Gauge Invariance of Radiative Jet function

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Outline

- Background and Motivation
  Why radiative jet functions and their gauge invariance?
- Gauge invariance of NLP SCET Lagrangians and radiative jet functions
- Conclusion

Background and Motivation:

Why do we study the radiative jet functions and their gauge invariance

NLP SCET Factorization and radiative jet functions

- In line with the progress in precision in the experiments, it becomes increasingly important to include next-to-leading power (NLP) contributions.

- However, new features emerge in NLP that are not present at LP. In particular, soft-quark fields start to play a role at NLP.

- In NLP SCET factorization, the hard-collinear parts are described by the **radiative jet functions** that contain couplings between soft quarks and hard-collinear quarks.

- The analyses of the radiative jet functions at $O(\lambda)$ have been carried out for $H \to (b\bar{b}) \to \gamma\gamma$ and leptonic *B* decay. Bosch, Hill, Lange, Neubert, 2003 Liu, Neubert, 2020

- In this work, we **extend** the study of the radiative jet functions **beyond** $O(\lambda)$, and, specifically, we discuss the property of **gauge invariance** of the radiative jet functions.

Why GI of radiative jet functions?

- To obtain the anomalous dimensions for the radiative jet functions, we need to compute the 1-loop corrections and identify the divergences.

- Since radiative jet functions and the operators in them are constructed to be gauge invariant from the start, it would be expected that one could choose a convenient gauge (e.g. the light-cone gauge), in which calculations are simpler.

- However, in the course of this study, we have found an unusual feature in the implementation of gauge invariance of the radiative jet functions.

- I show that, in order to render the radiative jet function gauge invariant, one must include certain contributions that contains **a soft quark propagator.**

- This soft propagator is canceled by a **gauge term** in the gluon propagator, leaving a purely hard-collinear contribution.

Standard SCET Notation

- In this analysis, we have chosen the **position-space** formalism of SCET.

$$r^\mu = \underbrace{\bar{n}\cdot r}_{r^+}\frac{n^\mu}{2} + \underbrace{n\cdot r}_{r^-}\frac{\bar{n}^\mu}{2} + r_\perp^\mu \qquad r^\mu = (r^+, r^-, r_\perp)$$

for *n*-collinear momenta $r^+ \gg r^-$

- The standard power counting for the *n*-hard-collinear and soft momenta:

$$r_{hc}^\mu \sim Q(1, \lambda^2, \lambda) \qquad r_s^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2) \qquad \lambda = \sqrt{m/Q}$$

Motivation for the appearance of hard-collinear mode:
A soft momentum flows through a hard-collinear momentum and holds it off shell by power $O(\lambda^2)$

m : a heavy quark mass

Q : a hard scale

- Standard notation and power counting for the quark and gluon fields:

$$\xi_n \sim O(\lambda) \qquad q \sim O(\lambda^3) \qquad G_n \sim r_{hc}$$

Gauge invariance of a subleading power SCET Lagrangian

An NLP SCET Lagrangian

$$\mathcal{L}^{(2)}_{q\xi_n}(x) = \bar{q}(x^+)\left(W_n^\dagger in\cdot D\frac{\not{\bar{n}}}{2}\xi_n\right)(x)$$

- This Lagrangian describes a coupling between a soft quark and a hard-collinear quark.

- This is power suppressed $O(\lambda^2)$.

  It is suppressed relative to the LP SCET Lagrangian because $\xi_n \sim O(\lambda) \longrightarrow q \sim O(\lambda^3)$

  Also, we can show that with $d^4x \sim O(\lambda^{-4})$ and $in\cdot D \sim O(\lambda^2) \quad \longrightarrow \quad \lambda^{3+2+1-4} = \lambda^2$

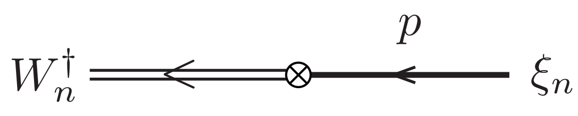
- In keeping with consistent power counting, the position argument of q is multipole expanded.

- This Lagrangian is formally soft and hard-collinear gauge invariant.

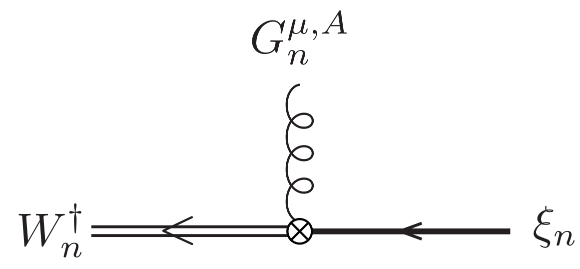
- However, we will find out that the gauge invariance must be implemented in an unusual way.

$q\xi_n G_n$ Interaction at $O(g_s)$

- The Feynman rule for the crossed vertex for $\mathcal{L}^{(2)}_{q\xi_n}$ is



$V_A^{(0)} = n\cdot p\frac{\bar{n}\!\!\!/}{2}$

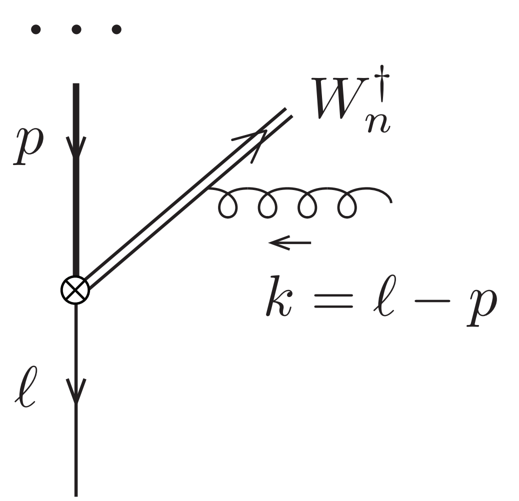


$V_A^{(1)} = g_s T^A n_\mu \frac{\bar{n}\!\!\!/}{2}$

$$\mathcal{L}^{(2)}_{q\xi_n}(x) = \bar{q}(x^+)\left(W_n^\dagger \underline{in\cdot D\frac{\bar{n}\!\!\!/}{2}}\xi_n\right)(x)$$

gives the crossed vertex V_A

- Let us consider the vertex for $\mathcal{L}^{(2)}_{q\xi_n}$ at $O(g_s)$



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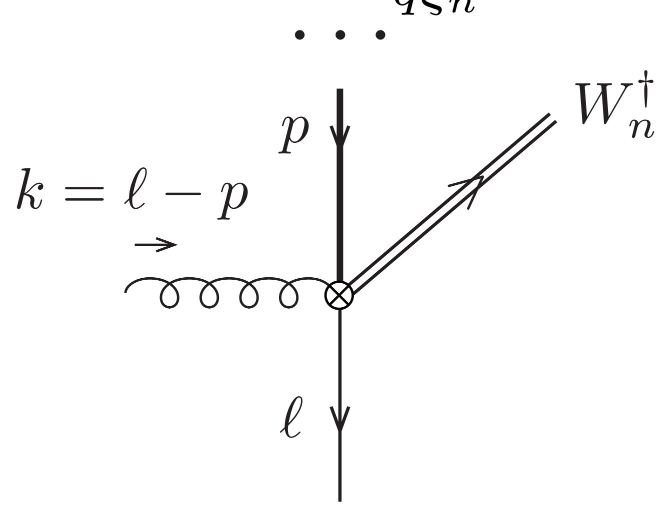


