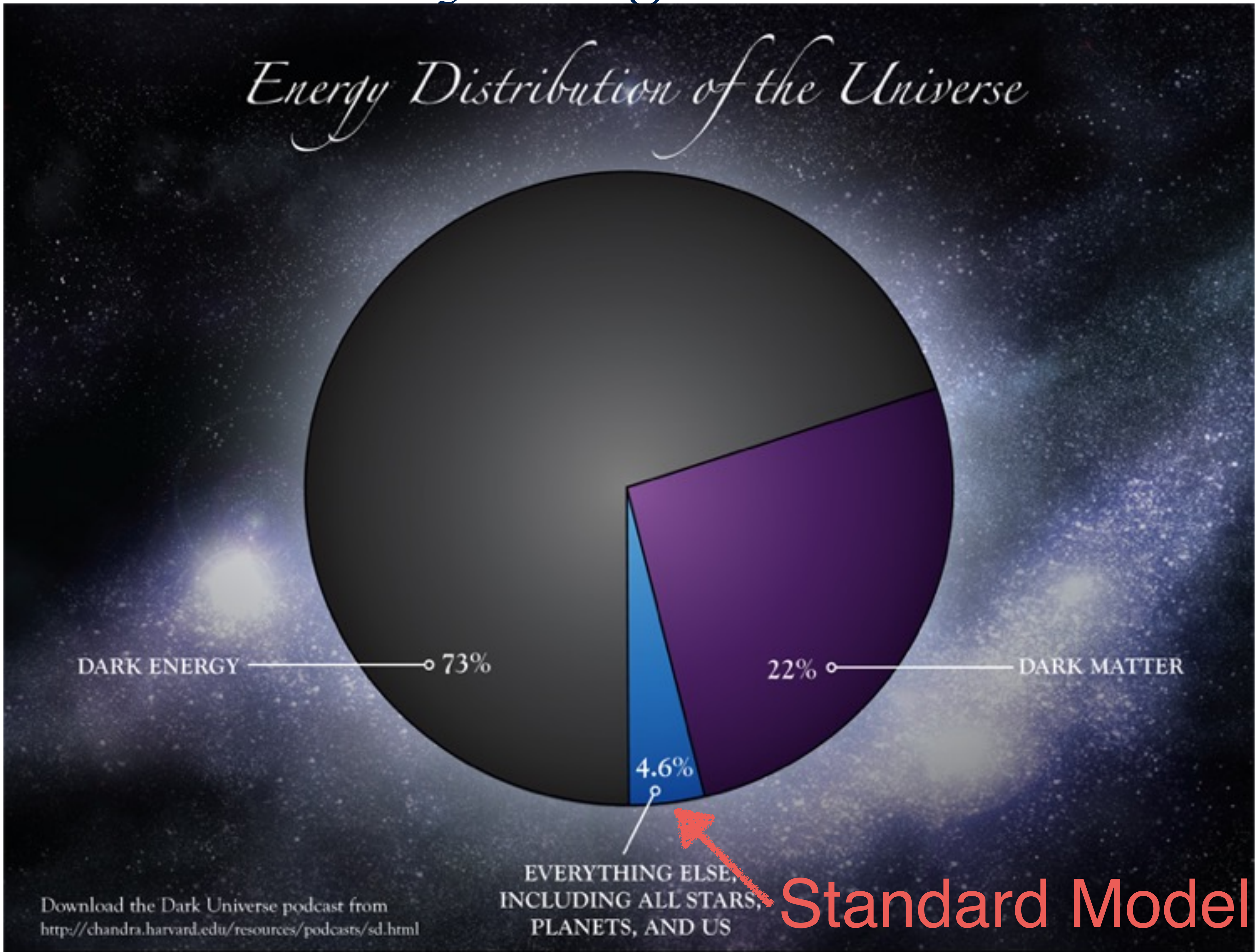
Testing the Standard Model at low-energy

- high-energy physics colliders are one way to search for BSM physics - but it is not clear this will be possible in the near future

  this helps emphasize the important role low-energy precision nuclear physics can play in searching for new physics (in addition to muon g-2 and proton size)

Testing the Standard Model at low-energy

- *While we have no direct confirmation of any BSM physics - we have very strong indirect evidence:*



Testing the Standard Model at low-energy

- To the best of our knowledge, the SM matter in the Universe is comprised entirely of matter and not anti-matter

A measure of the excess matter in the Universe is given by the primordial ratio of the number of baryons to photons - from the CMB, we know this number to be

$$\eta \equiv \frac{X_N}{X_\gamma} \simeq 6.2 \times 10^{-10}$$

However, the SM is nearly symmetric in matter and anti-matter. While this observed asymmetry is small, it is larger than predicted by the SM

Testing the Standard Model at low-energy

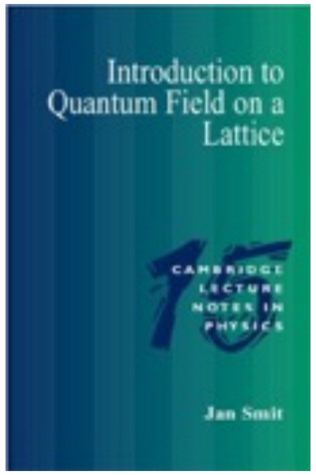
- To produce a matter/anti-matter asymmetry, we need the three Sakharov conditions:
  - baryon number violation
  - C-symmetry and CP-symmetry violation
  - interactions out of thermal equilibrium

  CP violation implies permanent electric dipole moments (EDMs) for SM fermions. There are significant experimental efforts to search for permanent electric dipole moments in electrons, protons, neutrons, deuterium, … ^{199}Hg, ^{225}Ra , …

  In order to relate constraints/measurements on permanent EDMs in nucleons/nuclei to BSM physics,

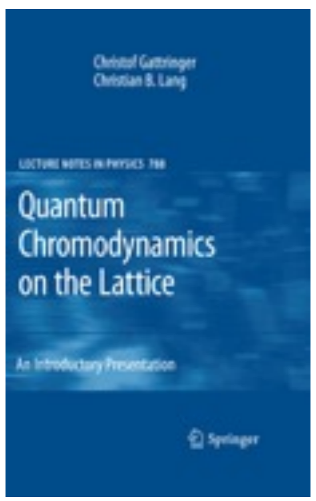
we must be able to solve QCD!

What is lattice QCD?



Introduction to Quantum Fields on a Lattice
Jan Smit
Cambridge Lecture Notes in Physics, 2002

my favorite formal introduction



Quantum Chromodynamics on the Lattice
Christof Gattringer & Christian B. Lang
Springer, 2010

good practical intro to lattice QCD

Lattice QCD for novices
Peter LePage
arxiv.org/abs/hep-lat/0506036

get your hands dirty with your laptop

Advanced Lattice QCD
Martin Lüscher
arxiv.org/abs/hep-lat/9802029

if you want to know more

QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_{f=u,d,s,\ldots} \overline{\psi}_f(x) \left\{ i\gamma^\mu \left[\partial_\mu + igA^a_\mu T^a \right] - m_f \right\} \psi_f(x)$$

$$\psi^A_{f\alpha}(x) \qquad A^{AB}_\mu(x) = \sum_{a=1}^{8} A^a_\mu(x) T^a_{AB} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

quarks | gluons | field-strength tensor

- The only free parameters are the gauge coupling g and the quark masses m_u,m_d,...
- QCD is thus an extremely predicting theory, if only we could solve it...

QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f(x) \left\{ i\gamma^\mu \left[\partial_\mu + igA^a_\mu T^a \right] - m_f \right\} \psi_f(x)$$

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quarks gluons field-strength tensor

- There are two important symmetries that will help understand the strong interactions
  - gauge symmetry of QCD
  - approximate chiral symmetry involving the light quarks

QCD

$$\mathcal{L}^{\psi}_{QCD} = \sum_{f=u,d,s,...} \overline{\psi}_f(x)\left[i\gamma^\mu D_\mu - m_f\right]\psi_f(x)$$

$$= \overline{\psi}_L(x) i\gamma^\mu D_\mu \psi_L(x) + \overline{\psi}_R(x) i\gamma^\mu D_\mu \psi_R(x) + \overline{\psi}_L(x)\, m\, \psi_R(x) + \overline{\psi}_R(x)\, m\, \psi_L(x)$$

(in the second line, I have suppressed the flavor labels, f)

$$\psi_L = \frac{1-\gamma_5}{2}\psi \qquad \psi_R = \frac{1+\gamma_5}{2}\psi$$

- In the limit the quark masses go to zero, m -> 0, QCD would have an exact chiral symmetry as the left and right handed modes would decouple from each other. For the lightest two quark flavors, u & d, QCD is perturbatively close to having this chiral symmetry. This would be an $SU(2)_L \times SU(2)_R$ GLOBAL symmetry.
  - GLOBAL = rotate all u to d and d to u quarks in the universe simultaneously and the physics is invariant

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \psi \to e^{i\frac{\tau_a}{2}\theta_a}\psi = \left[\cos(\theta/2) + i\sin(\theta/2)\hat{\theta}_a\tau_a\right]\psi$$

QCD

$$\mathcal{L}^{\psi}_{QCD} = \sum_{f=u,d,s,\ldots} \overline{\psi}_f(x)\left[i\gamma^\mu D_\mu - m_f\right]\psi_f(x)$$
$$= \overline{\psi}_L(x) i\gamma^\mu D_\mu \psi_L(x) + \overline{\psi}_R(x) i\gamma^\mu D_\mu \psi_R(x) + \overline{\psi}_L(x)\, m\, \psi_R(x) + \overline{\psi}_R(x)\, m\, \psi_L(x)$$

(in the second line, I have suppressed the flavor labels, f)

$$\psi_L = P_L \psi \qquad \psi_R = P_R \psi \qquad P_L = \frac{1-\gamma_5}{2}, P_R = \frac{1+\gamma_5}{2}$$

- If this approximate chiral symmetry were realized in nature, then we would observe a near degeneracy in the spectrum. The negative parity nucleon would have nearly the same mass as the nucleon, with small perturbative corrections due to the finite u,d quark masses, but:

$$m_N \simeq 940 \text{ MeV} \qquad m_{N^*} \simeq 1535 \text{ MeV}$$

- The expected degeneracy arises because the parity operator, which includes γ_4, flips the P_L <-> P_R projectors

QCD

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(in the second line, I have suppressed the flavor labels, f)

$$\psi_L = \frac{1-\gamma_5}{2}\psi \qquad \psi_R = \frac{1+\gamma_5}{2}\psi$$

- We also observe that all hadrons made of u,d quarks have masses >= 770 MeV, except for 3: π^+ π^- π^0

$$m_{\pi^0} \simeq 135 \text{ MeV} \qquad m_{\pi^\pm} \simeq 139 \text{ MeV}$$

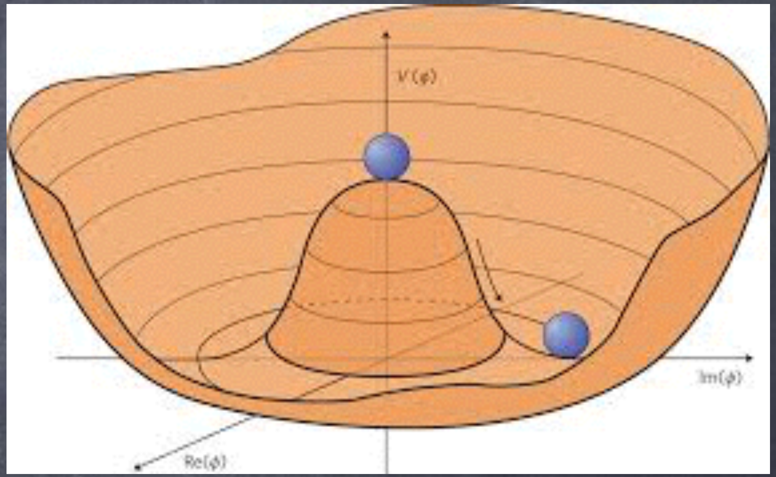
- What are these three light particles doing in the spectrum and why do we not observe a near degeneracy in the parity partners in the spectrum?

QCD

$$\mathcal{L}^{\psi}_{QCD} = \sum_{f=u,d,s,\ldots} \overline{\psi}_f(x)\,[i\gamma^\mu D_\mu - m_f]\,\psi_f(x)$$

$$= \overline{\psi}_L(x) i\gamma^\mu D_\mu \psi_L(x) + \overline{\psi}_R(x) i\gamma^\mu D_\mu \psi_R(x) + \overline{\psi}_L(x)\, m\, \psi_R(x) + \overline{\psi}_R(x)\, m\, \psi_L(x)$$

- This reminds us of spontaneous symmetry breaking. If a global symmetry is spontaneously broken, there must emerge a Nambu-Goldstone mode which is a massless excitation



- In our case, we have an approximate global symmetry. We postulate that the QCD vacuum spontaneously breaks this approximate global chiral symmetry down to the vector subgroup:

$$SU(2)_L \times SU(2)_R \underset{|\Omega_{QCD}\rangle}{\longrightarrow} SU(2)_V$$

- In our two flavor considerations, this SU(2)$_V$ group is the SU(2) of Isospin proposed by Heisenberg

QCD

$$\mathcal{L}^{\psi}_{QCD} = \sum_{f=u,d,s,\ldots} \overline{\psi}_f(x)\,[i\gamma^\mu D_\mu - m_f]\,\psi_f(x)$$

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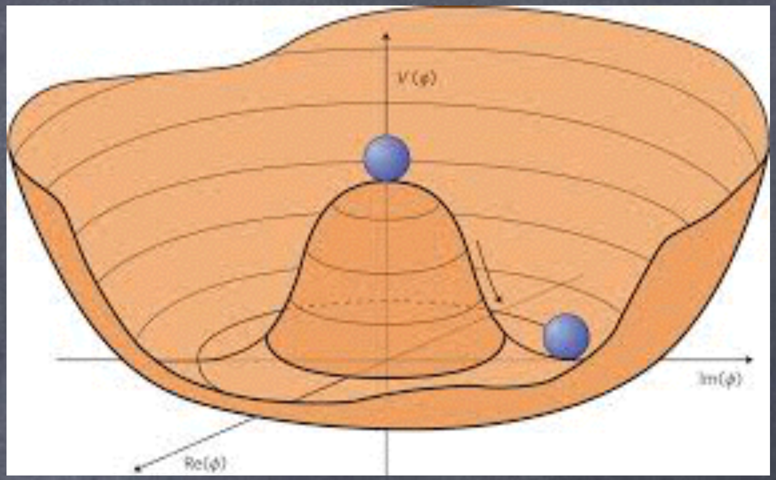
$$SU(2)_L \times SU(2)_R \underset{|\Omega_{QCD}\rangle}{\longrightarrow} SU(2)_V$$



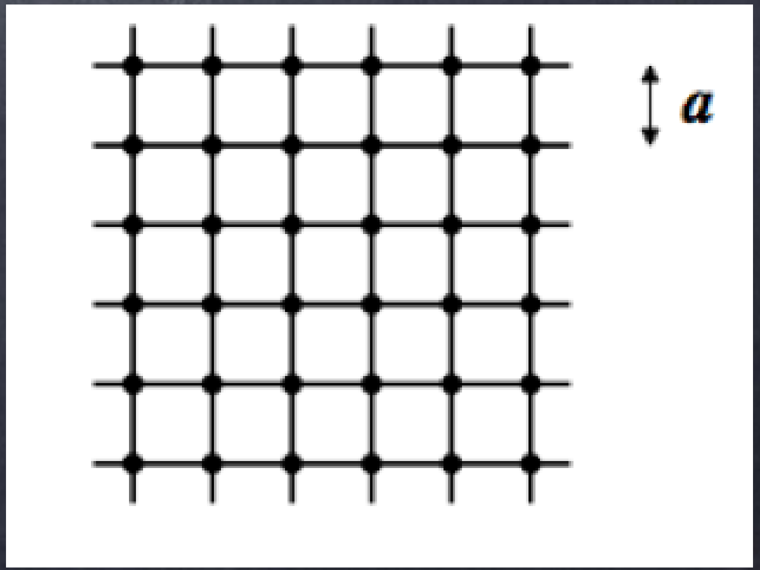
- We began with 3+3 generators of the symmetry (3 Pauli matrices for L and R SU(2)), but end with only 3 generators of the unbroken symmetry, $SU(2)_V$.
- The vector subgroup is parity-even, so we therefore expect to observe three nearly massless parity-odd spin-0 particles in the spectrum (the massless Nambu-Goldstone modes acquire a small mass from the non-zero values of the u,d quark masses) - these are the pions.
- Our understanding of low-energy QCD is heavily based upon this realization of the approximate chiral symmetry.

