Small-*x* Factorization from Effective Field Theory

Duff Neill, Aditya Pathak, Iain Stewart
(+ thank you, Ira Rothstein)
arXiv:2303.13710

Aditya Pathak
SCET 23, UC Berkeley, 27.03.2023

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Outline

Introduction

- The small-x region and the BFKL equation
- LL resummation by Catani and Hautmann

EFT modes and power counting

Small-x factorization from Glauber SCET

- Factorization formula
- IR divergences
- Collinear function & BFKL evolution

BFKL & DGLAP resummation

- Consistency with twist factorization
- BFKL resummation of F_2 and F_L
- Comparison with previous work

Backup slides

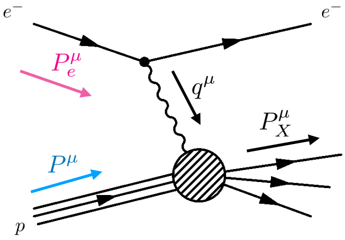
Outline

DIS review: Twist expansion

Consider unpolarized, inclusive DIS:

$$Q^2 = -q^2 > 0 \,, \qquad x_b = \frac{Q^2}{2P\cdot q} \,,$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_b\,\mathrm{d}Q^2}(e^- p \to e^- X) = \frac{2\pi y \alpha^2}{Q^4} L_{\mu\nu}(P_e, q)\, W^{\mu\nu}(P,q) \,.$$



$$W^{\mu\nu} = e_L^{\mu\nu}\frac{1}{x_b}F_L(x_b, Q^2) + e_2^{\mu\nu}\frac{1}{x_b}F_2(x_b, Q^2) \,.$$

Well-known twist-2 factorization:

$$\frac{1}{x_b}F_a(x_b, Q^2) = \sum_\kappa \int_{x_b}^1 \frac{\mathrm{d}\xi}{\xi}\, H_a^{(\kappa)}\left(\frac{x_b}{\xi}, Q, \mu\right) f_{\kappa/p}(\xi,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right) .$$

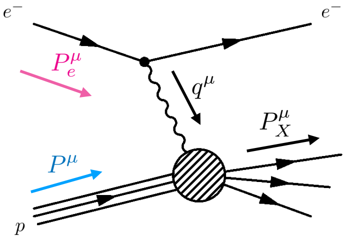
PDF absorbs all the IR divergences.

DIS review: Twist expansion

Consider unpolarized, inclusive DIS:

$$Q^2 = -q^2 > 0 \quad , \qquad x_b = \frac{Q^2}{2P\cdot q} \; ,$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_b\,\mathrm{d}Q^2}(e^- p \to e^- X) = \frac{2\pi y \alpha^2}{Q^4} L_{\mu\nu}(P_e, q)\, W^{\mu\nu}(P,q) \; .$$



Take Mellin Transform:

$$\bar{F}_p(N,Q^2) = \int_0^1 \frac{dx}{x} x^N \left(\frac{1}{x} F_p(x)\right), \qquad \Rightarrow \qquad \bar{F}_p^{(\kappa)}(N) = \sum_{\kappa'} \bar{H}_p^{(\kappa')}(N) \times \underbrace{\bar{\Gamma}_{\kappa'\kappa}(N,\epsilon)}_{\text{PDF}} \; .$$

IR divergences are exponentiated into PDFs (transition functions),

$$\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2), N, \epsilon) \equiv \mathrm{P}\exp\left(\int_0^{\alpha_s(\mu^2)} \frac{d\alpha}{\beta(\epsilon,\alpha)} \gamma^{s}(\alpha, N)\right)_{\kappa'\kappa} \; .$$

Mellin Transform

Mellin transform is very useful: (set $N = n + 1$)

$$\bar{f}(n) \equiv \int_0^1 \frac{dx}{x} x^{n+1} f(x)$$

Let us note that

$f(x)$	$\bar{f}(n)$	singularity in $x \to 0$
$\frac{1}{x} \ln^{\ell-1}(x)$	$\sim \frac{1}{n^\ell}$	pole at $n = 0$
$x^{p-1} \ln^{\ell-1}(x)$	$\sim \frac{1}{(n+p)^\ell}$	pole at $n = -p$

Small-x limit

Leading terms in $x_b \to 0$ limit:

$$\frac{\alpha_s^\ell}{x}\ln^{\ell-1}(x) \,,$$

Both coefficient function and the DGLAP anomalous dimension become singular in $x_b \to 0$ limit:

$$\bar{H}_a^{(\kappa)}(n) \sim \alpha_s \frac{\alpha_s}{n} + \alpha_s\left(\frac{\alpha_s}{n}\right)^2 + \alpha_s\left(\frac{\alpha_s}{n}\right)^3 + \dots \,,$$
$$\gamma_{gg}(n) \sim \frac{\alpha_s}{n} + \left(\frac{\alpha_s}{n}\right)^2 + \left(\frac{\alpha_s}{n}\right)^3 + \dots \,,$$
$$\gamma_{qg}(n) \sim \alpha_s \frac{\alpha_s}{n} + \alpha_s\left(\frac{\alpha_s}{n}\right)^2 + \alpha_s\left(\frac{\alpha_s}{n}\right)^3 + \dots$$

Our goal is to resum these leading logarithmic series.