p : hard-collinear quark momentum

ℓ : soft quark momentum

$k = p - \ell$: hard-collinear gluon momentum

- The resulting amplitude is

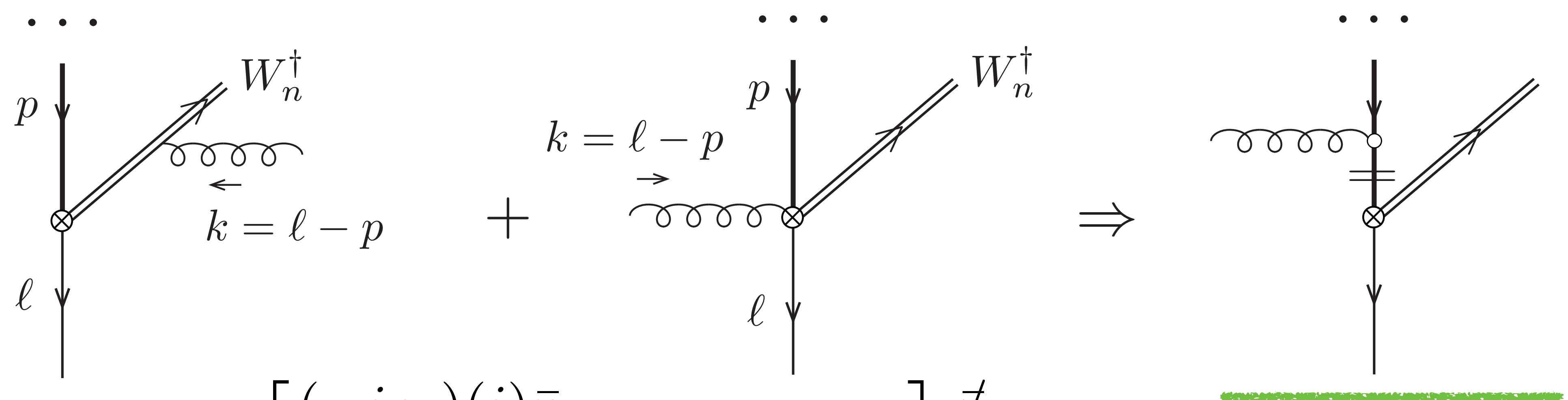
$$A_1 = \epsilon^\mu(k)\left[\frac{(-ig_s)(i)\bar{n}_\mu}{\bar{n}\cdot k + i\varepsilon}n\cdot p + g_s n_\mu\right]\frac{\bar{n}\!\!\!/}{2}P_n\ldots$$

Ward identities

- We can work out the Ward identities for these amplitudes by

$$\epsilon^\mu(k) \to \epsilon^\mu(k) + k^\mu$$

The contributions of the gauge term should vanish to ensure gauge invariance



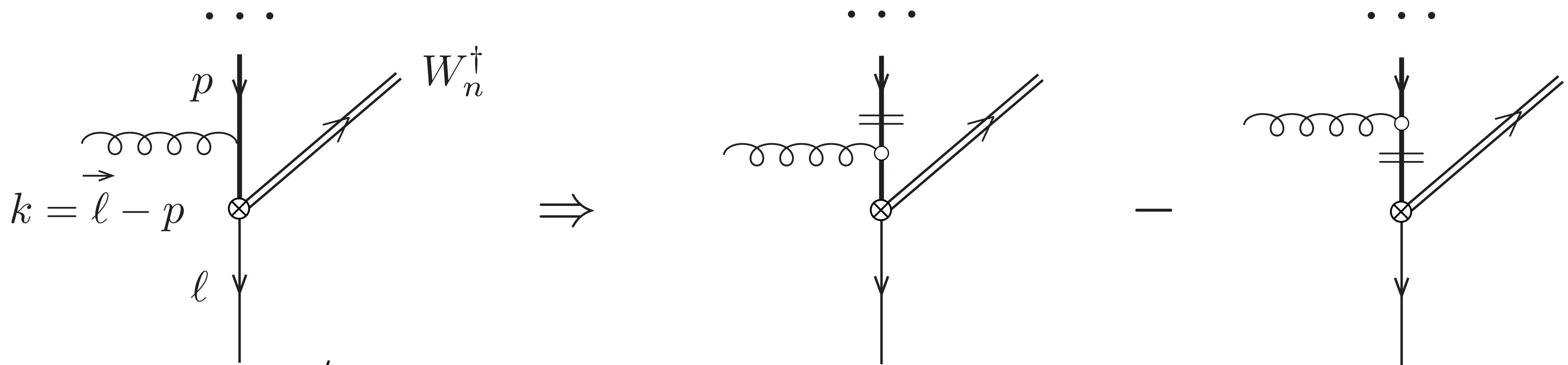
$$A_1 = \epsilon^\mu(k)\left[\frac{(-ig_s)(i)\bar{n}_\mu}{\bar{n}\cdot k + i\varepsilon} n\cdot p + g_s n_\mu\right]\frac{\bar{\not{n}}}{2}P_n\ldots,$$

$$\Downarrow \quad \boxed{\epsilon^\mu(k) \to k^\mu}$$

$$A_1^{\text{gauge}} = g_s(n\cdot p + n\cdot k)\frac{\bar{\not{n}}}{2}P_n\ldots = \underline{g_s n\cdot\ell\frac{\bar{\not{n}}}{2}P_n}\ldots$$

- The gauge term does not vanish, and it is proportional to the soft quark momentum at $\ell \sim \lambda^2$.
- It is not valid to discard ℓ^- compared to p^- in k because both terms are of power λ^2.

Resolution: Soft propagator diagram



$$A_2 = n \cdot \ell \frac{\bar{n}\!\!\!/}{2} P_n \frac{i}{\ell\!\!\!/ - m + i\varepsilon} \epsilon^\mu(k)(ig_s)\gamma_\mu \ldots ,$$

$\Downarrow$ $\boxed{\epsilon^\mu(k) \to k^\mu}$

$$A_2^{\text{gauge}} = n \cdot \ell \frac{\bar{n}\!\!\!/}{2} P_n \frac{i}{\ell\!\!\!/ - m + i\varepsilon} (ig_s) k\!\!\!/ \ldots$$

$$= g_s n \cdot \ell \frac{\bar{n}\!\!\!/}{2} P_n \frac{1}{\ell\!\!\!/ - m + i\varepsilon} [(p\!\!\!/ - m) - \underline{(\ell\!\!\!/ - m)}] \ldots ,$$

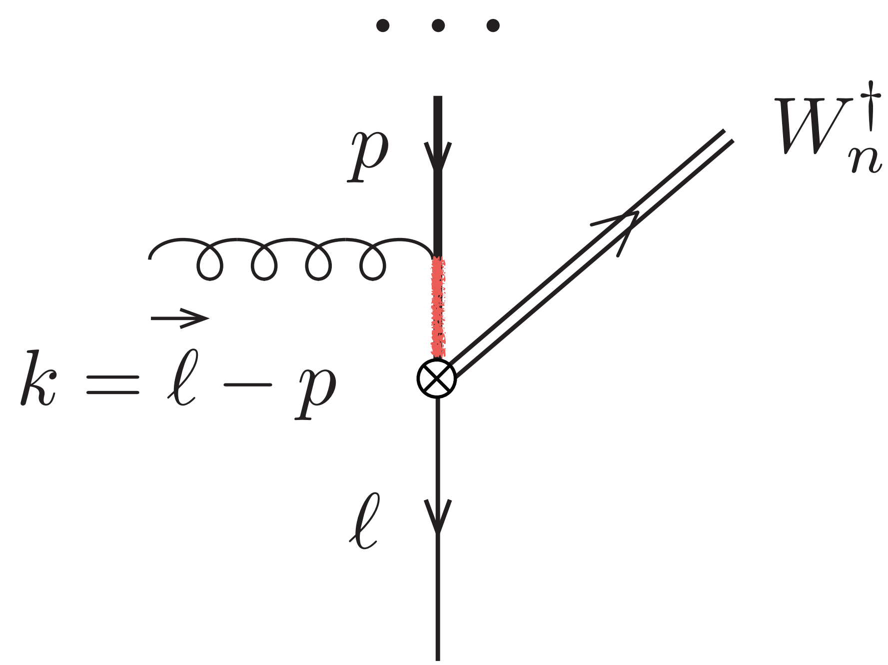
this term cancels A_1^{gauge}

- The second term in the square brackets cancels A_1^{gauge}, and the first term can be shown to vanish through Dirac equation for the external quark spinor (and additional Ward identities).

$$A_1^{\text{gauge}} = \underline{g_s n \cdot \ell \frac{\bar{n}\!\!\!/}{2} P_n} \ldots$$

Therefore, $\boxed{A_1^{\text{gauge}} + A_2^{\text{gauge}} = 0}$

Soft quark propagator diagram



- In diagram for A_2, the momentum ℓ on the red propagator is soft.
- Therefore, it seems not to belong properly to the collinear function.
- However, we have shown that the soft propagator can be canceled in the application of Ward identities, resulting in **a purely hard-collinear contribution**.
- Let us see how this is realized in the radiative jet functions.