Feynman Path Integrals

$$\mathcal{Z} = \int DA_\mu D\psi D\overline{\psi} e^{iS_{QCD}} \qquad S_{QCD} = \int d^4x \mathcal{L}_{QCD}$$

$$\langle \Omega | \hat{\mathcal{O}}(y) \hat{\mathcal{O}}^\dagger(x) | \Omega \rangle = \frac{1}{\mathcal{Z}} \int DA_\mu D\psi D\overline{\psi} e^{iS_{QCD}} \mathcal{O}(y)\mathcal{O}^\dagger(x)$$

- The path-integral gives us a relation between matrix elements of operators and a high dimensional integral over field configurations.
- We know how to do the integral on the right (in principle at least). The beginning of lattice QFT is to discretize the universe so that we can compute the path-integral representation directly with a computer.
- Suppose we chop the universe into size $32 \times 32 \times 32 \times 64 = 2^{21}$
- our path integral goes over all field configurations on all sites, $n^{2^{21}}$ terms!



Feynman Path Integrals

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$$\langle \Omega | \hat{\mathcal{O}}(y) \hat{\mathcal{O}}^\dagger(x) | \Omega \rangle = \frac{1}{\mathcal{Z}} \int DA_\mu D\psi D\overline{\psi} e^{iS_{QCD}} \mathcal{O}(y) \mathcal{O}^\dagger(x)$$

- How can we actually perform this integral?
- If we Wick-rotate to Euclidean time, $t \to it_E$, then we have

$$\langle \Omega | \hat{\mathcal{O}}(y_E) \hat{\mathcal{O}}^\dagger(x_E) | \Omega \rangle = \frac{1}{\mathcal{Z}} \int DA_\mu D\psi D\overline{\psi} e^{-S^E_{QCD}} \mathcal{O}(y_E) \mathcal{O}^\dagger(x_E)$$

- For zero quark chemical-potential (zero baryon chemical potential)

$$e^{-S^E_{QCD}} \in \mathbb{R}$$

- We can use this factor as a probability measure to importance sample the integral with Monte-Carlo methods for those field configurations that minimize S^E_{QCD}

Feynman Path Integrals

$$\langle\Omega|\hat{\mathcal{O}}(y_E)\hat{\mathcal{O}}^\dagger(x_E)|\Omega\rangle = \frac{1}{\mathcal{Z}}\int DA_\mu D\psi D\overline{\psi} e^{-S^E_{QCD}}\mathcal{O}(y_E)\mathcal{O}^\dagger(x_E)$$

- We can make N_{cfg} different samples of the field configurations and then our correlation functions are approximated with finite statistics

$$\langle\Omega|\hat{\mathcal{O}}(y_E)\hat{\mathcal{O}}^\dagger(x_E)|\Omega\rangle = \lim_{N_{cfg}\to\infty}\frac{1}{N_{cfg}}\sum_{i=1}^{N_{cfg}}\langle\Omega|\hat{\mathcal{O}}(y_E)[A^i_\mu,\psi_i,\overline{\psi}_i]\hat{\mathcal{O}}^\dagger(x_E)[A^i_\mu,\psi_i,\overline{\psi}_i]|\Omega\rangle$$

$[A^i_\mu,\psi_i,\overline{\psi}_i]$ = the i^{th} value of the fields on "configuration" i

- At finite statistics (finite N_{cfg}) we will have an approximation to the correlation functions with some computable statistical uncertainty that can be systematically improved (with more computing time)

Feynman Path Integrals

$$\langle\Omega|\hat{\mathcal{O}}(y_E)\hat{\mathcal{O}}^\dagger(x_E)|\Omega\rangle = \frac{1}{\mathcal{Z}}\int DA_\mu D\psi D\overline{\psi} e^{-S^E_{QCD}}\mathcal{O}(y_E)\mathcal{O}^\dagger(x_E)$$

- What do we expect our Euclidean spacetime correlation functions to look like? Let us take x_E=0 (without loss of generality - translation invariance lets us do this) and $\vec{y}_E = 0$ for simplicity

$$C(t) = \langle\Omega|\hat{\mathcal{O}}(t,\vec{0})\hat{\mathcal{O}}^\dagger(0,\vec{0})|\Omega\rangle$$

- Insert a complete set of states

$$1 = \sum_n |n\rangle\langle n|$$

$$C(t) = \sum_n \langle\Omega|\hat{\mathcal{O}}(t)|n\rangle\langle n|\hat{\mathcal{O}}^\dagger(0)|\Omega\rangle$$

$$= \sum_n \langle\Omega|e^{\hat{H}t}\hat{\mathcal{O}}(0)e^{-\hat{H}t}|n\rangle\langle n|\hat{\mathcal{O}}^\dagger(0)|\Omega\rangle$$

$$= \sum_n Z_n Z_n^\dagger e^{-E_n t}$$

$$Z_n = \langle\Omega|\hat{\mathcal{O}}(0)|n\rangle$$

Feynman Path Integrals

$$\langle\Omega|\hat{\mathcal{O}}(y_E)\hat{\mathcal{O}}^\dagger(x_E)|\Omega\rangle = \frac{1}{\mathcal{Z}}\int DA_\mu D\psi D\overline{\psi} e^{-S^E_{QCD}}\mathcal{O}(y_E)\mathcal{O}^\dagger(x_E)$$

$$C(t) = \langle\Omega|\hat{\mathcal{O}}(t,\vec{0})\hat{\mathcal{O}}^\dagger(0,\vec{0})|\Omega\rangle = \sum_n Z_n Z_n^\dagger e^{-E_n t}$$

$$= A_0 e^{-E_0 t}\left[1 + \sum_{n>0}\frac{A_n}{A_0}e^{-\Delta_{n0}t}\right]$$

$$A_n \equiv Z_n Z_n^\dagger \qquad \Delta_{n0} \equiv E_n - E_0$$

- In the long Euclidean time limit, the excited tower of states becomes exponentially suppressed compared with the ground state, E_0. For simple quantities like the spectrum, we do not need to worry about calculating in Euclidean time rather than Minkowski time.

Feynman Path Integrals

$$C(t) = \langle \Omega | \hat{\mathcal{O}}(t,\vec{0}) \hat{\mathcal{O}}^\dagger(0,\vec{0}) | \Omega \rangle = \sum_n Z_n Z_n^\dagger e^{-E_n t}$$

$$= A_0 e^{-E_0 t}\left[1 + \sum_{n>0} \frac{A_n}{A_0} e^{-\Delta_{n0} t}\right]$$

- We use a derived quantity, the effective mass, to help understand our numerical calculations:

$$m_{eff}(t,\tau) = \frac{1}{\tau} \ln\left(\frac{C(t)}{C(t+\tau)}\right)$$

$$= \frac{1}{\tau} \ln\left(\frac{A_0 e^{-E_0 t}(1 + A_1/A_0 e^{-\Delta_{10} t} + \cdots}{A_0 e^{-E_0 (t+\tau)}(1 + A1/A0 e^{-\Delta_{10}(t+\tau)} + \ldots}\right)$$

$$= E_0 + \frac{1}{\tau} \ln\left(\frac{1 + A_1/A_0 e^{-\Delta_{10} t} + \cdots}{1 + A_1/A_0 e^{-\Delta_{10}(t+\tau)} + \ldots}\right)$$

Quark fields

$$[\not{\partial} + m]\, \underbrace{\psi(x)\overline{\psi}}(0) = \delta^{(4)}(x) \qquad \underbrace{\psi(x)\overline{\psi}}(0) \equiv S(x,0)$$

Euclidean free-quark two-point function

$$\underbrace{\psi(x)\overline{\psi}}(0) = \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{e}^{ipx}}{i\,\not{p} + m}$$

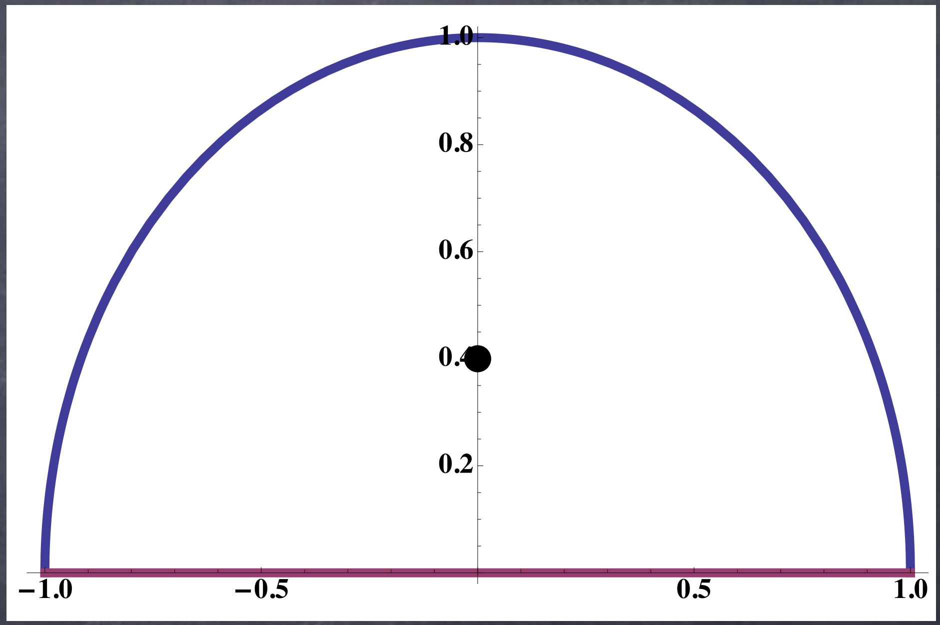
$$\gamma_\mu^\dagger = \gamma_\mu \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

$$p \cdot x = p_0 x_0 + p_k x_k \qquad \not{p} = \gamma_0 p_0 + \gamma_k p_k$$

Quark correlator

$$\int d^3x\; S(x,0) = \int d^3x \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi}\; \frac{-i\, \not{p} + m}{p^2 + m^2} \mathrm{e}^{ipx}$$

$$x_0 > 0$$

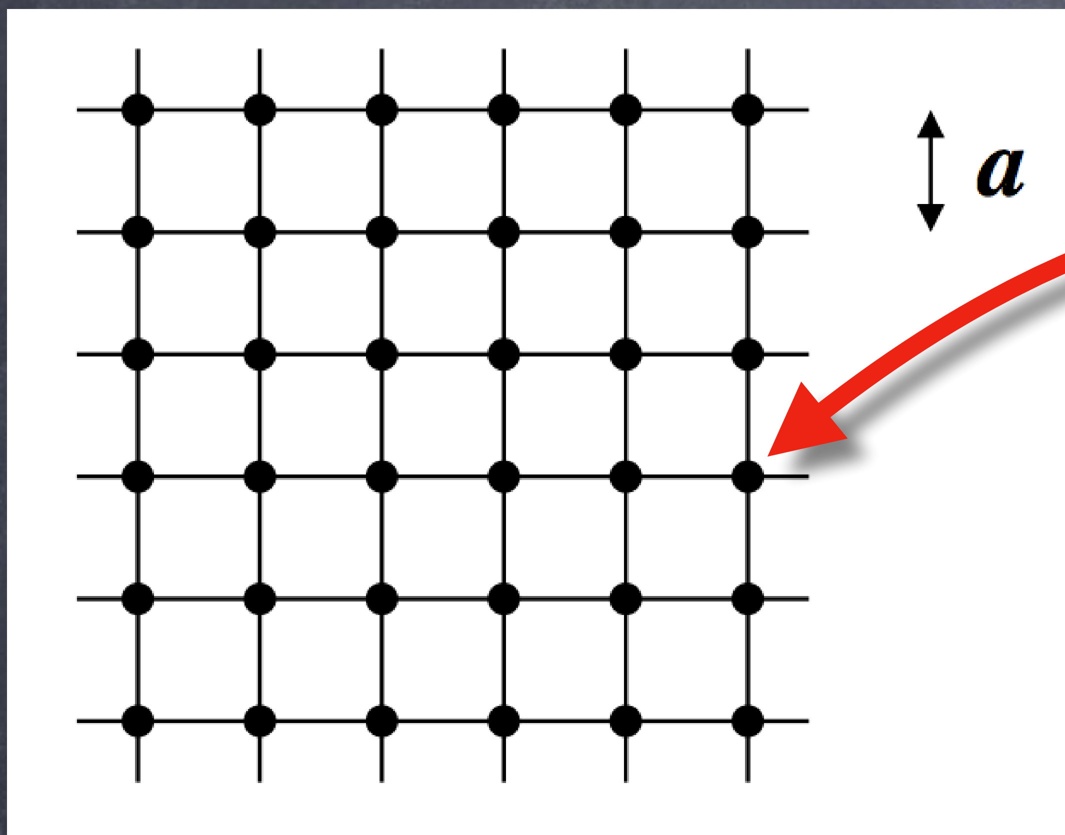


$$\int d^3x\; S(x,0) = P_+ \mathrm{e}^{-m x_0}$$

$$P_+ = \frac{1}{2}(1 + \gamma_0)$$

Lattice Path Integrals

- Now we need to construct discrete versions of our fields.



$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{2\pi^4} \mathrm{e}^{ipx} \tilde{\psi}(p)$$

Cutoff $|p_\mu| \leq \dfrac{\pi}{a}$

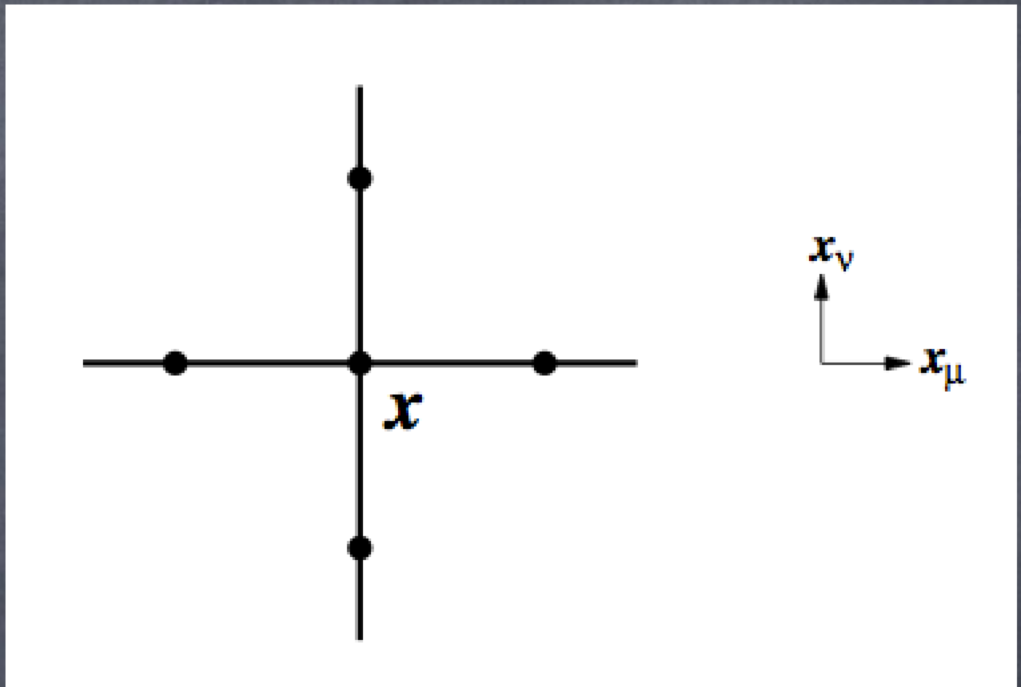
$$x = a(n_0, n_1, n_2, n_3) \qquad n_\mu \in \mathbb{Z}$$

$$\widetilde{\psi}(p) = a^4 \sum_x \mathrm{e}^{-ipx} \psi(x)$$

Derivatives

$$\partial_\mu \psi(x) = \frac{1}{a}\left[\psi(x+a\hat{\mu}) - \psi(x)\right]$$

$$\partial^*_\mu \psi(x) = \frac{1}{a}\left[\psi(x) - \psi(x-a\hat{\mu})\right]$$



$$\partial_\mu \longrightarrow \frac{1}{a}\left[\mathrm{e}^{iap_\mu} - 1\right] = ip_\mu\left[1 + \mathrm{O}(ap)\right]$$

$$\frac{1}{2}\left(\partial^*_\mu + \partial_\mu\right) \longrightarrow \frac{i}{a}\sin\left(ap_\mu\right) \equiv i\mathring{p}_\mu \qquad \mathring{p}_\mu = p_\mu + \mathrm{O}(a^2)$$

$$\partial^*_\mu \partial_\mu \longrightarrow -\hat{p}_\mu \hat{p}_\mu \qquad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

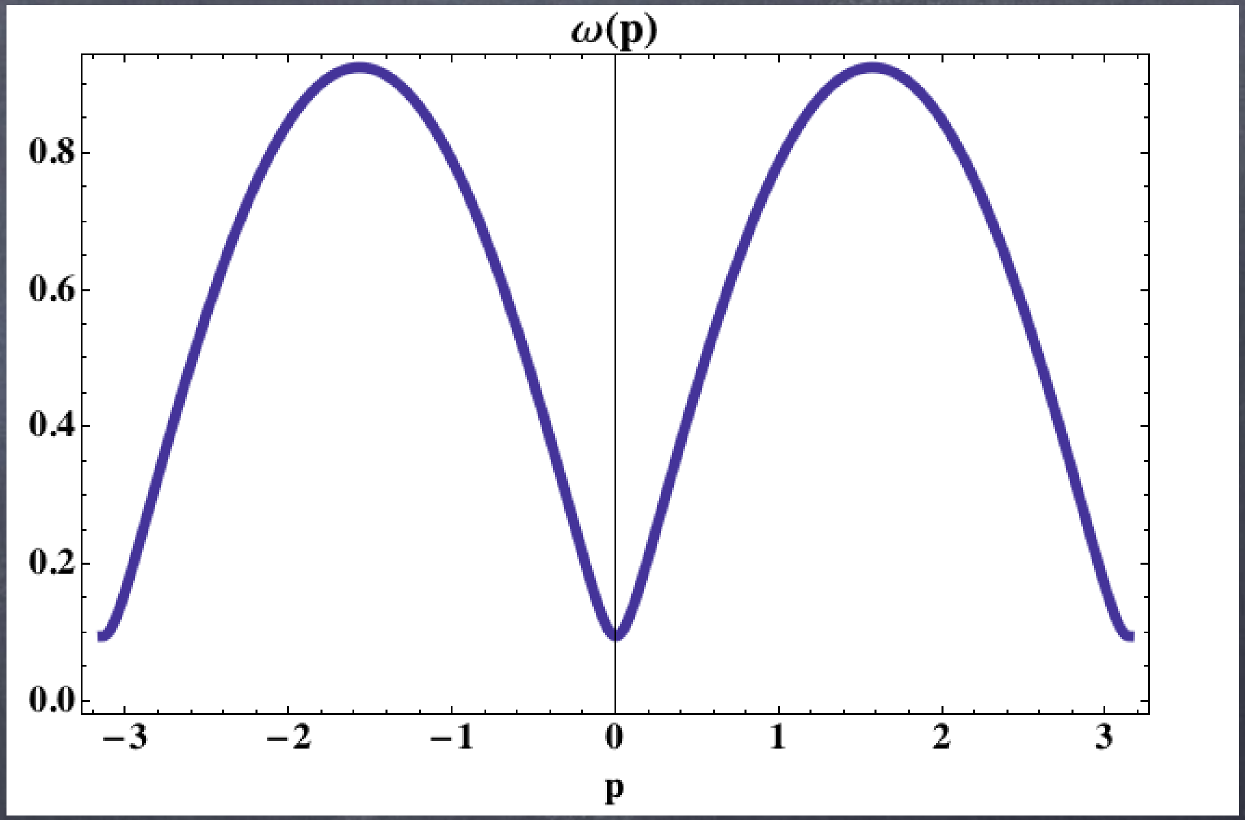
Doublers

$$\frac{1}{i\ \mathring{\not p} + m} = \frac{-i\ \mathring{\not p} + m}{\mathring{p}^2 + m^2}$$

$$= \frac{-i\mathring{\not p} + m}{\frac{\sin^2(ap_\mu)}{a^2} + m^2}$$

$$\omega(\underline{p}) = \frac{2}{a}\text{asinh}\left\{\frac{a}{2}\sqrt{\frac{\mathring{\underline{p}}^2 + m^2}{1 + am}}\right\}$$

Poles $p_0 = \pm i\omega(\underline{p})$



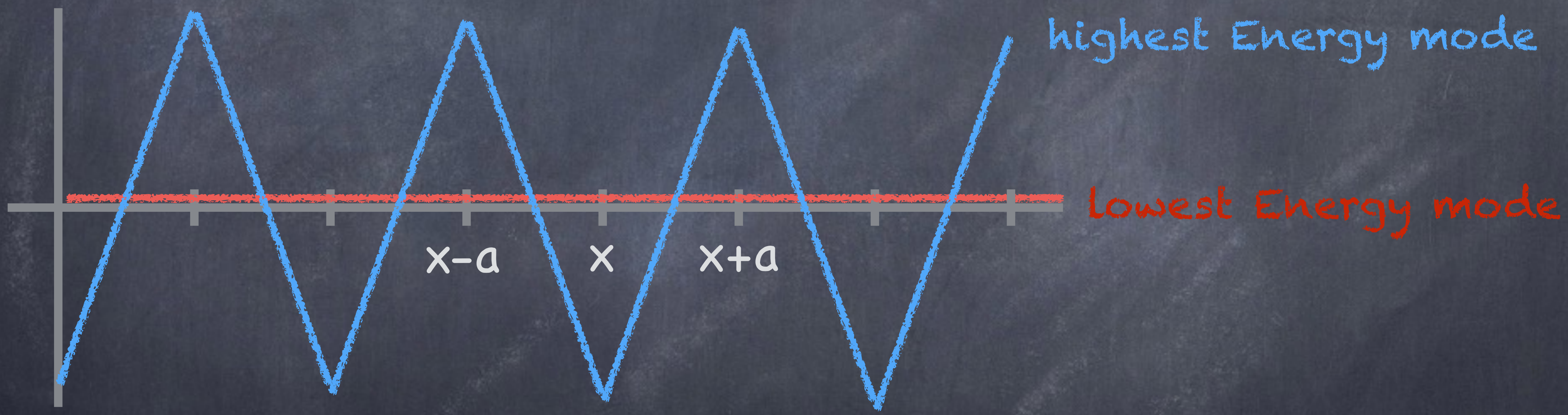
Additional states with energy = mass!

- each naive fermion we add really has 2^d states in d dimensions

Doublers

- Why do we get these doublers?
- It is because of the Dirac equation having only a single derivative for fermions:

$$\nabla_\mu = \frac{1}{2}(\partial^*_\mu + \partial_\mu) \qquad \nabla_\mu \psi = \frac{\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})}{2a}$$



- Our difference operator can not distinguish between the lowest and highest energy modes allowed. This does not happen for bosons

$$\partial^*_\mu \partial_\mu \phi = \frac{\phi(x + a\hat{\mu}) + \phi(x - a\hat{\mu}) - 2\phi(x)}{a^2}$$

Wilson–Dirac operator

$$D_W = \sum_{\mu=0}^{3} \frac{1}{2}\left[\gamma_\mu\left(\partial_\mu^* + \partial_\mu\right) - \underline{a\partial_\mu^*\partial_\mu}\right] \longrightarrow i\,\tilde{\not{p}} + \underline{\frac{1}{2}a\hat{p}^2}$$

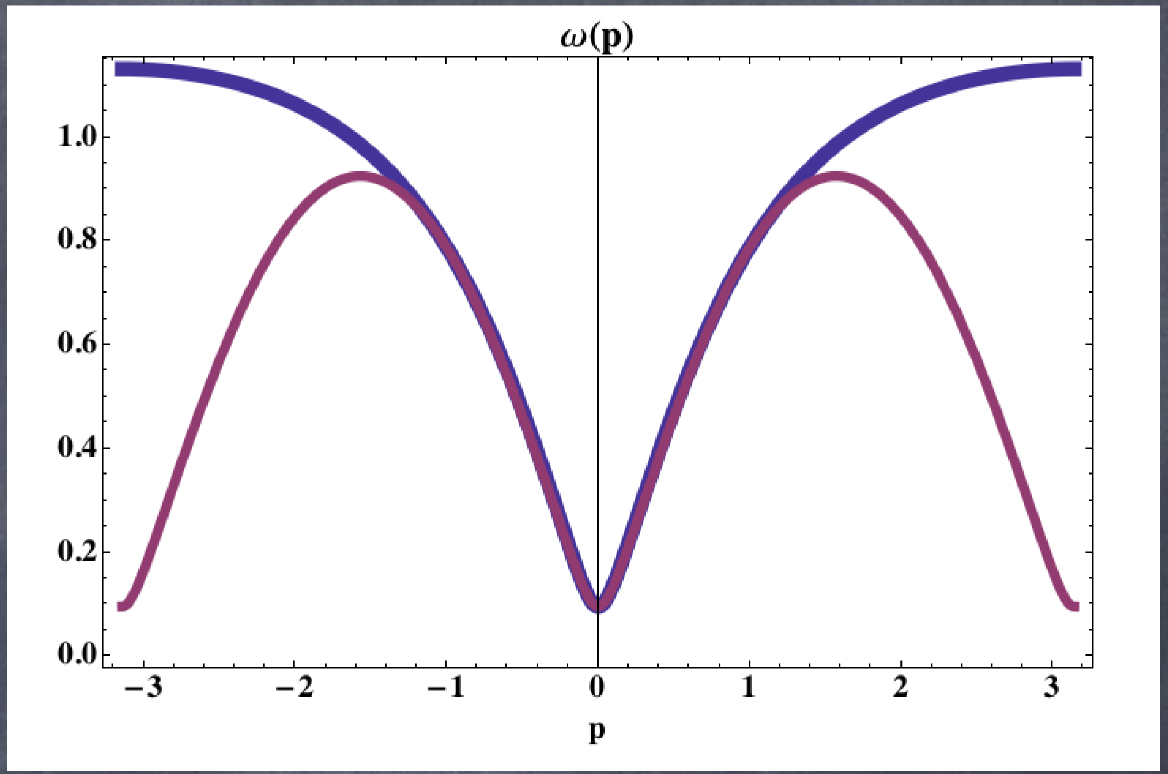
Free-quark two-point functions

$$(D_W + m)\,\underbrace{\psi(x)\overline{\psi}}(0) = \frac{1}{a^4}\delta_{x,0}$$

$$\underbrace{\psi(x)\overline{\psi}}(0) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\,\frac{\mathrm{e}^{ipx}}{i\,\tilde{\not{p}} + \frac{1}{2}a\hat{p}^2 + m}$$

Energy

$$\frac{1}{i\,\mathring{\not p} + \frac{1}{2}a\hat{p}^2 + m} = \frac{-i\,\mathring{\not p} + M(p)}{\mathring{p}^2 + M(p)^2}$$



Poles $p_0 = \pm i\omega(\underline{p})$

$$\omega(\underline{p}) = \frac{2}{a}\mathrm{asinh}\left\{\frac{a}{2}\sqrt{\frac{\underline{\mathring{p}}^2 + M(\underline{p})^2}{1 + aM(\underline{p})}}\right\}$$

$$M(\underline{p}) \equiv m + \frac{1}{2}a\underline{\hat{p}}^2$$

Wilson term

$$D_W = \sum_{\mu=0}^{3} \frac{1}{2}\left[\gamma_\mu\left(\partial^*_\mu + \partial_\mu\right) - a\partial^*_\mu\partial_\mu\right]$$

- The Wilson term is irrelevant in the continuum limit

$a\overline{\psi}\partial^*_\mu\partial_\mu\psi$

  - dimension 5 operator, so coefficient must have dimension -1 to include ([a] = -1) in the Lagrangian
  - irrelevant = vanishes in continuum limit

- The Wilson term breaks chiral symmetry!

$$\overline{\psi}\gamma_\mu\psi = \overline{\psi_L}\gamma_\mu\psi_L + \overline{\psi_R}\gamma_\mu\psi_R \qquad \overline{\psi}\partial^*_\mu\partial_\mu\psi = \overline{\psi_L}\partial^*_\mu\partial_\mu\psi_R + \overline{\psi_R}\partial^*_\mu\partial_\mu\psi_L$$

- The Wilson operator will mix non-perturbatively with the quark mass operator

$$\overline{\psi}m\psi = \overline{\psi_L}m\psi_R + \overline{\psi_R}m\psi_L$$

- The input quark mass receives LARGE additive correction from non-perturbative effects from Wilson operator - FINE TUNING BY LQCD practitioner to get light physical quark masses

Chiral Symmetry on the lattice

- Constructing a lattice action that respects chiral symmetry is challenging (1-2 orders of magnitude more expensive)
    - define lattice-chiral symmetry: Ginsparg Wilson relation
    - Domain-Wall Fermions
    - Overlap Fermions