Gauge invariance of a radiative jet function at $O(\lambda^2)$

Radiative jet function A

$$A(\ell^-) = \int d^D x\, e^{-i\frac{\ell^- x^+}{2}} \langle \mathcal{Q}\bar{\mathcal{Q}}({}^3S_1^{[1]}, p, p)| T \left(W_n^\dagger i n\cdot D_n \frac{\bar{\not n}}{2}\xi_n \right)^{\beta,b} (x) (\bar{\xi}_n W_n)^{\alpha,a}(0)|0\rangle$$

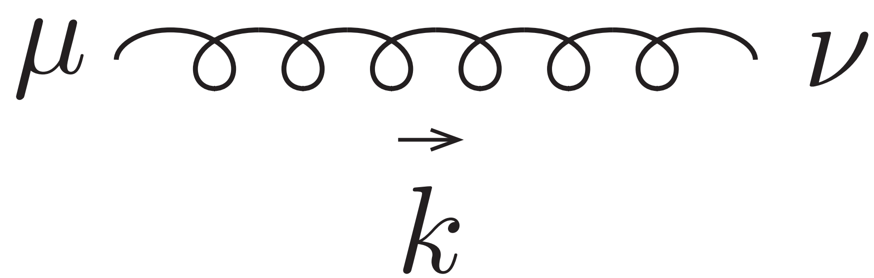
- First operator arises from soft-to-hard-collinear transition Lagrangian $\mathcal{L}^{(2)}_{q\xi_n}$
- Second operator arises from the hard-to-*n*-hard-collinear transition.
- The exponential factor injects the flow of the soft momentum ℓ into the radiative jet function.
- We choose a $\mathcal{Q}\bar{\mathcal{Q}}$ external state because it allows us to see the problem **at the Born level.**
- For definiteness, we take the $\mathcal{Q}\bar{\mathcal{Q}}$ state to be spin triplet and color singlet.

$$(v^c \otimes \bar{u}^d) \to \Pi^{cd}_{{}^3S_1^{[1]}} = -\frac{\not\epsilon^*(\not p + m)}{2\sqrt{2}m} \times \frac{\delta^{cd}}{\sqrt{N_c}}$$ spin-triplet, color-singlet projector

$$p = (p^+, p^-, p_\perp) = (p^+, m^2/p^+, 0) \sim Q(1, \lambda^4, 0)$$ on-shell condition

Feynman and light-cone gauges

- In this discussion, we will investigate the gauge invariance of the radiative jet functions by computing them in the two gauges:
  the Feynman and the light-cone gauges



- Feynman-gauge gluon propagator

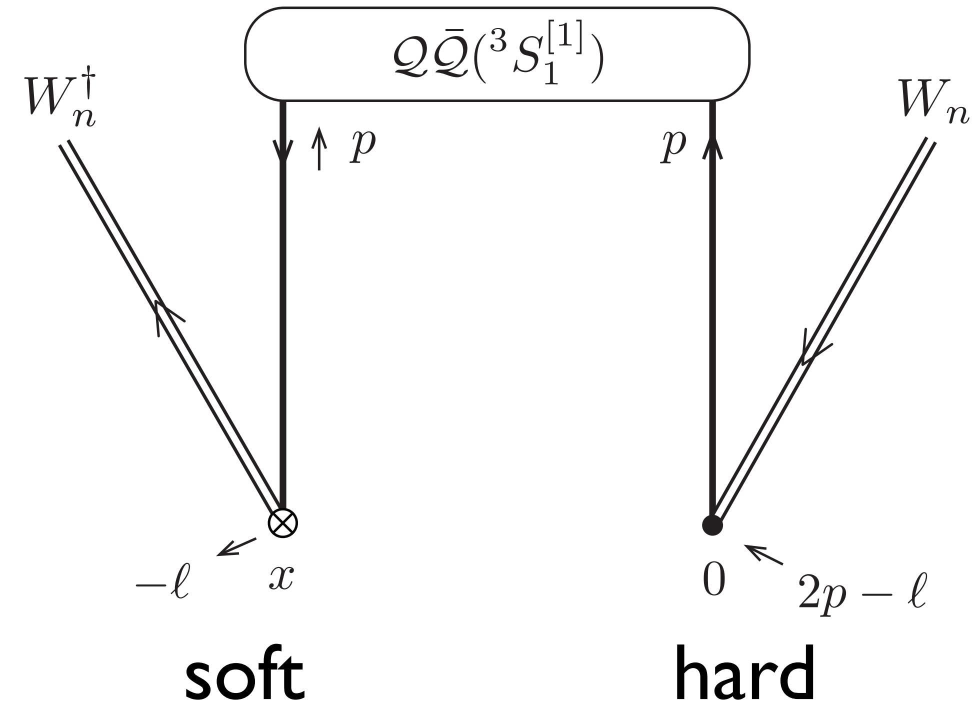
$$\frac{i}{k^2}(-g^{\mu\nu})$$

- Light-cone-gauge gluon propagator

$$\frac{i}{k^2}\left(-g^{\mu\nu}+\frac{k^\mu\bar{n}^\nu+\bar{n}^\mu k^\nu}{k\cdot\bar{n}}\right)$$