Gauge fields

Gauge transformations

$$\psi(x) \longrightarrow \Lambda(x)\psi(x) \qquad \Lambda(x) \in SU(3)$$

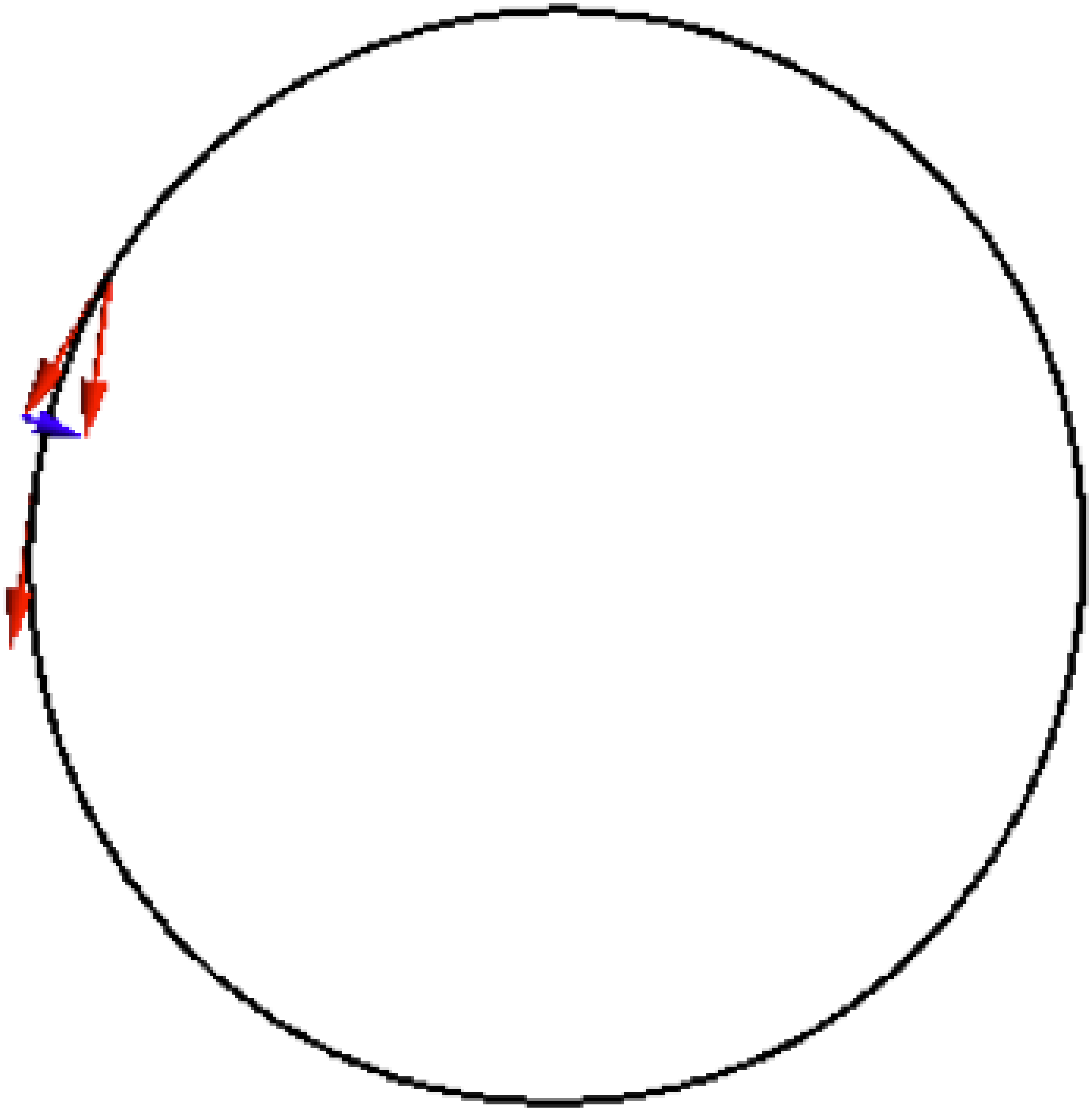
Covariant derivative in the continuum

$$D_\mu \psi(x) = \left(\partial_\mu + A_\mu(x)\right)\psi(x)$$

$$A_\mu(x) \longrightarrow \Lambda(x) A_\mu(x) \Lambda(x)^{-1} + \Lambda(x)\partial_\mu \Lambda(x)^{-1}$$

$$\Downarrow$$

$$D_\mu \psi(x) \longrightarrow \Lambda(x)\left[D_\mu \psi(x)\right]$$



Lattice covariant derivatives

$$\partial_\mu \psi(x) = \frac{1}{a}\left[\psi(x+a\hat{\mu}) - \psi(x)\right]$$

$$\longrightarrow \frac{1}{a}\left[\Lambda(x+a\hat{\mu})\psi(x+a\hat{\mu}) - \Lambda(x)\psi(x)\right]$$

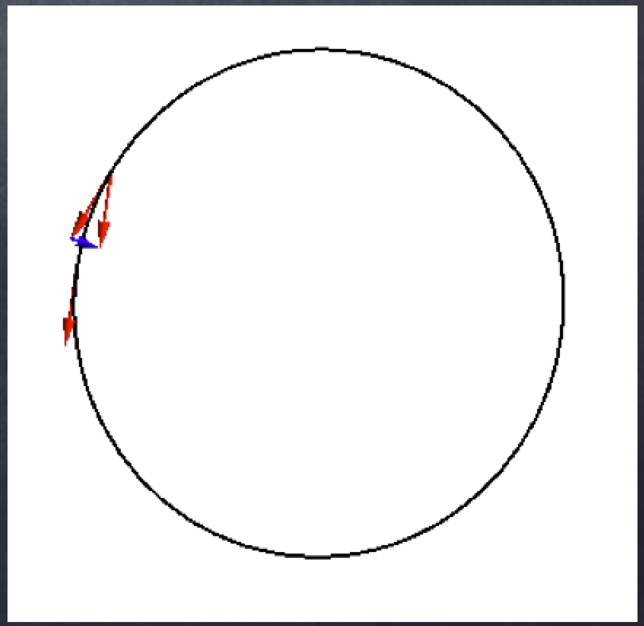
Need gauge connection

$$U(x,\mu) \in SU(3) \qquad U(x,\mu) \longrightarrow \Lambda(x)U(x,\mu)\Lambda(x+a\hat{\mu})^{-1}$$

Covariant derivatives

$$\nabla_\mu \psi(x) = \frac{1}{a}\left[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right]$$

$$\nabla_\mu \psi(x) \longrightarrow \Lambda(x)\nabla_\mu \psi(x)$$



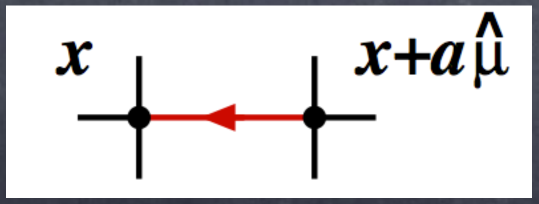
$$\nabla_\mu^* \psi(x) = \frac{1}{a}\left[\psi(x) - U(x-a\hat{\mu},\mu)^{-1}\psi(x-a\hat{\mu})\right]$$

=> gauge covariant Wilson-Dirac operator

$$D_W = \sum_{\mu=0}^{3} \frac{1}{2}\left[\gamma_\mu\left(\nabla_\mu^* + \nabla_\mu\right) - a\nabla_\mu^*\nabla_\mu\right]$$

An SU(3) lattice gauge field is an assignment of a matrix

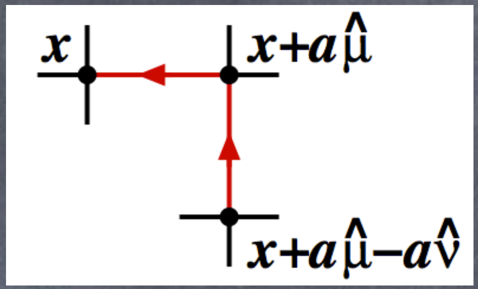
$$U(x,\mu) \in SU(3)$$



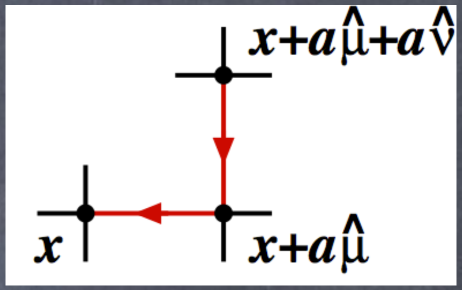
to every link $(x, x+a\hat{\mu})$ on the lattice

Wilson lines

$$U(x,\mu)U(x+a\hat{\mu}-a\hat{\nu},\nu)^{-1}$$

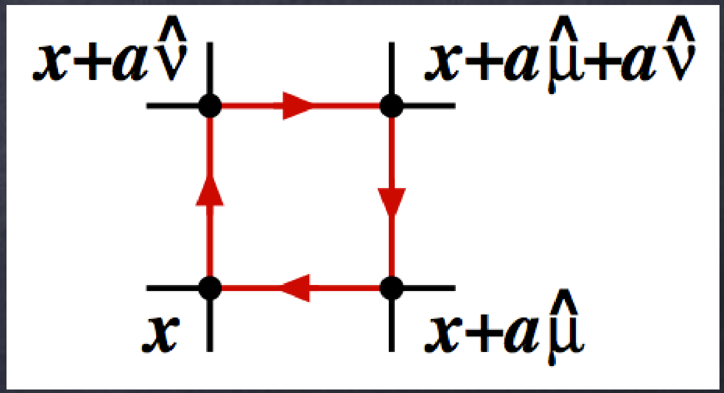


$$U(x,\mu)U(x+a\hat{\mu},\nu)$$



Plaquette

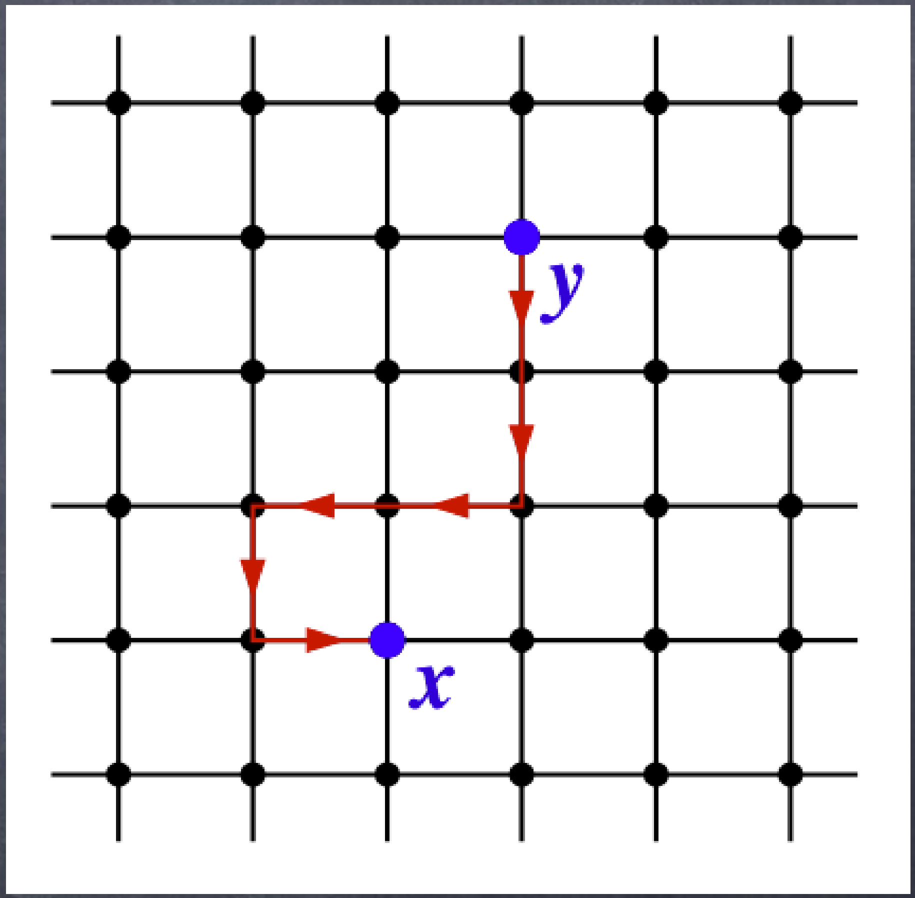
$$U(x,\mu)U(x+a\hat{\mu},\nu)U(x+a\hat{\nu},\mu)^{-1}U(a,\nu)^{-1}$$



Wilson lines

$U(x, y; \mathcal{C})$ Ordered products of U's

$$U(x, y; \mathcal{C}) \longrightarrow \Lambda(x) U(x, y; \mathcal{C}) \Lambda(y)^{-1}$$



For any closed curve the Wilson loop

$$W(\mathcal{C}) = \mathrm{tr}\left[U(x, x; \mathcal{C})\right]$$

is gauge invariant and independent of x

Lattice and continuum gauge fields

How do we approximate a continuum gauge field by a lattice gauge field?

$$U(x, y; \mathcal{C}) \longrightarrow \Lambda(x) U(x, y; \mathcal{C}) \Lambda(y)^{-1}$$

In the continuum the "gauge transporter"

$$G(x, x + a\hat{\mu}) = \mathcal{T} \exp \left\{ a \int_0^1 d\tau A_\mu \left(x + (1 - \tau) a\hat{\mu} \right) \right\}$$

$$G(x, x + a\hat{\mu}) \rightarrow \Lambda(x) G(x, x + a\hat{\mu}) \Lambda^{-1}(x + a\hat{\mu})$$

Lattice gauge field = gauge transporter

Lattice and continuum gauge fields

$$U(x, x + a\hat{\mu}) = G(x, x + a\hat{\mu}) + \mathrm{O}(a)$$

Introduce algebra-valued gauge field

$$U(x, \mu) = \exp\{aA_\mu\} \simeq 1 + aA_\mu(x) + \mathrm{O}(a^2)$$

==>

$$\nabla_\mu \psi(x) = \frac{1}{a}\left[(1 + aA_\mu(x))\psi(x + 1\hat{\mu}) - \psi(x)\right] + \mathrm{O}(a) =$$

$$\left(\partial_\mu + A_\mu(x)\right)\psi(x) + \mathrm{O}(a)$$

Gauge invariant local fields

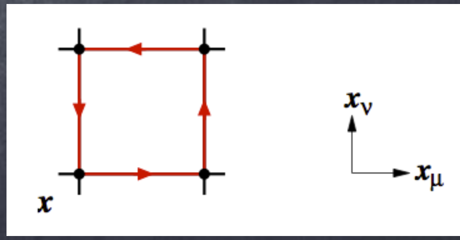
Quark bilinears

$$\overline{\psi}(x)\psi(x) \quad \overline{\psi}(x)\gamma_5\tau^a\psi(x) \quad \overline{\psi}(x)\gamma_\mu\psi(x)$$

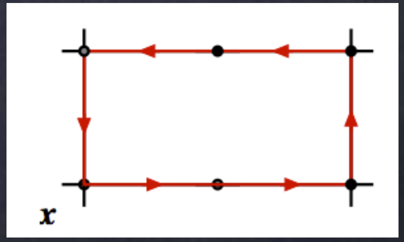
$$\overline{\psi}(x)\gamma_\mu\nabla_\nu\psi(x) \quad \overline{\psi}(x)\nabla_\mu\nabla_\nu\psi(x) \quad \overline{\psi}(x)U(x,\mu)\psi(x+a\hat{\mu})$$

Plaquette and rectangle fields

$$P_{\mu\nu}(x) = \mathrm{Re}\ \mathrm{tr}\,[1 - U(x,x;\square)]$$



$$R_{\mu\nu} = \mathrm{Re}\ \mathrm{tr}\,\{1 - U(x,x;\boxed{})\}$$



Classical continuum limit

$$\mathcal{O}(x) \underset{a\to 0}{\sim} \sum_{n\geq 0} a^n \mathcal{O}_n(x)$$

$\mathcal{O}_n(x)$ Gauge invariant polynomial of $\psi(x),\ \overline{\psi}(x),\ A_\mu(x)$ and their derivatives of dim=n

Examples

$$U(x,x;\Box) = -\frac{1}{2}a^4 \mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] - \frac{1}{2}a^5 \mathrm{tr}\left[F_{\mu\nu}(x)\left(D_\mu + D_\nu\right)F_{\mu\nu}(x)\right] + \cdots$$

$$R_{\mu\nu}(x) = -2a^4 \mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \cdots$$

- Lattice fields can be classified by their leading behavior in the classical continuum limit
- Any gauge-invariant, local continuum field can be represented on the lattice
- The representation is not unique <==> many lattice representations for a local continuum field

Lattice QCD action

Wilson 1974

$$S = S_G + S_F$$

$$S_G = \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu} P_{\mu\nu}(x)$$

$$S_F = a^4 \sum_x \overline{\psi}(x) \left[D_W + M \right] \psi(x)$$

$$D_W = \frac{1}{2} \left[\gamma_\mu \left(\nabla_\mu + \nabla_\mu^* \right) - a \nabla_\mu^* \nabla_\mu \right]$$

Other lattice actions

$$S_G = \frac{1}{g_0^2} \sum_x \sum_{\mu\nu} [c_0 P_{\mu\nu}(x) + c_1 R_{\mu\nu}(x)] \qquad c_0 + 4c_1 = 1$$

- The differences are of order a^p in the classical continuum limit
- Additional terms can be added and the coefficients tuned to improve the convergence to the continuum limit

Integrating the fermion fields

$$\mathcal{Z}_F = \int \mathcal{D}[\psi]\,\mathcal{D}[\overline{\psi}]\,\exp\{-S_W[U,\overline{\psi},\psi]\} \qquad S_W = a^4 \sum_x \overline{\psi}(x)\,[D_W + M]\,\psi(x)$$

This integral is quadratic in the fermions, so we can directly do the integral

$$= \det[D_W + M] = \prod_{q=1}^{N_f} \det[D_W + m_q]$$

$$\mathcal{Z}_{QCD} = \int DU_\mu \det[D_W + M]\, e^{-S_G[U_\mu]}$$

Quark contractions

$$[D_W + M]\, S(x,y;U) = \frac{1}{a^4}\delta_{xy}$$

Quark propagator
and
correlation function

$$\left\langle \psi(x)\overline{\psi}(y)\right\rangle_F = S(x,y;U)$$

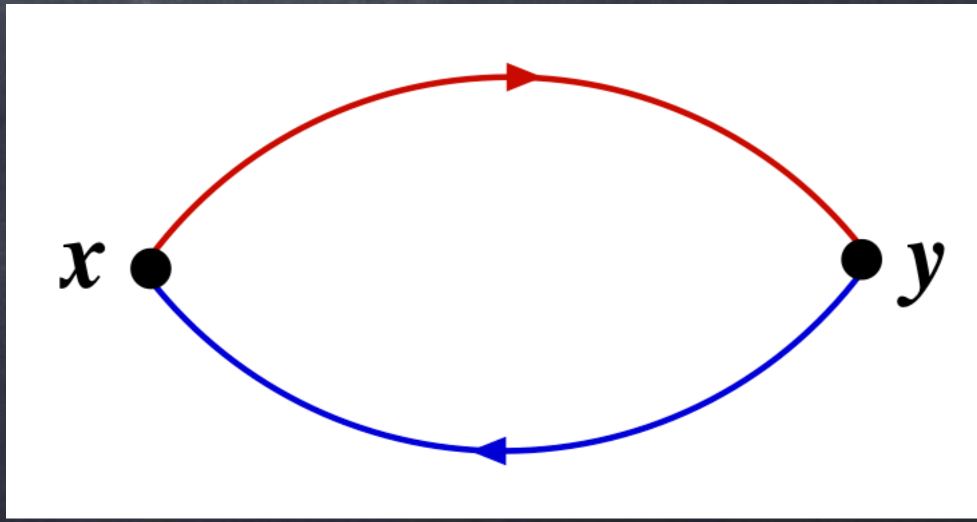
$$\left\langle \psi(x_1)\overline{\psi}(y_2)\psi(x_2)\overline{\psi}(y_2)\right\rangle_F = S(x_1,y_1;U)S(x_2,y_2;U) - \ldots$$

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\,[U]\,\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle_F \prod_{q=1}^{N_f} \det\,[D_W + m_q] \exp\{-S_G[U]\}$$

Pion correlation function

$$\left\langle \overline{u}(x)\gamma_5 d(x)\overline{d}(y)\gamma_5 u(y)\right\rangle_F = -\mathrm{tr}\,\{\gamma_5 S_{dd}(x,y;U)\gamma_5 S_{uu}(y,x;U)\}$$

==> now only bosonic integral



Regularity

In a finite volume

- The space of gauge fields is compact
- After fermion fields are integrated out one is normally left with a bosonic integral

==> the correlation functions are well defined
==> lattice QCD provides a non-perturbative regularization of QCD

Gauge invariance

$$\langle \mathcal{O} \rangle = \left\langle \mathcal{O}^{\Lambda} \right\rangle$$

$$\mathcal{O}^{\Lambda}\left[U, \overline{\psi}, \psi\right] = \mathcal{O}\left[U^{\Lambda}, \overline{\psi}^{\Lambda}, \psi^{\Lambda}\right]$$

$$U^{\Lambda}(x,\mu) = \Lambda(x) U(x,\mu) \Lambda(x + a\hat{\mu})^{-1}, \ldots$$

Example

$$\left\langle \psi(x)\overline{\psi}(y) \right\rangle = \Lambda(x) \left\langle \psi(x)\overline{\psi}(y) \right\rangle \Lambda(y)^{-1}$$

$$= 0 \qquad x \neq y$$

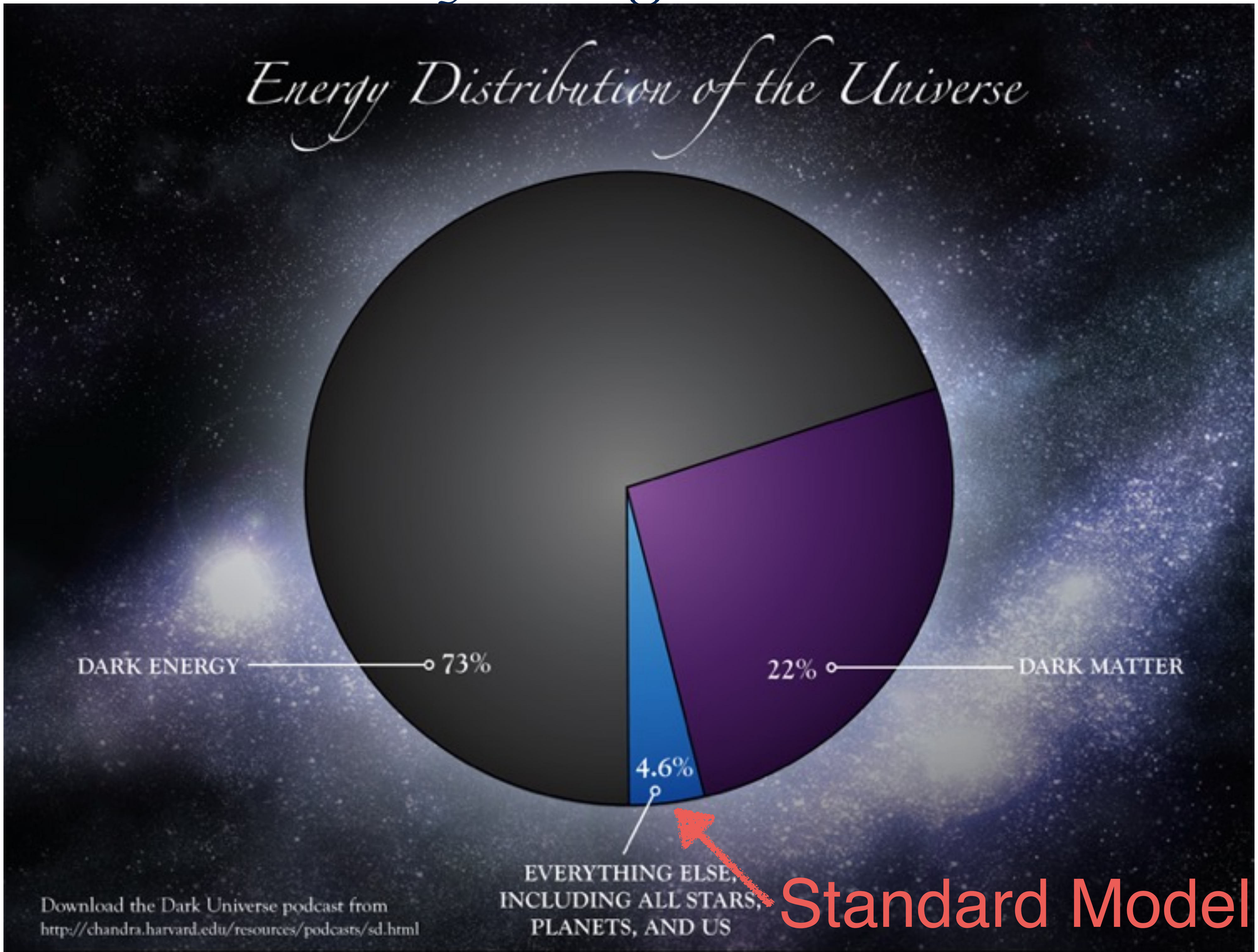
Space-time symmetries

Correlation functions are invariant

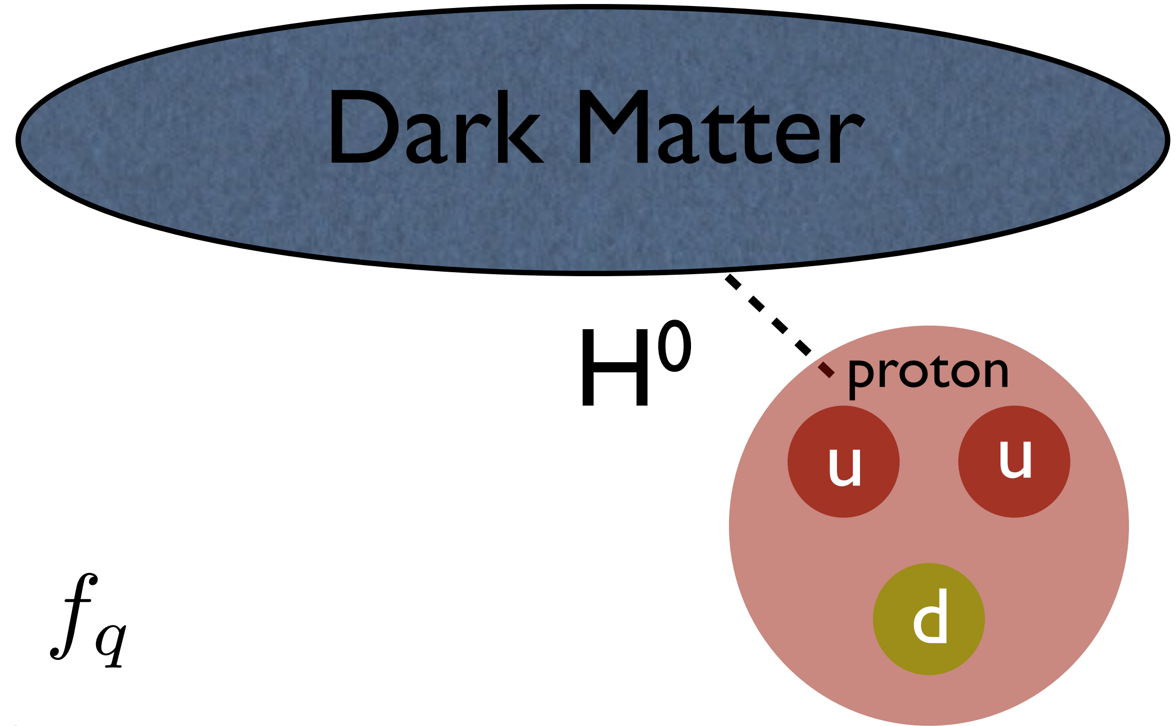
- Translation by lattice vectors
- Space-time rotations H(4)
  - continuous rotational symmetry O(4) is broken down to hypercubic rotations
- Charge conjugation, [parity, time-reversal]

Testing the Standard Model at low-energy

- *While we have no direct confirmation of any BSM physics - we have very strong indirect evidence:*



If Dark Matter couples to the scalar current of the nucleon (eg via Higgs) Spin Independent cross section



$$\sigma \propto |f|^2 \qquad f = \frac{2}{9} + \frac{7}{9} \sum_{q=u,d,s} f_q$$

$$f_q \equiv \frac{\langle N | m_q \bar{q} q | N \rangle}{m_N}$$

see eg. Cheung, Hall, Pinner, Ruderman arXiv:1211.4873

with enhancement of A^2 for nucleus (Xenon)

scalar current difficult to measure experimentally

$f_{u,d}$ estimated from pion-nucleon scattering

f_s uncertainty dominates estimates of cross section

Ellis, Olive, Savage Phys.Rev. D77 (2008)