$$\bar{n}\cdot G_n\big|_{\text{light-cone gauge}}=0,$$

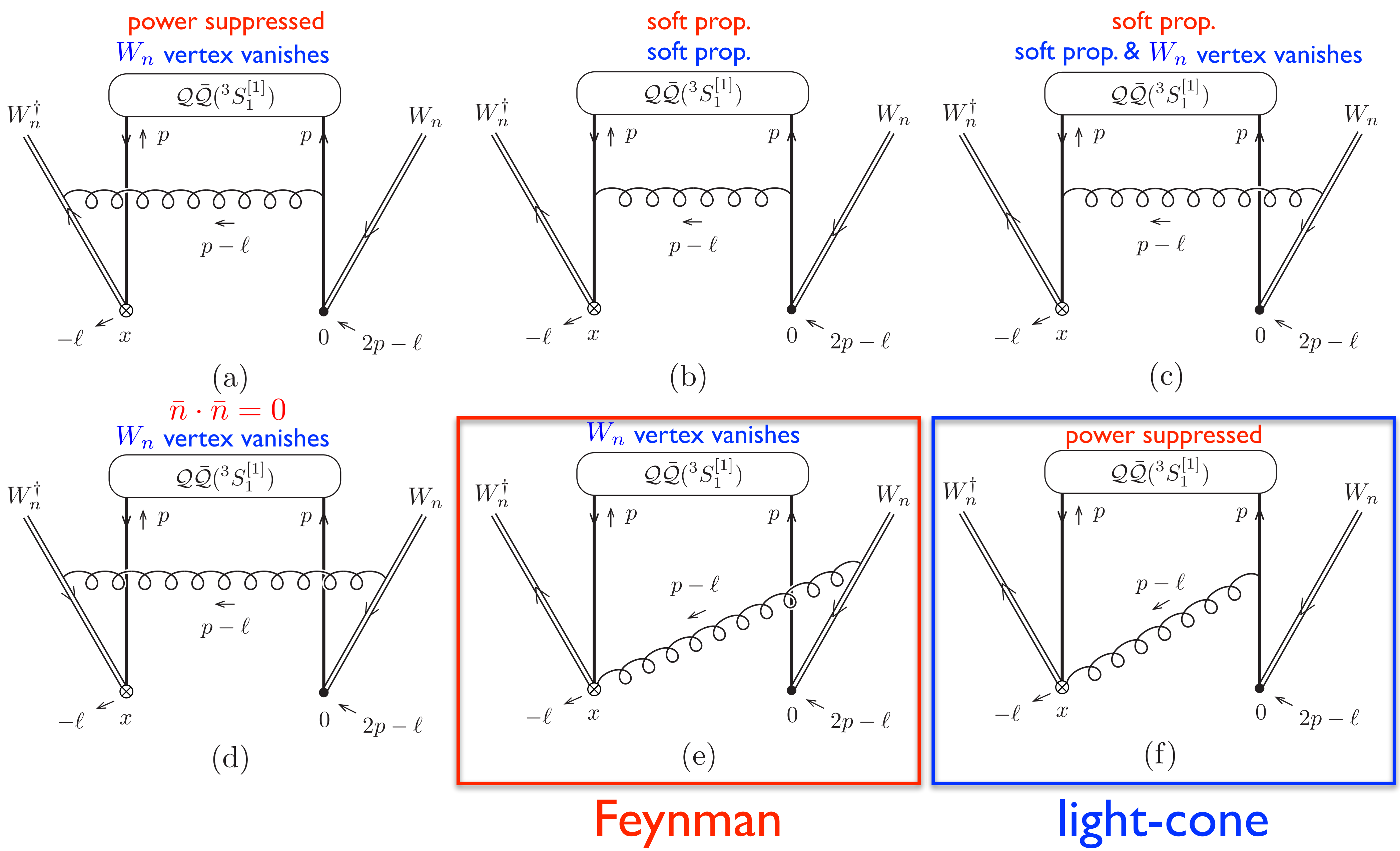
Diagram for radiative jet function A



$$A(\ell^-) = \int d^D x\, e^{-i\frac{\ell^- x^+}{2}} \underbrace{\langle \mathcal{Q}\bar{\mathcal{Q}}(^3S_1^{[1]}, p, p)}_{\text{top}} | T \underbrace{\left(W_n^\dagger in\cdot D_n \frac{\bar{n}\!\!\!/}{2}\xi_n \right)^{\beta,b} (x)}_{\text{left}} \underbrace{(\bar{\xi}_n W_n)^{\alpha,a}(0)}_{\text{right}} |0\rangle,$$

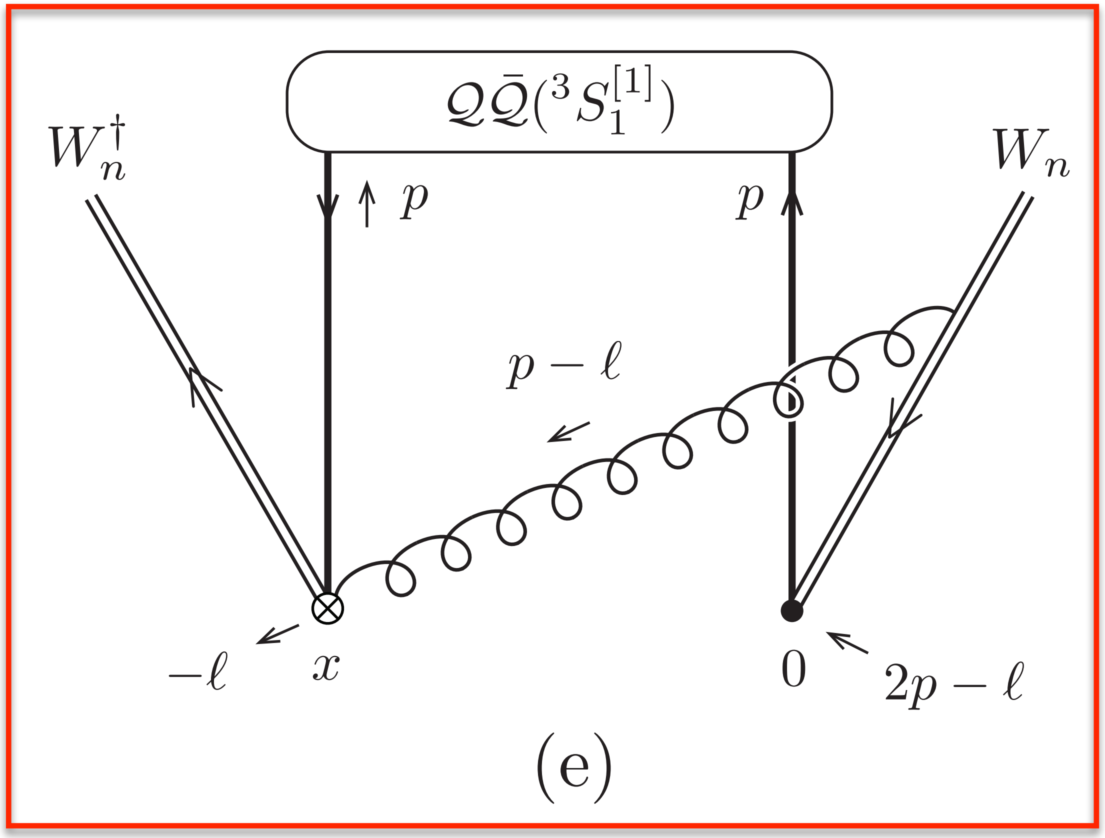
- The $O(g_s^0)$ diagram for radiative jet function A.
- $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ each carry momentum *p*, and they form $^3S_1^{[1]}$ $\mathcal{Q}\bar{\mathcal{Q}}$ state.
- Soft momentum ℓ flows in through the vertex x, and the hard-collinear momentum $2p-\ell$ flows in through the vertex 0.
- Momentum conservation requires at least one gluon connection between the left and right parts of the diagram.

LO Diagrams

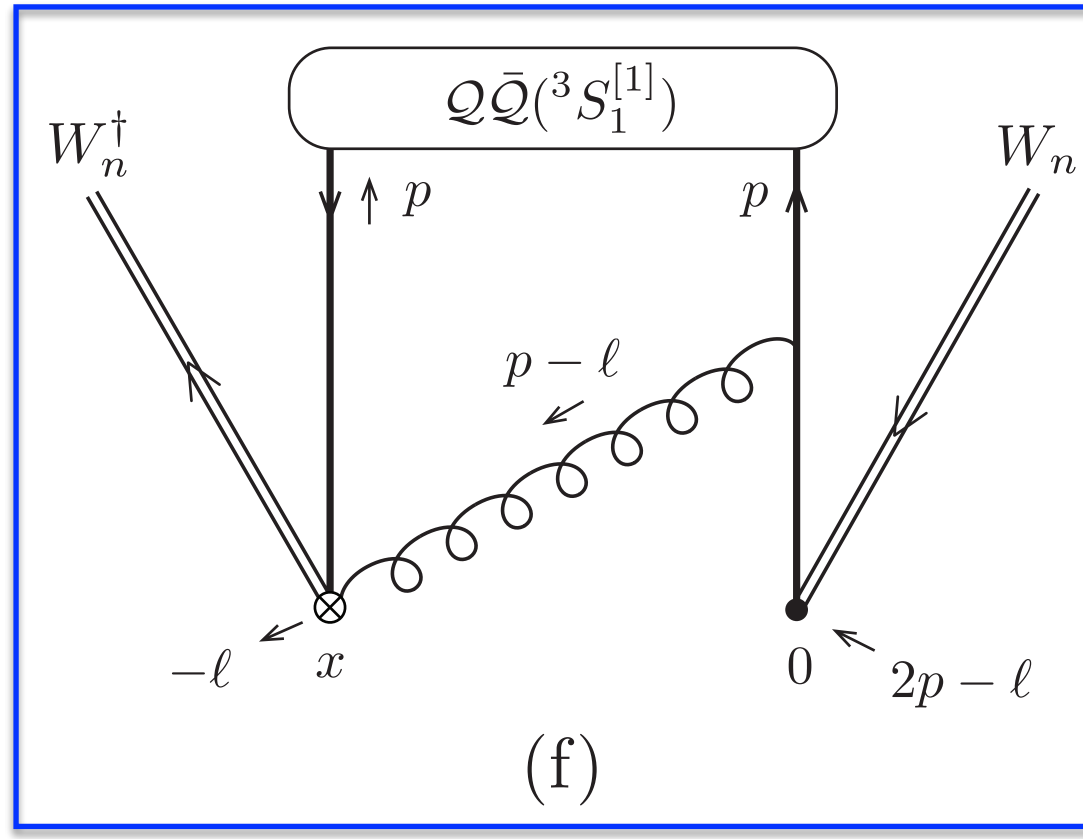


- In the Feynman gauge, (a) and (f) are power suppressed, (d) is 0 by $\bar{n} \cdot \bar{n} = 0$, and (b) and (c) include a soft propagator, and so only (e) contributes.
- In the light-cone gauge, only (f) contributes; the other diagrams involve a Wilson line vertex, which is 0 in light-cone gauge, or a soft propagator.

LO in different gauges



Feynman



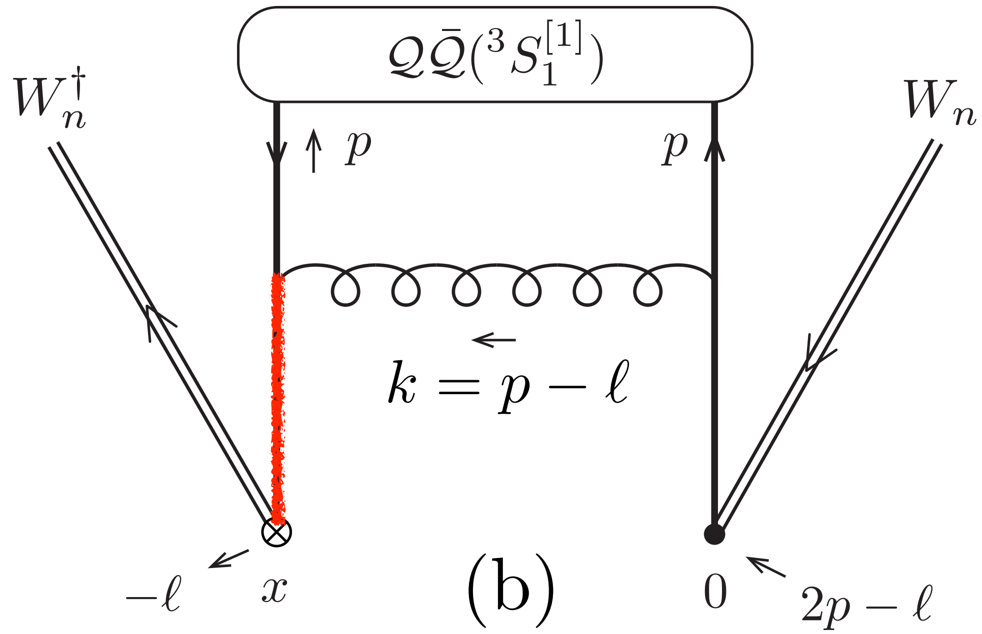
light-cone

$$A_{(\mathrm{e}),\mathrm{Feynman}}(\ell^-) = -\frac{ig_s^2 C_F \delta^{ba}}{\sqrt{2N_c} m p^+} \frac{1}{(-\ell^- + i\varepsilon)} \left(\frac{\not{\bar{n}}\not{n}\not{\epsilon}^*}{4}\right)^{\beta\alpha}$$

$$A_{(\mathrm{f}),\text{light-cone}}(\ell^-) = -\frac{2ig_s^2 C_F \delta^{ba}}{\sqrt{2N_c} m p^+} \frac{1}{(-\ell^- + i\varepsilon)} \left(\frac{\not{\bar{n}}\not{n}\not{\epsilon}^*}{4}\right)^{\beta\alpha}$$

- The light-cone gauge result deviates from the Feynman gauge result by a factor of 2!

Soft propagator diagram in light-cone gauge



- In Feynman and light-cone gauges, we discarded (b) because it involves a soft propagator with momentum ℓ (red-colored).

- The red-colored propagator carries soft momentum, and so it usually is not regarded as a proper diagram of radiative jet function.

$$\frac{1}{\ell^2 - m^2 + i\varepsilon}$$

- In the light-cone gauge, the gluon propagator is

$$\frac{i}{k^2 + i\varepsilon}\left[-g^{\mu\nu} + \frac{k^\mu \bar{n}^\nu + \bar{n}^\mu k^\nu}{k \cdot \bar{n}}\right]$$

- Let us consider separately the contributions of each of the three terms:

$$-g^{\mu\nu} \qquad k^\mu \bar{n}^\nu \qquad \bar{n}^\mu k^\nu$$

Soft propagator diagram in light-cone gauge

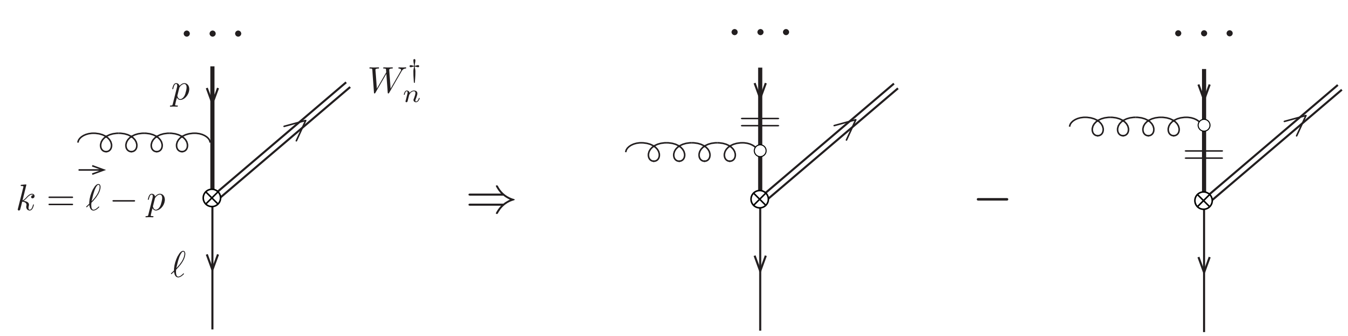
- The three terms in the light-cone gauge gluon propagator give

$$A^{(-g^{\mu\nu})}_{\text{(b),light-cone}}(\ell^-) = \frac{ig_s^2 C_F \delta^{ba}}{\sqrt{2N_c}mp^+}\frac{1}{(-\ell^- + i\varepsilon)}\left(\frac{\not{\bar{n}}\not{n}\not{\epsilon}^*}{4}\right)^{\beta\alpha} \times \frac{m^2}{(\ell^2 - m^2 + i\varepsilon)}.$$

$$\boxed{A^{(k^\mu \bar{n}^\nu)}_{\text{(b),light-cone}}(\ell^-) = \frac{ig_s^2 C_F \delta^{ba}}{\sqrt{2N_c}mp^+}\frac{1}{(-\ell^- + i\varepsilon)}\left(\frac{\not{\bar{n}}\not{n}\not{\epsilon}^*}{4}\right)^{\beta\alpha}.}$$

$$A^{(\bar{n}^\mu k^\nu)}_{\text{(b),light-cone}}(\ell^-) = \frac{ig_s^2 C_F \delta^{ba}}{\sqrt{2N_c}mp^+}\frac{1}{(-\ell^- + i\varepsilon)}\left(\frac{\not{\bar{n}}\not{n}\not{\epsilon}^*}{4}\right)^{\beta\alpha} \times \frac{\ell^+\ell^-}{(\ell^2 - m^2 + i\varepsilon)}.$$

- We can find that the soft propagator cancels only in the second term, and this is the precisely the gauge term we considered in the Ward identities (k^μ in $\epsilon^\mu(k) \to \epsilon^\mu(k) + k^\mu$)



Soft propagator diagram in light-cone gauge

- Then, keeping only the term without a soft propagator, we can finally find

$$A_{(\mathrm{e}),\mathrm{Feynman}}(\ell^-) = A_{(\mathrm{f}),\text{light-cone}}(\ell^-) + \underbrace{A^{(k^\mu \bar{n}^\nu)}_{(\mathrm{b}),\text{light-cone}}(\ell^-)}_{\text{from the soft propagator diagram}}$$