If Dark Matter couples to the scalar current of the nucleon (eg via Higgs) Spin Independent cross section

$$\sigma \propto |f|^2 \qquad f = \frac{2}{9} + \frac{7}{9} \sum_{q=u,d,s} f_q$$

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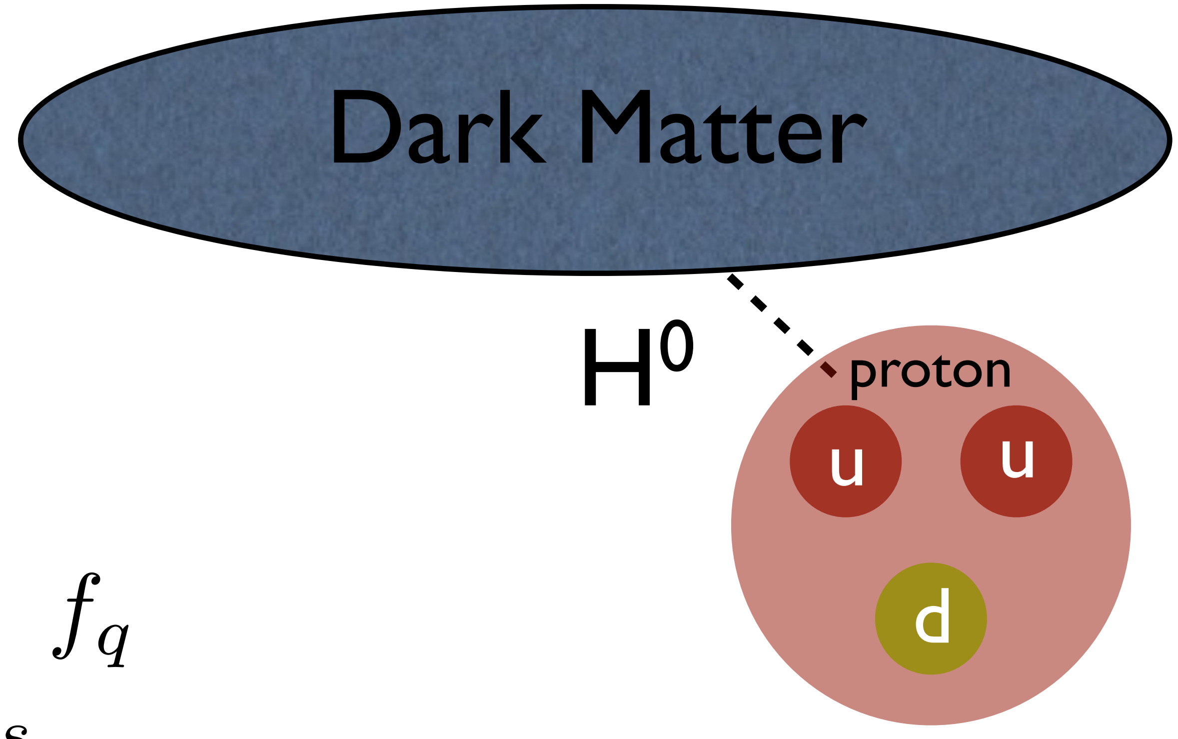
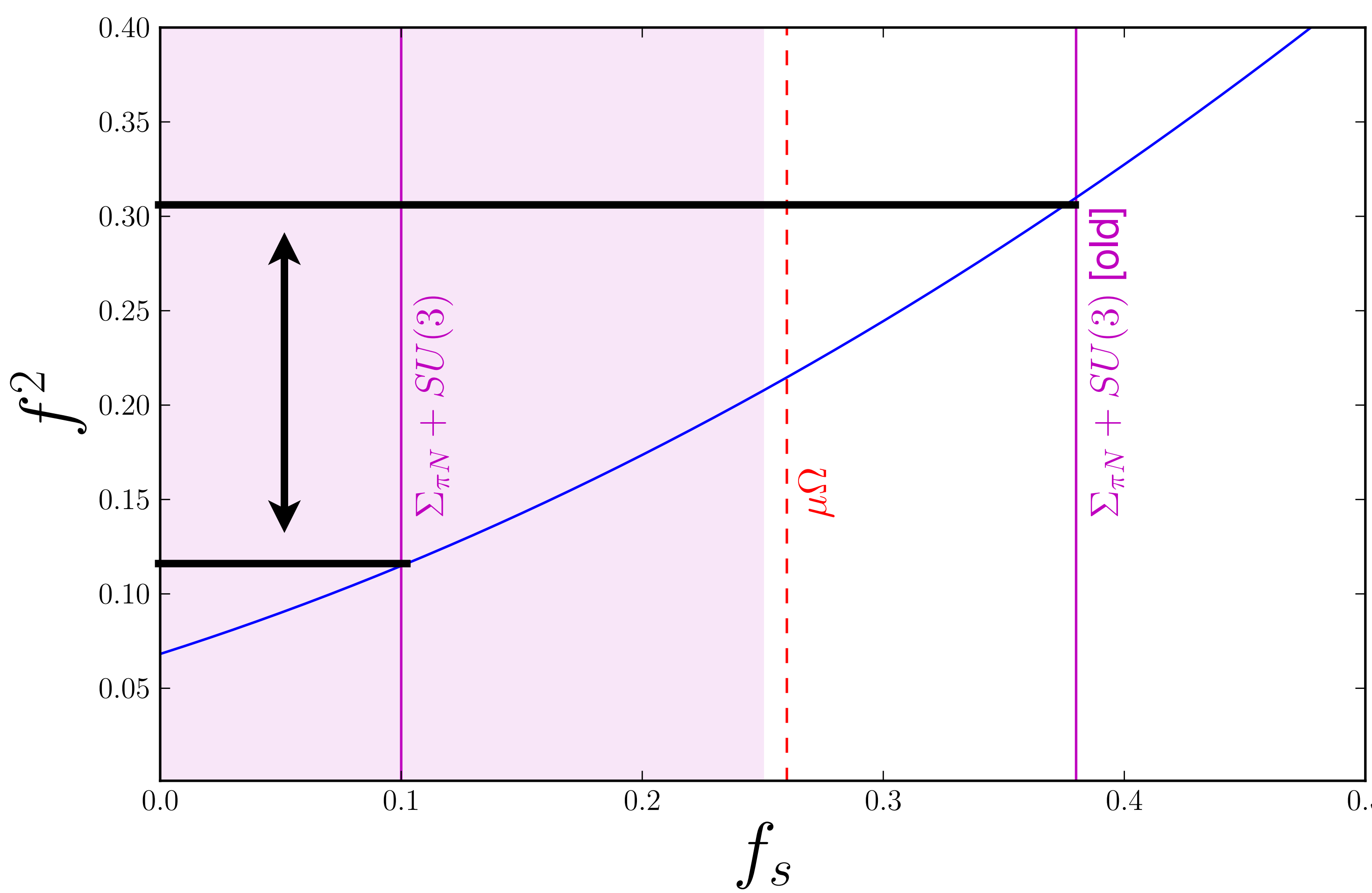


figure adapted from arXiv:1211.4873
thanks to J. Ruderman and collaborators

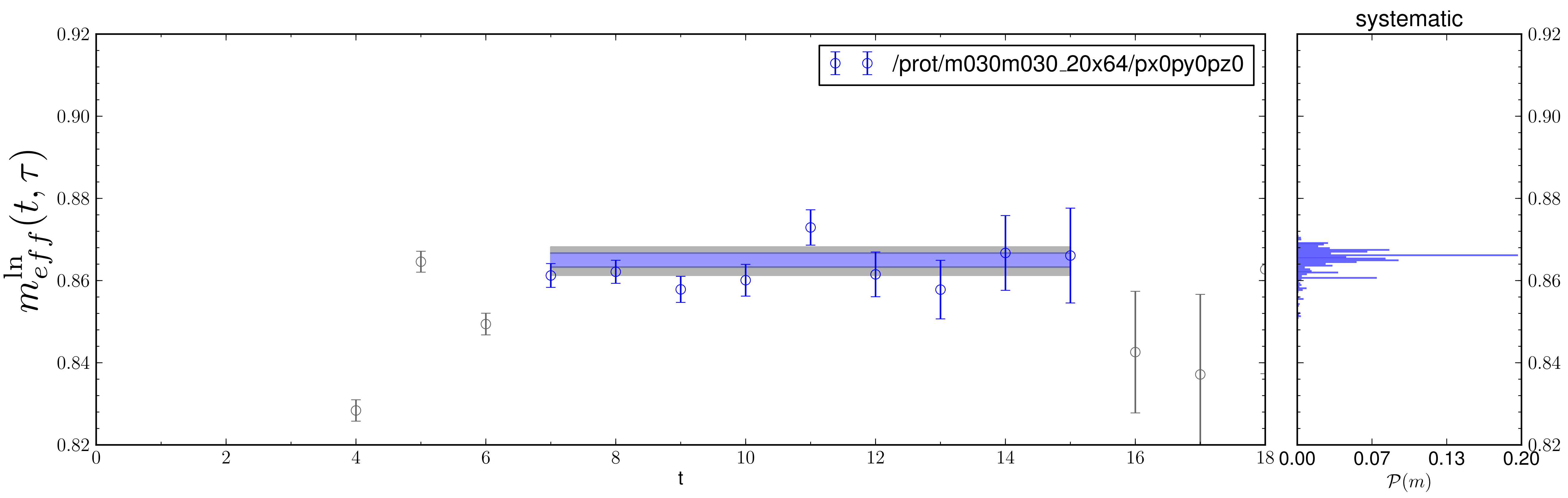


strange content of the nucleon

P. Junnarkar and AWL
arXiv:1301.1114

Lattice QCD perfect tool to compute strange content of nucleon $m_s\langle N|\bar{s}s|N\rangle$

Feynman-Hellmann Theorem $$m_q\langle N|\bar{q}q|N\rangle = m_q \frac{\partial}{\partial m_q} m_N$$



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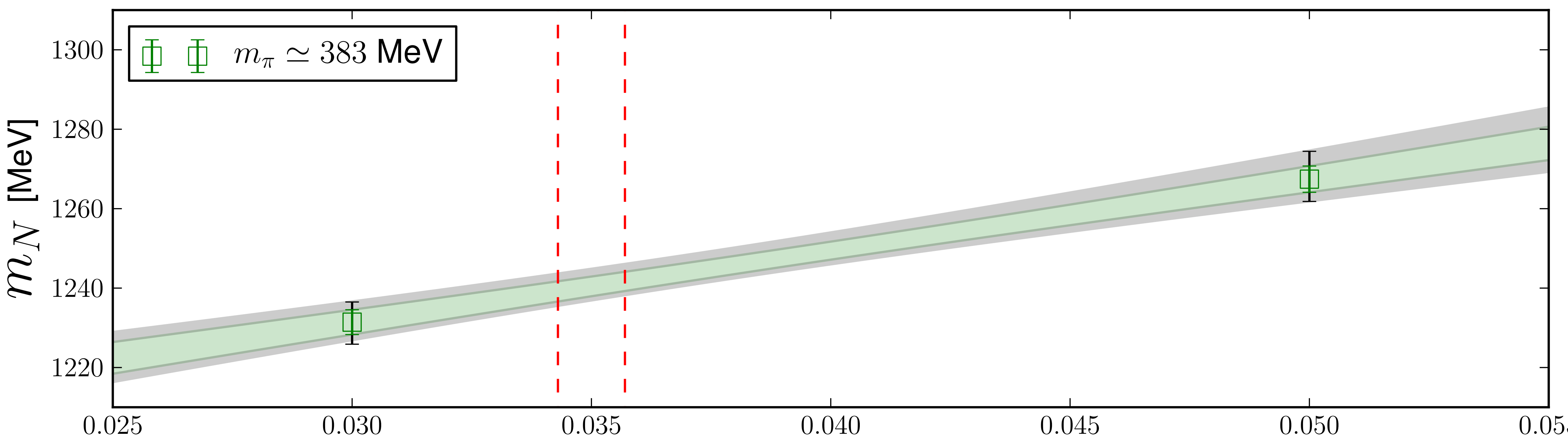
am_s^{phys}



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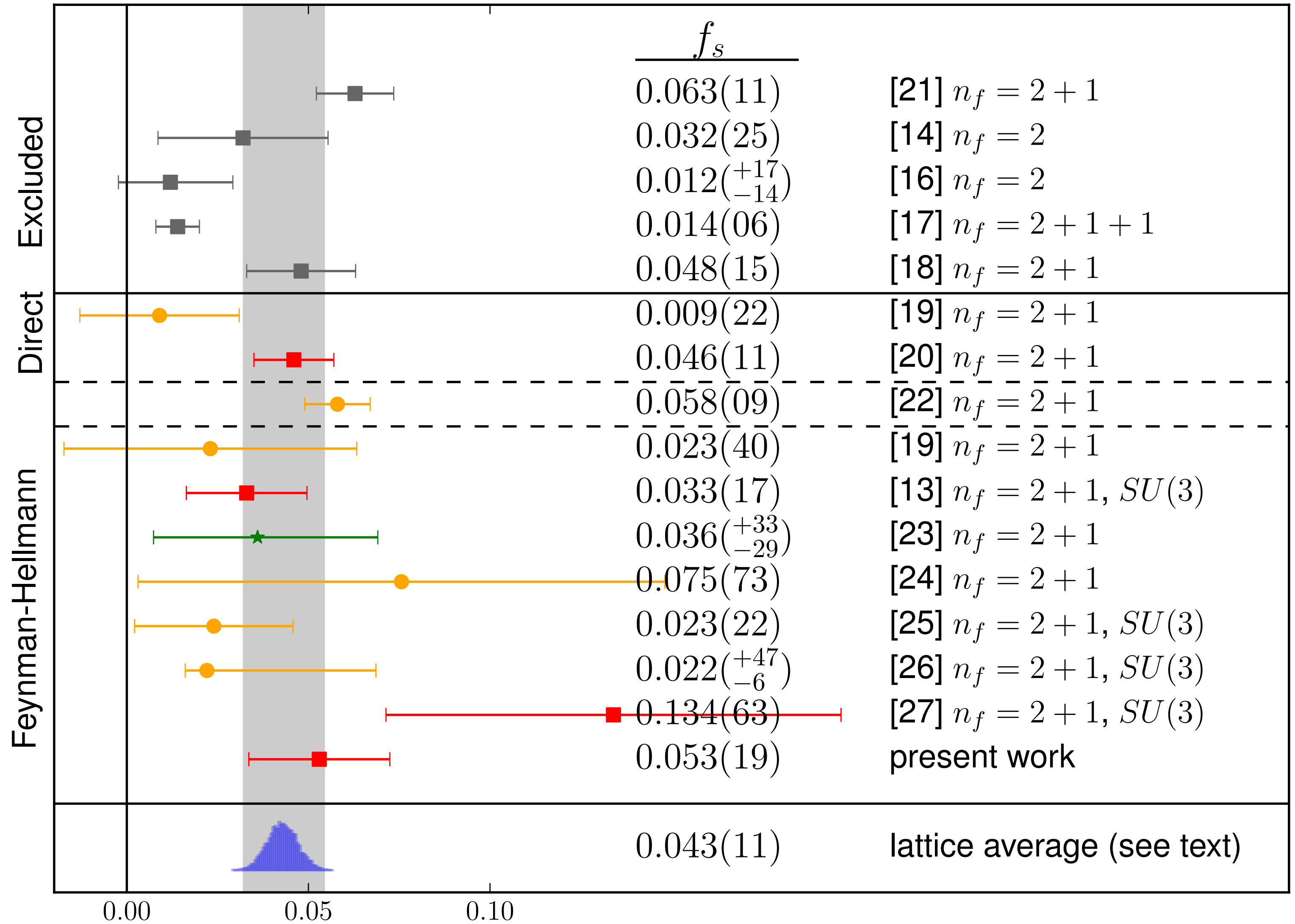
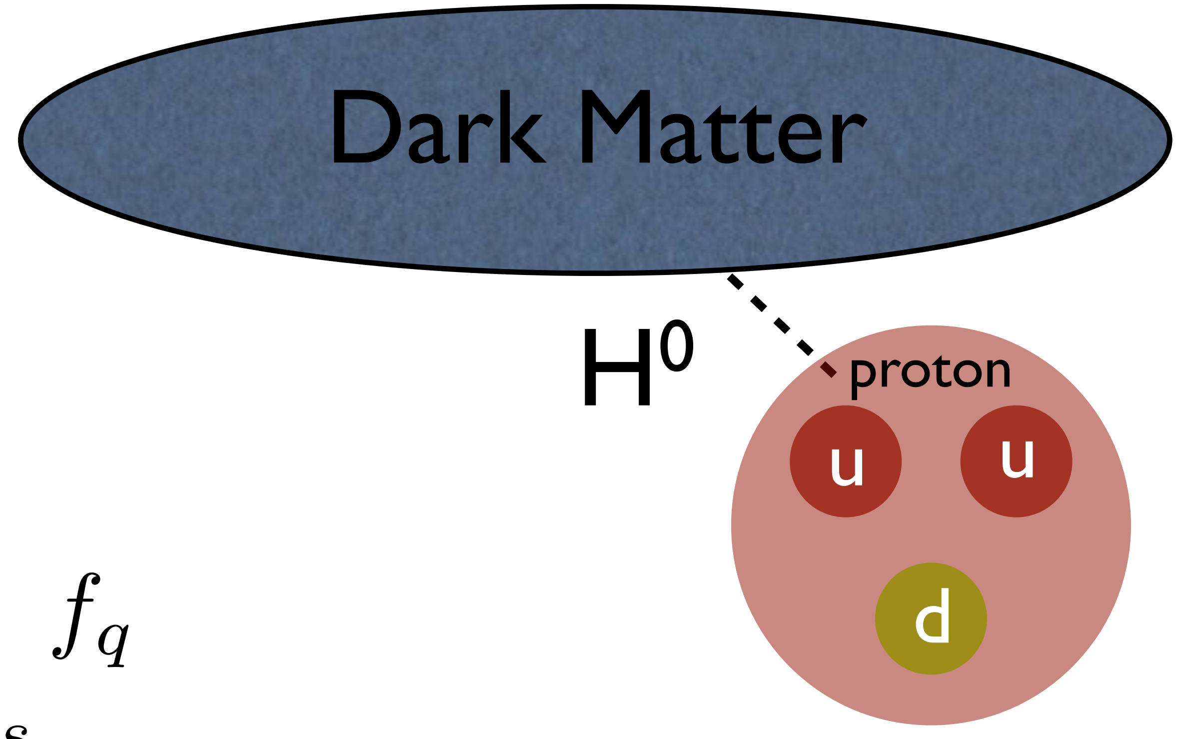