- In order for the radiative jet function to be gauge invariant,
  - ✓ we must include a soft-propagator diagram in conjunction with a gauge term, in which
  - ✓ k contracts to a vertex between the *n*-hard-collinear and soft line.
- This guarantees the cancellation of the soft propagator, resulting in the pure hard-collinear contributions in the radiative jet functions.
- In Feynman gauge, there is no gauge term to cancel the soft propagator, and so there is no contribution to the radiative jet function from the soft-propagator diagrams.
- We can observe a similar feature in $\mathcal{L}^{(2)}_{mq\xi_n}(x) = \bar{q}(x^+)\left(-mW_n^\dagger \xi_n\right)(x)$

Gauge invariance of a radiative jet function at $O(\lambda)$

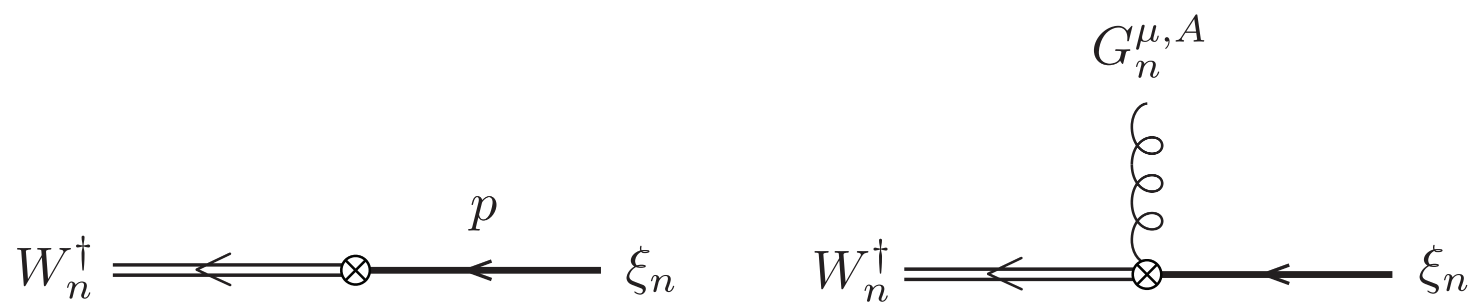
NLP SCET Lagrangian at $O(\lambda)$

$$\mathcal{L}^{(1)}_{q\xi_n}(x) = \bar{q}(x^+)\left(W_n^\dagger i\not{D}_{n\perp}\xi_n\right)(x)$$

$d^4x \sim O(\lambda^{-4})$

$\lambda^{3+1+1-4} = \lambda$

- The Feynman rule for the crossed vertex for $\mathcal{L}^{(1)}_{q\xi_n}$ is



$V_C^{(0)} = \not{p}_\perp$ $\qquad$ $V_C^{(1)} = g_s T^A \gamma_{\perp\mu}$

- The radiative jet function constructed from this Lagrangian $\mathcal{L}^{(1)}_{q\xi_n}$ arises in radiative leptonic *B*-meson decay, and in Higgs-boson decay to two photons through a *b*-quark loop.

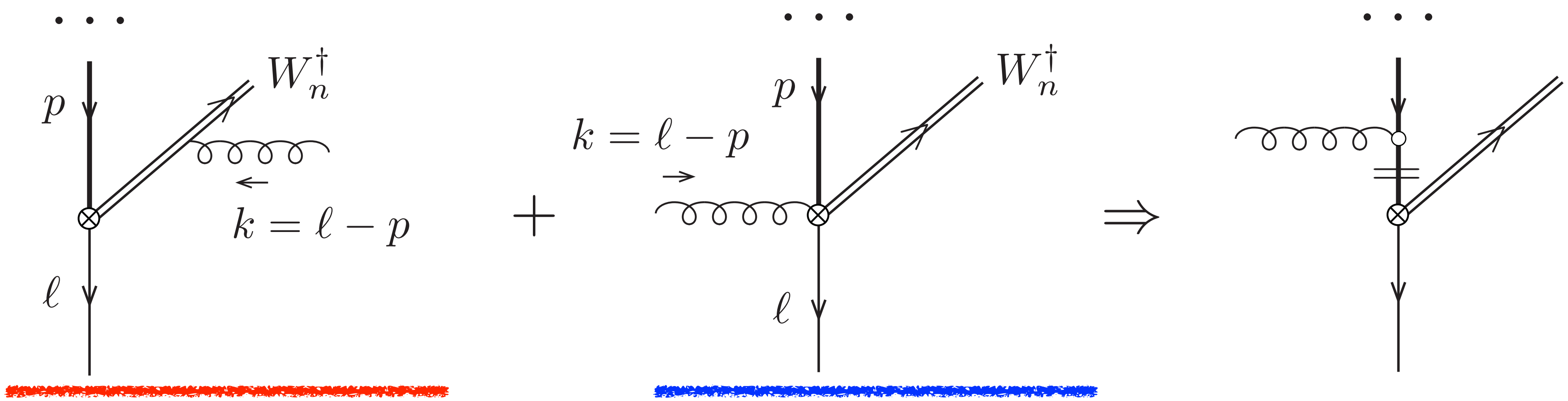
Bosch, Hill, Lange, Neubert, 2003
Liu, Neubert, 2020

- In some previous works, light-cone gauge calculations were performed without including the soft-propagator diagrams that we consider.

- Let us investigate whether this is justified.

NLP SCET Lagrangian at $O(\lambda)$

- Let us consider the vertex for $\mathcal{L}^{(1)}_{q\xi_n}$ at $O(g_s)$



- The corresponding amplitude and its application of the Ward identities are

$$C_1 = \epsilon^\mu(k)\left[\frac{(-ig_s)(i)\bar{n}_\mu}{\bar{n}\cdot k + i\varepsilon}\not{p}_\perp + g_s\gamma_{\perp\mu}\right]P_n \ldots$$

$$\Downarrow \quad \boxed{\epsilon^\mu(k) \to k^\mu}$$

$$C_1^{\rm gauge} = g_s(\not{p}_\perp + \not{k}_\perp)P_n \ldots = g_s\not{\ell}_\perp P_n \ldots$$

- In this case, we can discard the residual soft momentum $\ell_\perp$ because it is power suppressed: $p_\perp \sim k_\perp \sim O(\lambda)$ while $\ell_\perp \sim O(\lambda^2)$.
- Therefore, we do not need to consider the soft-propagator diagrams. The existing calculation of $O(\lambda)$ radiative jet functions in the light-cone gauge are justified.

Conclusion

- We have pointed out an unusual feature in the implementation of gauge invariance in SCET at $O(\lambda^2)$.
- The feature appears in the radiative jet functions that arise from the coupling of hard-collinear quarks and soft quarks.
- We have shown that one must include Feynman diagrams that contain a soft propagator in order to obtain a gauge-invariant result.
- The criterion for the inclusion of a soft propagator diagram is that a gluon momentum from “gauge term” in the gluon polarization sum must be contracted into a vertex that is adjacent to the soft propagator.
- We observed this feature at the Born level, but we expect more sophisticated realization of it at higher order corrections in α_s.
- As practical manner, it might be simpler to avoid this issue by calculating the radiative jet functions in Feynman gauge.