$$f_s = m_s\langle N|\bar{s}s|N\rangle / m_N$$

If Dark Matter couples to the scalar current of the nucleon (eg via Higgs) Spin Independent cross section



$$\sigma \propto |f|^2 \qquad f = \frac{2}{9} + \frac{7}{9} \sum_{q=u,d,s} f_q$$

$$f_q \equiv \frac{\langle N | m_q \bar{q} q | N \rangle}{m_N}$$

see eg. Cheung, Hall, Pinner, Ruderman
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figure adapted from arXiv:1211.4873
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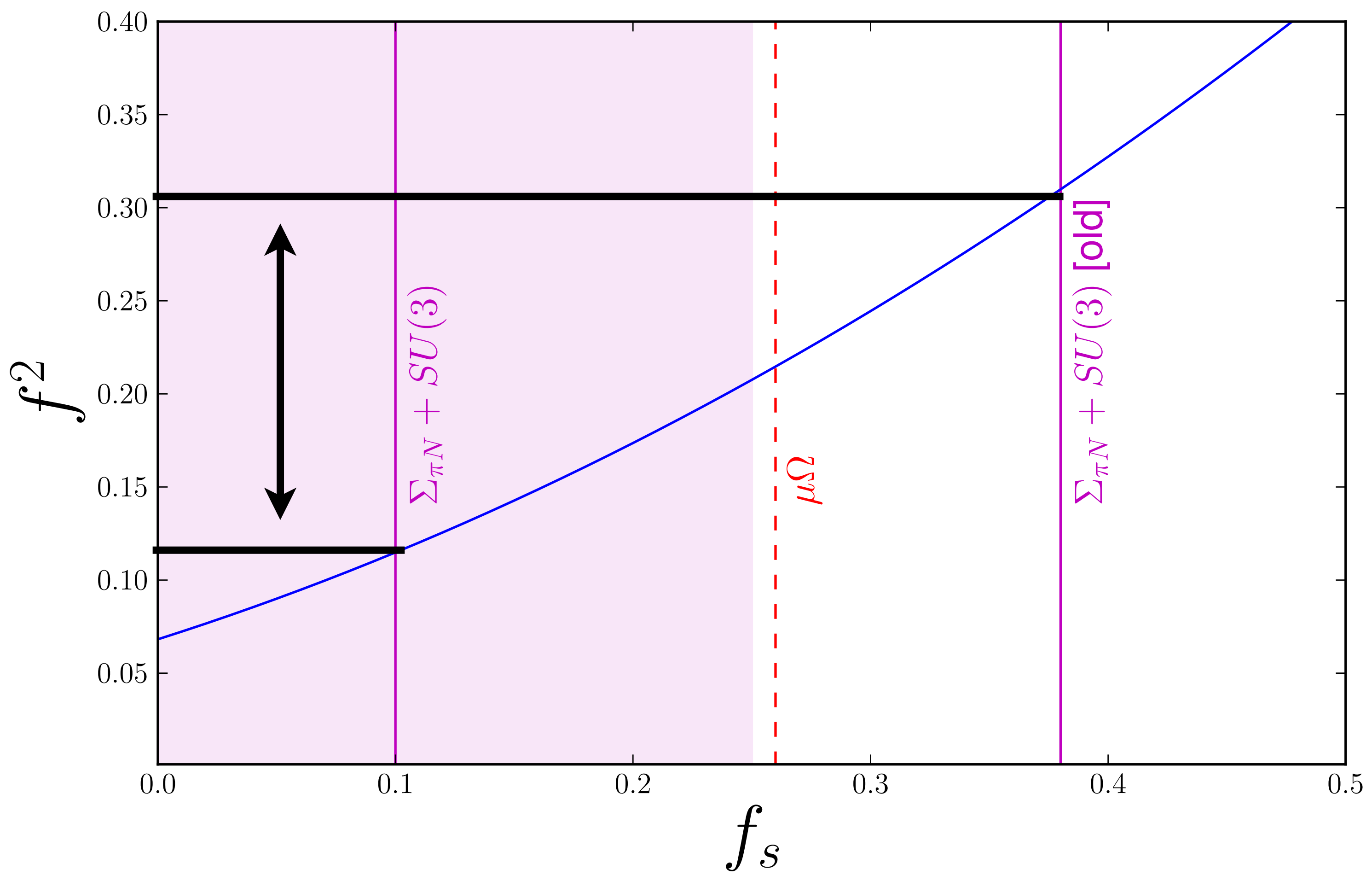

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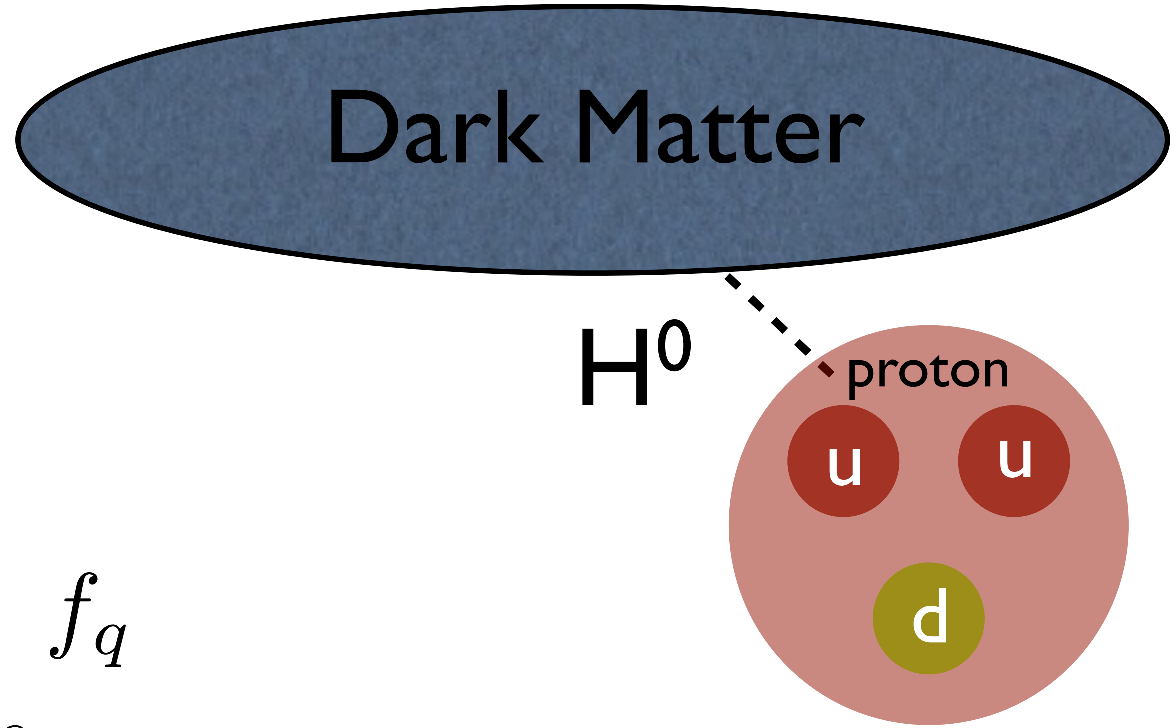

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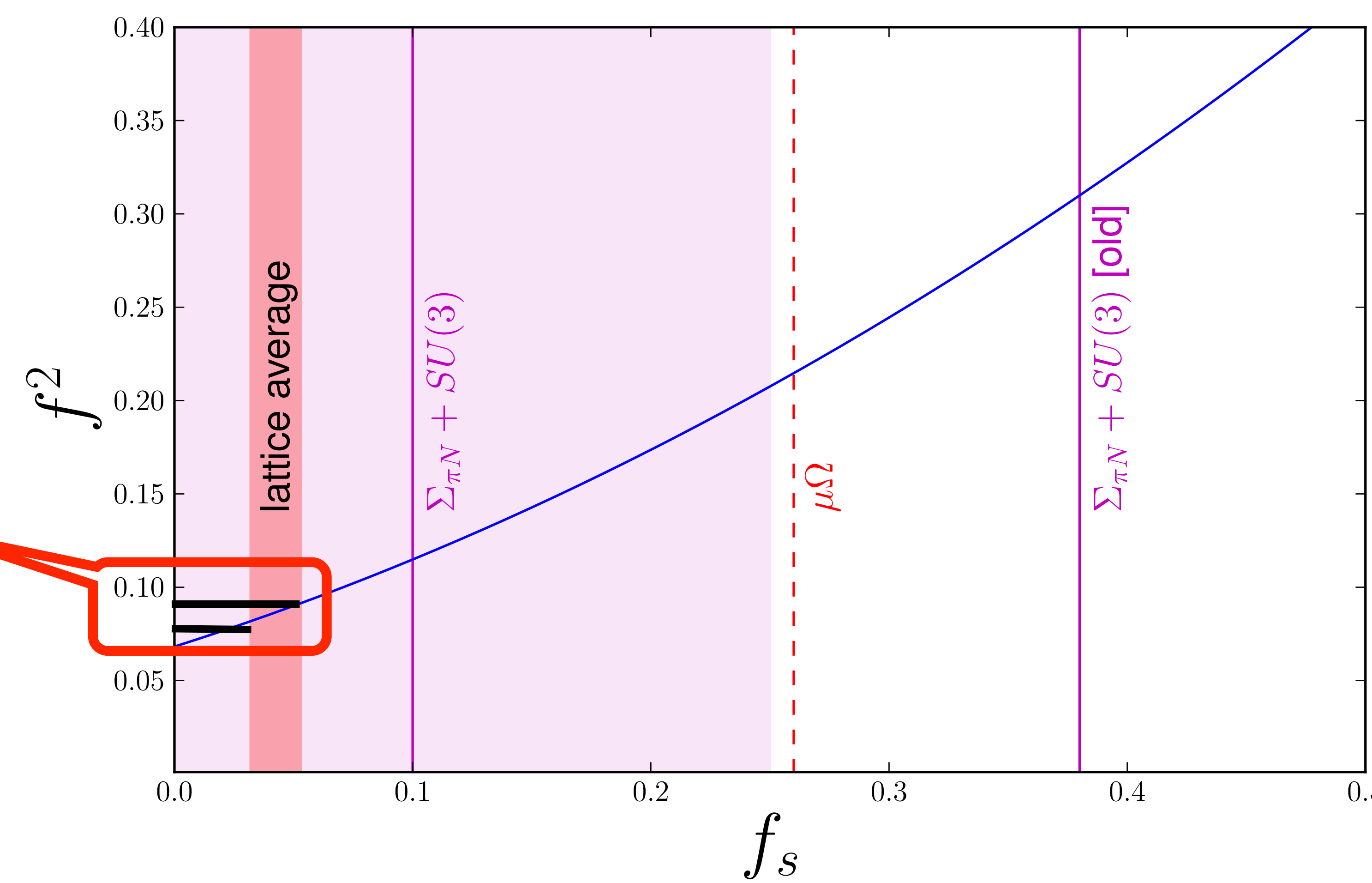


dramatic reduction in uncertainty of cross section

now $f_{u,d}$ gives larger uncertainty - but harder

Light quark mass dependence of M_B

$f_{u,d}$ can be determined from the pion mass dependence of the nucleon mass

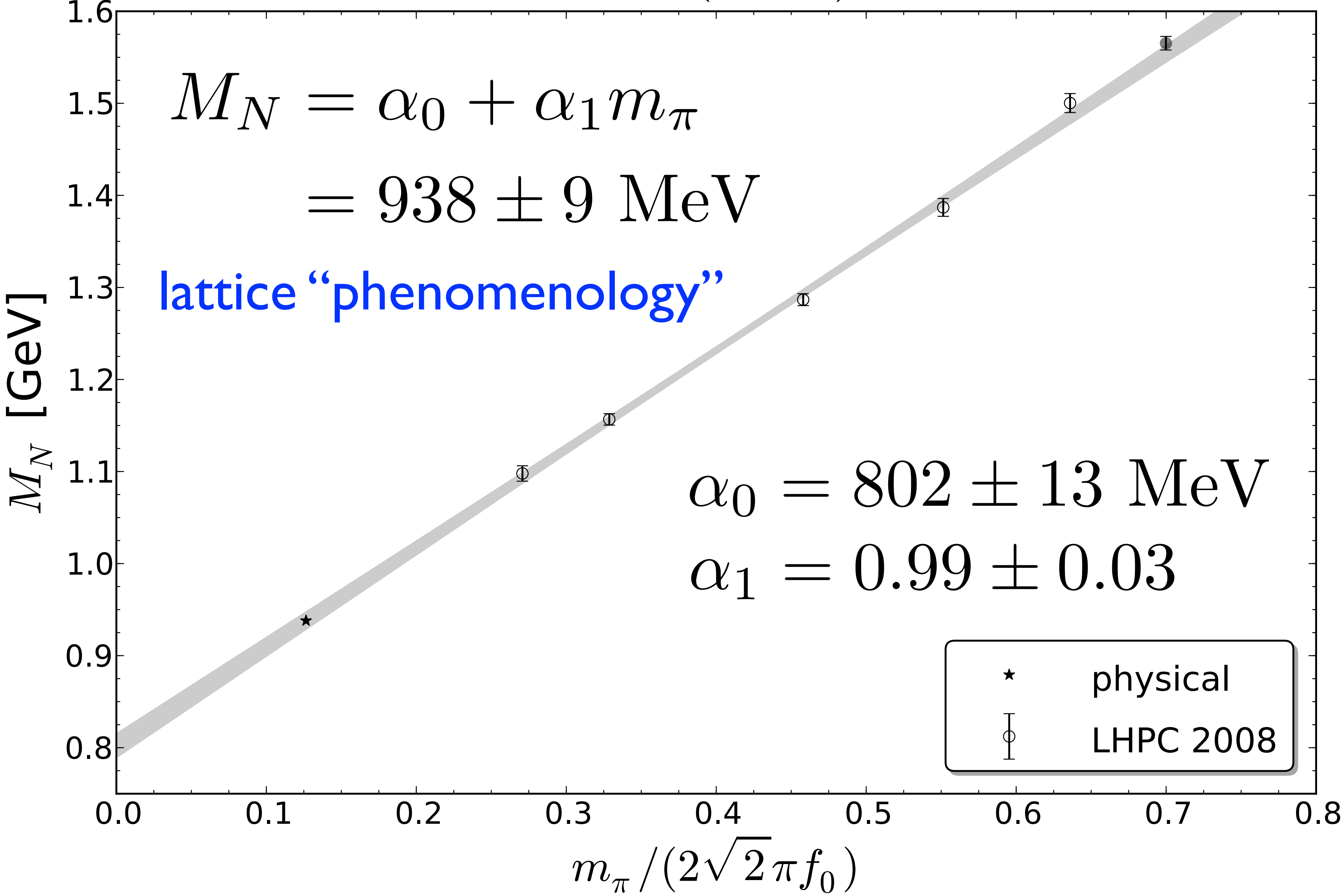
$$M_N = M_0 + \alpha_N m_\pi^2 - \frac{3\pi g_A^2}{(4\pi f_\pi)^2} m_\pi^3 + \cdots$$

$$m_\pi^2 = B_0(m_u + m_d) + \cdots$$

(these expressions are derived from chiral perturbation theory, the low-energy effective field theory of QCD whose construction is based upon the approximate chiral symmetry of QCD)

Light quark mass dependence of M_B

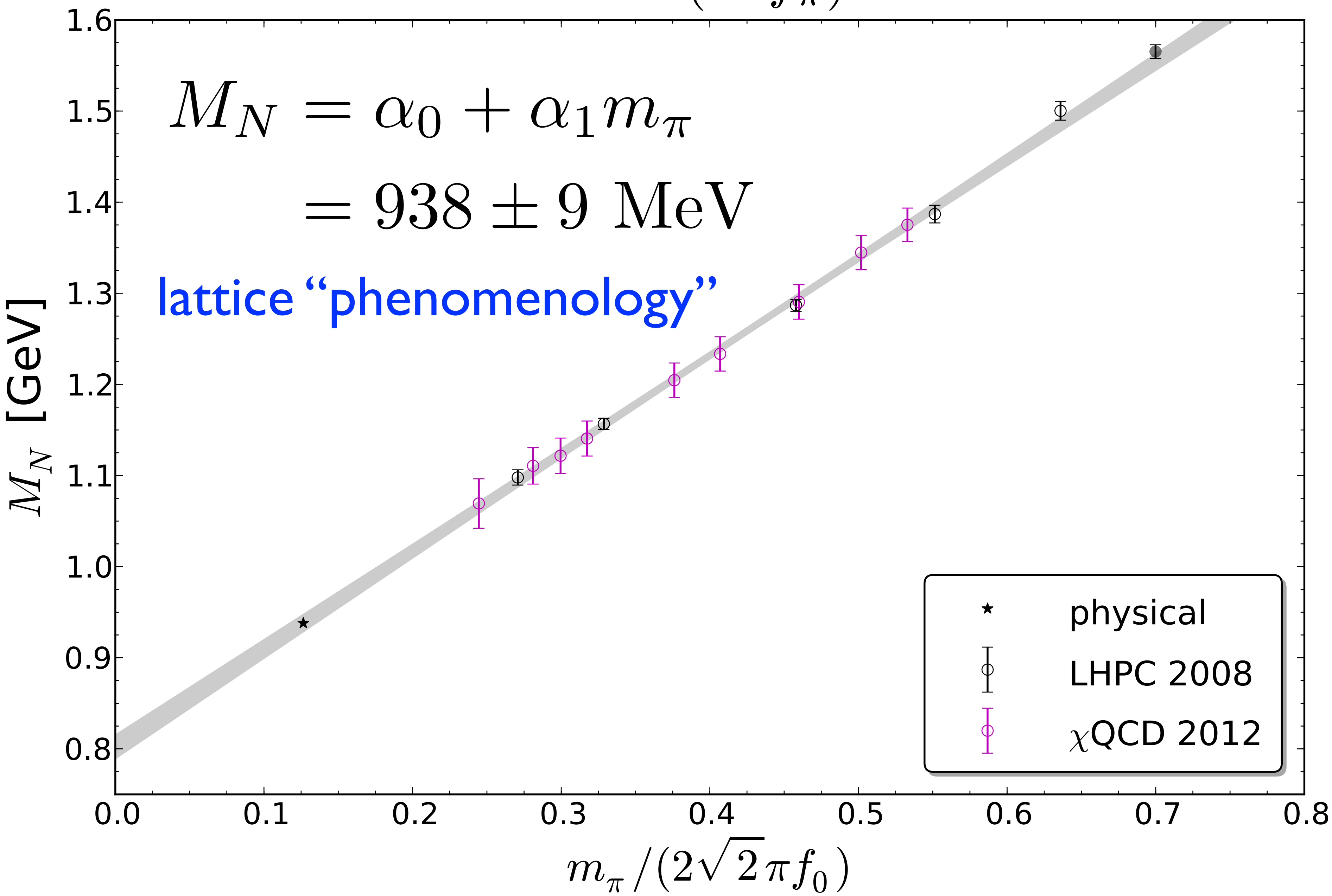
$$M_N = M_0 + \alpha_N m_\pi^2 - \frac{3\pi g_A^2}{(4\pi f_\pi)^2} m_\pi^3 + \cdots$$ theory



Physical point NOT included in fit

Light quark mass dependence of M_B

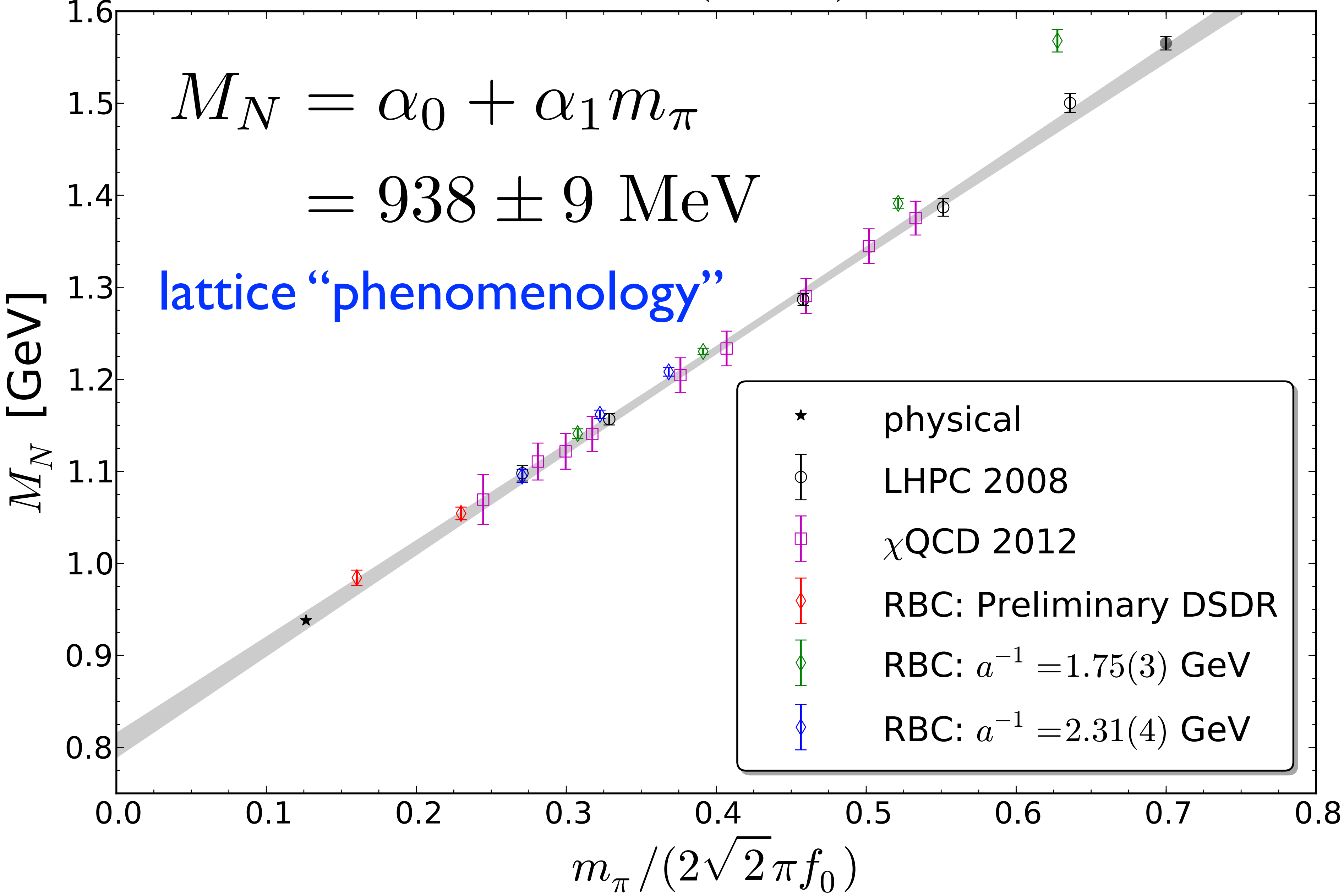
$$M_N = M_0 + \alpha_N m_\pi^2 - \frac{3\pi g_A^2}{(4\pi f_\pi)^2} m_\pi^3 + \cdots$$ theory



χQCD Collaboration uses Overlap Valence fermions on Domain-Wall (RBC-UKQCD) sea fermions

Light quark mass dependence of M_B

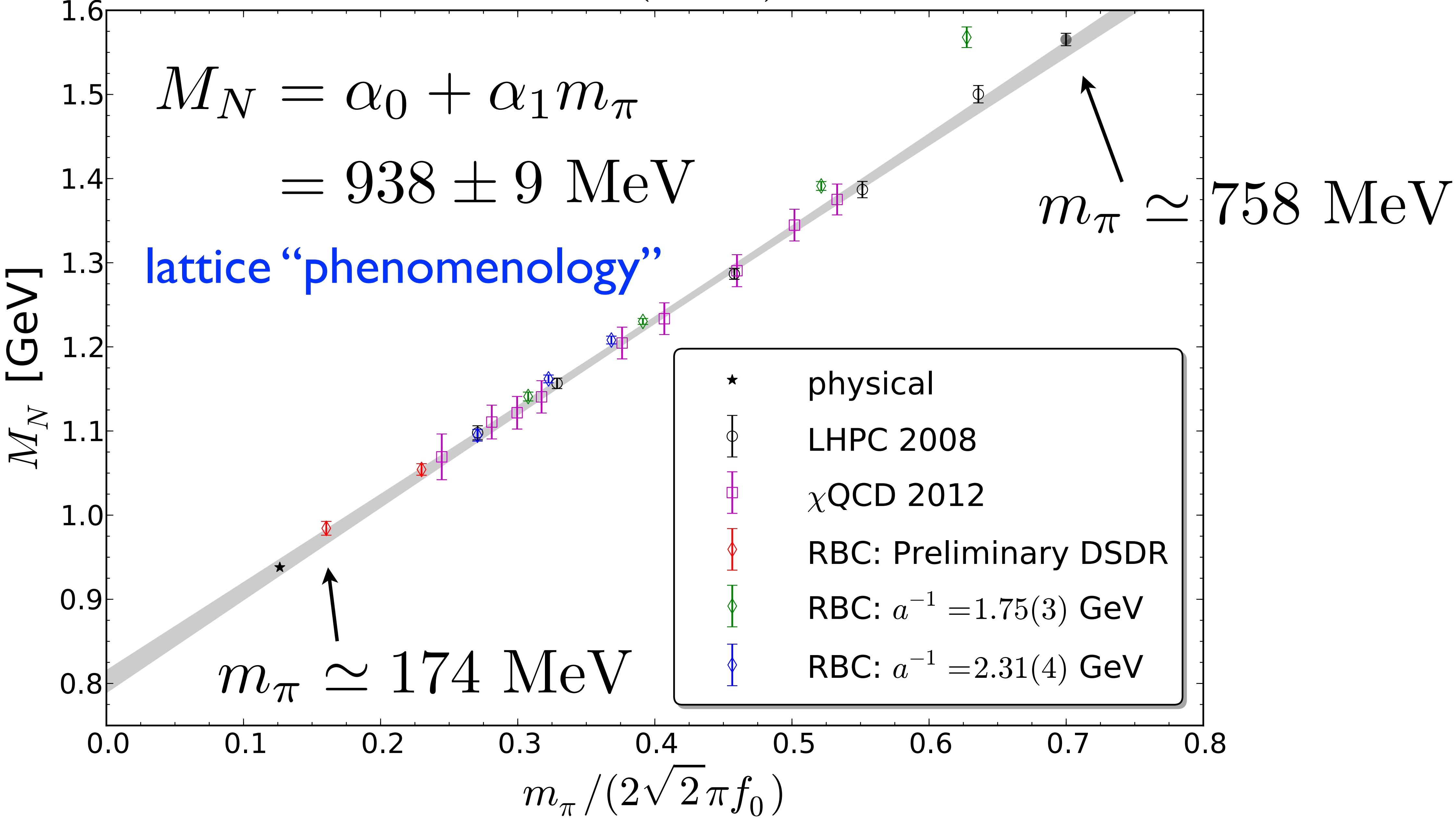
$$M_N = M_0 + \alpha_N m_\pi^2 - \frac{3\pi g_A^2}{(4\pi f_\pi)^2} m_\pi^3 + \cdots$$ theory



RBC-UKQCD Collaboration uses Domain-Wall valence and sea fermions

Light quark mass dependence of M_B

$$M_N = M_0 + \alpha_N m_\pi^2 - \frac{3\pi g_A^2}{(4\pi f_\pi)^2} m_\pi^3 + \cdots$$ theory



Taking this seriously yields

$$\sigma_{\pi N} = 67 \pm 4 \text{ MeV}$$

I am not advocating this as a good model for QCD!

Conclusions

- Understanding nuclear physics from the fundamental theory of strong interactions, QCD, is exciting and important for these and other reasons:
  - Quantitative connection between QCD and the rich nuclear phenomenology
  - Understanding precision low-energy nuclear physics to constrain the SM and searches for BSM physics
- The growth of computing power and algorithms means that TODAY is the beginning of a renaissance in nuclear physics where these exciting things are just becoming possible!

These were just a few select examples!

Thank You