Backup slides

Graphical Ward identities

Collins,
Foundation of PQCD

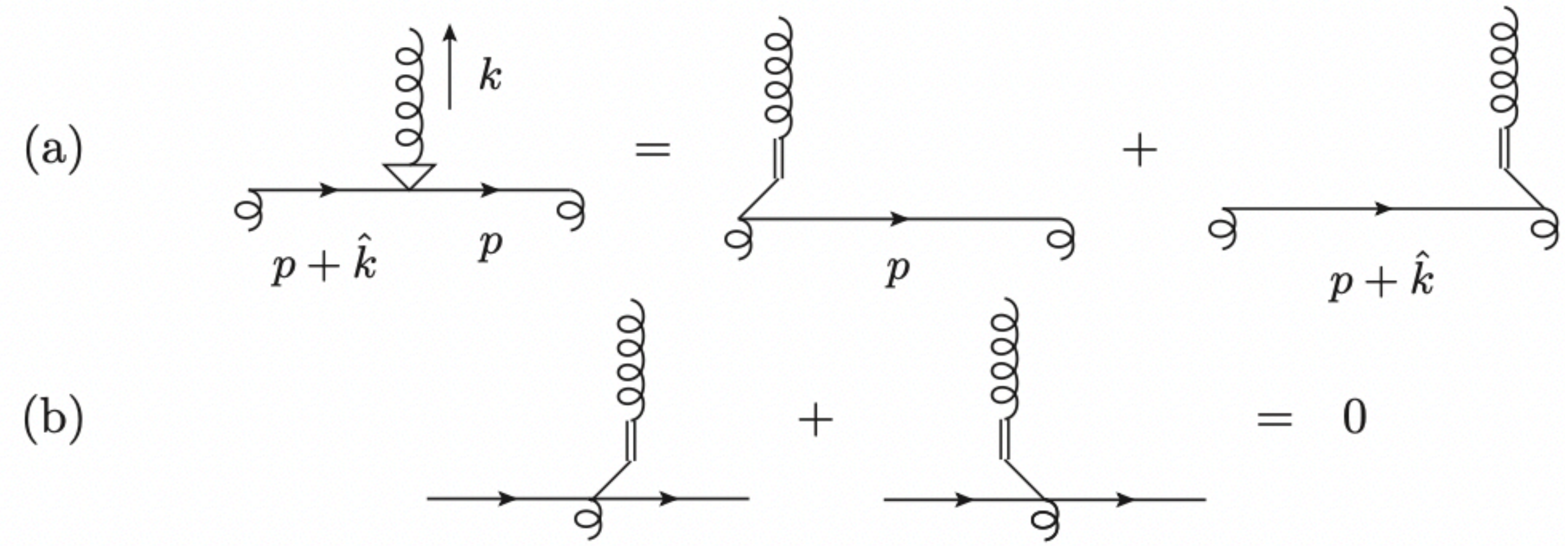


Fig. 10.12. Graphical elements of Ward identity: (a) application to line, (b) sum at vertex (in abelian gauge theory).

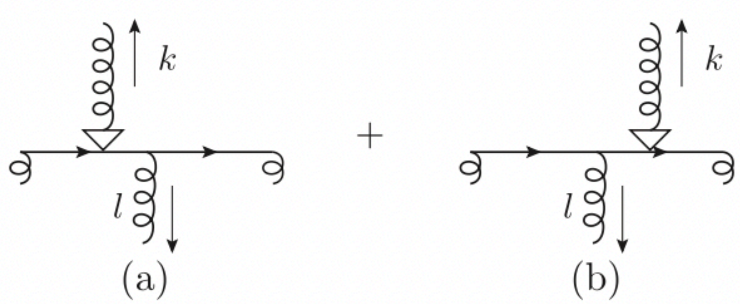


Fig. 10.13. Example of graphical structure which leads to the canceling terms in Fig. 10.12(b).

$$\frac{n_1^{\mu}}{k\cdot n_1 + i0}\,\frac{i}{\not p - m + i0}\,(-ig\not{\hat k})\,\frac{i}{\not p + \not{\hat k} - m + i0}$$
$$= \frac{i(-ign_1^{\mu})}{k\cdot n_1 + i0}\left[\frac{i}{\not p - m + i0} - \frac{i}{\not p + \not{\hat k} - m + i0}\right].$$

Feynman rules in Position space formalism

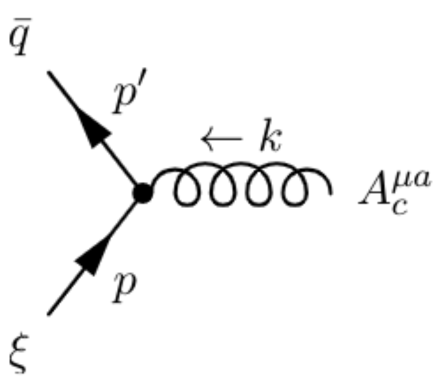
Beneke, Garny, Szafron, Wang, 2018

$$\mathcal{L}^{\mathrm{pos.}(1)}_{q\xi_n}(x) = \bar{q}(x^+)\left(W_n^\dagger i\not{D}_{n\perp}\xi_n\right)(x) + \mathrm{h.c.}$$

$$\mathcal{L}^{\mathrm{pos.}(2a)}_{q\xi_n}(x) = \bar{q}(x^+)\left\{W_n^\dagger\left[in\cdot D + i\not{D}_{n\perp}\frac{1}{i\bar{n}\cdot D_n}i\not{D}_{n\perp}\right]\frac{\not{\bar{n}}}{2}\xi_n\right\}(x) + \mathrm{h.c.}$$

$$\mathcal{L}^{\mathrm{pos.}(2b)}_{q\xi_n}(x) = \left[\bar{q}(x_\perp\cdot\overleftarrow{D}_{\mathrm{s}})\right](x^+, x_\perp)\left(W_n^\dagger i\not{D}_{n\perp}\xi\right)(x) + \mathrm{h.c.}$$

Soft-quark-collinear-quark-collinear-gluon vertex



$$ig_s t^a \begin{cases} 0 & \mathcal{O}(\lambda^0) \\ \Gamma_\mu(p) & \mathcal{O}(\lambda) \\ \left[n_{-\mu} + \gamma_{\perp\mu}\frac{\not{p}_\perp}{n_+p} + \frac{n_{+\mu}}{n_+k}\frac{p^2}{n_+p}\right]\frac{\not{n}_+}{2} - (p'X_\perp)\Gamma_\mu(p) & \mathcal{O}(\lambda^2) \end{cases} \tag{A.35}$$

where

$$\Gamma^\mu(p) = \gamma_\perp^\mu - \frac{\not{p}_\perp}{\bar{n}\cdot p}\bar{n}^\mu \qquad\qquad X^\mu = \partial^\mu\left[(2\pi)^4\delta^{(4)}\left(\sum p_{\mathrm{in}} - \sum p_{\mathrm{out}}\right)\right]$$

Gauge invariance in Position space formalism

$$
\begin{aligned}
i\mathcal{M}^{\text{gauge}}_{\text{pos.(1)}} &= ig_s T^a \left(\gamma_\perp^\mu - \frac{\not{p}_\perp}{\bar{n}\cdot p}\bar{n}^\mu\right) k_\mu \\
&= ig_s T^a \left[\left(\not{p}'_\perp - \not{p}_\perp\right) - \frac{\not{p}_\perp}{\bar{n}\cdot p}\bar{n}\cdot(p'-p)\right] \\
&= ig_s T^a \left[-\not{p}_\perp - \frac{\not{p}_\perp}{\bar{n}\cdot p}\bar{n}\cdot(-p)\right] + O(\lambda^2) \\
&= O(\lambda^2),
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{\text{gauge}}_{\text{pos.(2a)}} &= ig_s T^a \left(n_\mu + \gamma_{\perp\mu}\frac{\not{p}_\perp}{\bar{n}\cdot p} + \frac{\bar{n}_\mu}{\bar{n}\cdot k}\frac{p^2}{\bar{n}\cdot p}\right)\frac{\not{\bar{n}}}{2}k^\mu \\
&= ig_s T^a \left[n\cdot(p'-p) + (\not{p}'_\perp - \not{p}_\perp)\frac{\not{p}_\perp}{\bar{n}\cdot p} + \frac{p^2}{\bar{n}\cdot p}\right]\frac{\not{\bar{n}}}{2} \\
&= \boxed{ig_s T^a (n\cdot p')\frac{\not{\bar{n}}}{2}} + O(\lambda^3),
\end{aligned}
$$

$O(\lambda^2)$ so not gauge invariant without soft propagator

$$
\begin{aligned}
i\mathcal{M}^{\text{gauge}}_{\text{pos.(2b)}} &= ig_s T^a \left[-(p'\cdot X_\perp)\left(\gamma_{\perp\mu} - \frac{\not{p}_\perp}{\bar{n}\cdot p}\bar{n}_\mu\right)\right] k^\mu \\
&= ig_s T^a \left[-(p'\cdot X_\perp)\times O(\lambda^2)\right] \\
&= O(\lambda^3)
\end{aligned}
$$

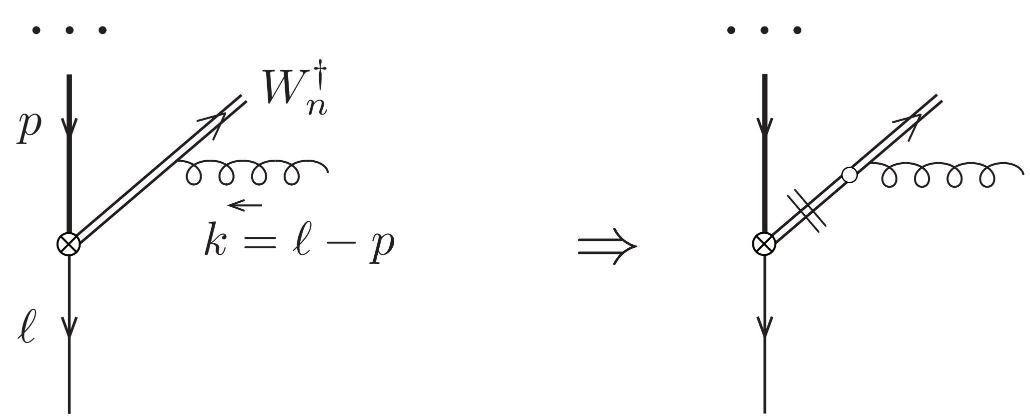
Gauge invariance of another radiative jet function at $O(\lambda^2)$

Another NLP SCET Lagrangian

$$\mathcal{L}^{(2)}_{mq\xi_n}(x) = \bar{q}(x^+)\left(-mW_n^\dagger \xi_n\right)(x) \qquad \begin{array}{l} d^4x \sim O(\lambda^{-4}) \\ \lambda^{3+2+1-4} = \lambda^2 \end{array}$$

- This Lagrangian describes a coupling between a soft and hard-collinear quark through the quark mass at $O(\lambda^2)$, with the vertex $(-m)$.

- The $O(g_s)$ diagram for this vertex is given by

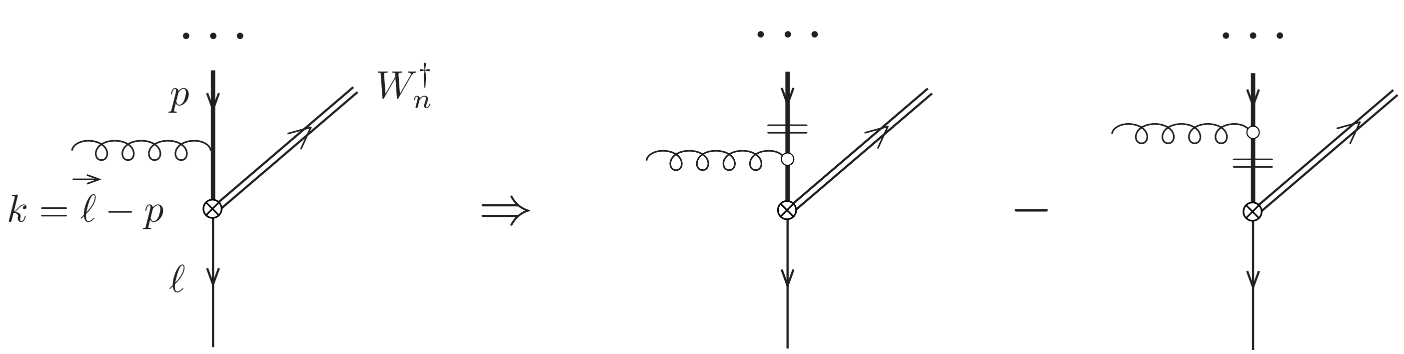


$$B_1 = \epsilon^\mu(k)\frac{(-ig_s)(i)\bar{n}_\mu}{\bar{n}\cdot k + i\varepsilon}(-m)P_n \dots,$$

$$\Downarrow \quad \boxed{\epsilon^\mu(k) \to k^\mu}$$

$$B_1^{\text{gauge}} = g_s(-m)P_n$$

- The soft propagator diagram gives



$$B_2 = (-m)P_n \frac{i}{\not{\ell} - m + i\varepsilon}\epsilon^\mu(k)(ig_s)\gamma_\mu$$

$$\Downarrow \quad \boxed{\epsilon^\mu(k) \to k^\mu}$$

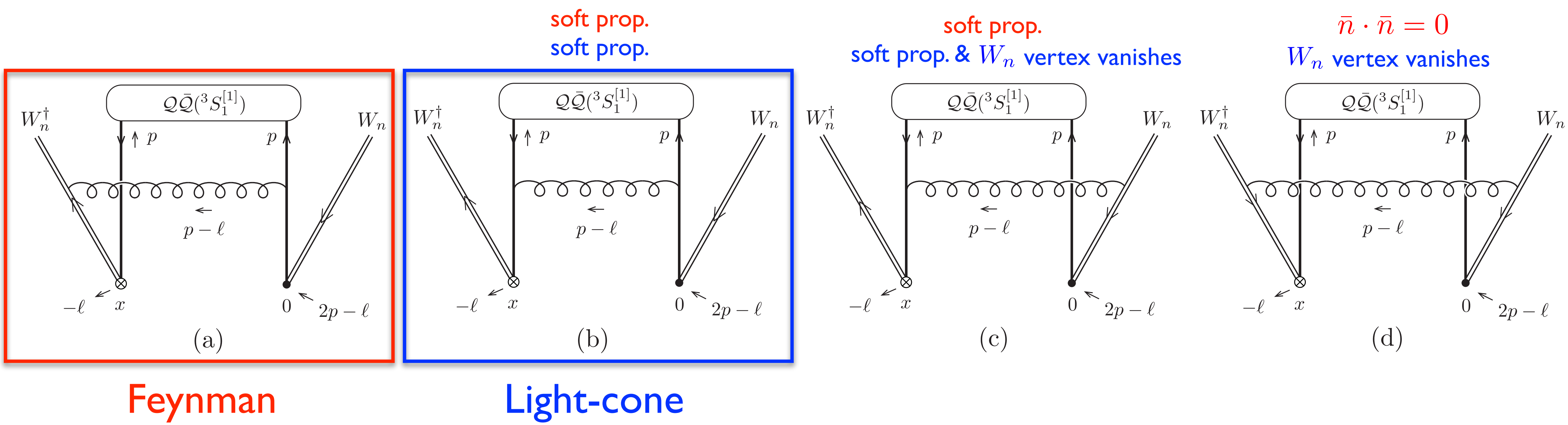
$$B_2^{\text{gauge}} = g_s(-m)P_n \frac{1}{\not{\ell} - m + i\varepsilon} \times [(\not{p} - m) - (\not{\ell} - m)] \dots$$

$$\boxed{B_1^{\text{gauge}} + B_2^{\text{gauge}} = 0}$$

B radiative jet function

$$B(\ell^-) = \int d^D x\, e^{-i\frac{\ell^- x^+}{2}} \langle Q\bar{Q}(^3S_1^{[1]}, p, p)|T\left(-mW_n^\dagger \xi_n\right)^{\beta,b}(x)(\bar{\xi}_n W_n)^{\alpha,a}(0)|0\rangle$$

- The first operator arises from soft-hard-collinear interaction $\mathcal{L}^{(2)}_{mq\xi_n}(x)$



- In the Feynman gauge, only diagram (a) contributes. (b) and (c) involve soft propagators, and (d) is 0 by $\bar{n}\cdot\bar{n}=0$.
- In the light-cone gauge, only diagram (b) is nonzero. It includes a soft propagator.
- From an analysis that is similar to that for the radiative jet function *A*, we can show that $B^{k^\mu \bar{n}^\nu}_{(\mathrm{b}),\text{light-cone}}(\ell^-) = B_{(\mathrm{a}),\text{Feynman}}(\ell^-)$