- Notice that the leading terms in the coefficient function start at $\mathcal{O}(\alpha_s^2)$

The BFKL equation

Resummation of small-x_b logs invovles solving the BFKL equation. For a function $f(x, \boldsymbol{q}_\perp)$

$$f(x, \boldsymbol{q}_\perp) \sim x^{p-1} (\text{logs of } x) \,,$$

that satisfies BFKL equation in 4 dimensions:

$$x \frac{d}{dx} f(x, \boldsymbol{q}_\perp) = (p-1) f(x, \boldsymbol{q}_\perp) + c \big[K \otimes_\perp f\big](\boldsymbol{q}_\perp)$$

where

$$\big[K \otimes_\perp f\big](\boldsymbol{q}_\perp) \equiv (2\pi) \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \left\{ \frac{2 f(\boldsymbol{k}_\perp)}{(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2} - \frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2 (\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2} f(\boldsymbol{q}_\perp) \right\} ,$$

In the n-space we have an iterative equation

$$\bar{f}(n, \boldsymbol{q}_\perp) = \frac{1}{n+p} \times \underbrace{f(x=1, \boldsymbol{q}_\perp)}_{\text{Boundary condition}} - \frac{c}{n+p} \big[K \otimes_\perp \bar{f}(n)\big](\boldsymbol{q}_\perp)$$

The BFKL equation

Resummation of small-x_b logs invovles solving the BFKL equation. For a function $f(x,\boldsymbol{q}_\perp)$

$$f(x,\boldsymbol{q}_\perp) \sim x^{p-1}\big(\text{logs of } x\big)\,,$$

that satisfies BFKL equation in 4 dimensions:

$$x\frac{d}{dx}f(x,\boldsymbol{q}_\perp) = (p-1)f(x,\boldsymbol{q}_\perp) + c\big[K\otimes_\perp f\big](\boldsymbol{q}_\perp)$$

where

$$\big[K\otimes_\perp f\big](\boldsymbol{q}_\perp) \equiv (2\pi)\int\frac{\mathrm{d}^2k_\perp}{(2\pi)^2}\left\{\frac{2f(\boldsymbol{k}_\perp)}{(\boldsymbol{q}_\perp-\boldsymbol{k}_\perp)^2} - \frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp-\boldsymbol{k}_\perp)^2}f(\boldsymbol{q}_\perp)\right\}\,,$$

Eigenfunctions of BFKL Kernel:

$$\left[K\otimes_\perp\left(\frac{1}{\boldsymbol{k}_\perp^{2(1-\gamma)}}e^{\mathrm{i}n\phi}\right)\right](\boldsymbol{q}_\perp) = \chi(n,\gamma)\frac{1}{\boldsymbol{q}_\perp^{2(1-\gamma)}}e^{\mathrm{i}n\phi}\,, \qquad 0<\mathrm{Re}\,\gamma<1\,.$$

Bad boundary condition = IR divergence!

What happens for $\gamma = 0$?

$$\gamma = 0: \qquad \left[K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^2}\right](\boldsymbol{q}_\perp) = \frac{1}{\boldsymbol{q}_\perp^2}(2\pi)\int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2}$$

This Integral is divergent! But we can make sense of it in dimensional regularization:

$$\begin{aligned}(2\pi)I_\epsilon[\boldsymbol{q}_\perp^2] &\equiv (2\pi)\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{\mathrm{d}^{2-2\epsilon} k_\perp}{(2\pi)^{2-2\epsilon}}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2} \\ &= \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\Gamma(-\epsilon)e^{\epsilon\gamma_E}\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ &= -\frac{1}{\epsilon} + \log\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\,.\end{aligned}$$

Bad boundary condition = IR divergence!

What happens for $\gamma = 0$?

$$\gamma = 0: \qquad \left[K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^2}\right](\boldsymbol{q}_\perp) = \frac{1}{\boldsymbol{q}_\perp^2}(2\pi)\int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2}$$

This Integral is divergent! But we can make sense of it in dimensional regularization:

$$\begin{aligned}(2\pi) I_\epsilon[\boldsymbol{q}_\perp^2] &\equiv (2\pi)\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{\mathrm{d}^{2-2\epsilon} k_\perp}{(2\pi)^{2-2\epsilon}}\frac{\boldsymbol{q}_\perp^2}{\boldsymbol{k}_\perp^2(\boldsymbol{q}_\perp - \boldsymbol{k}_\perp)^2} \\ &= \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\Gamma(-\epsilon)e^{\epsilon\gamma_E}\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ &= -\frac{1}{\epsilon} + \log\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\,.\end{aligned}$$

This is relevant: Nature produces bad boundary conditions for the BFKL equation and these IR divergences go into the PDF, but *not every* IR divergence is generated this way.

LL small-x resummation by Catani and Hautmann

LL small-x_b resummation consistent with twist factorization by Catani and Hautmann [CH94]:

$$\bar{F}_L^{(g)}(n) = h_L(\gamma_{gg}) \times R(n) \times \left(\frac{Q^2}{\mu^2}\right)^{\gamma_{gg}} \times \bar{\Gamma}_{gg}\,,$$

- h_L: describes coupling with photon, IR finite, defined via an *off-shell* cross section.
- R: scheme chosen to factorize the IR divergences.

[CH94] resummed $\gamma_{gg}(\alpha_s, n)$ using a separate calculation of *gluon Green's function* that satisfies

$$\bar{\mathcal{F}}_g^{(0)}(n, \boldsymbol{q}_\perp) = \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) + \frac{\bar{\alpha}_s}{n}\big[K \otimes_\perp \bar{\mathcal{F}}_g^{(0)}(n)\big](\boldsymbol{q}_\perp)\,, \qquad \bar{\alpha}_s \equiv \frac{\alpha_s C_A}{\pi}$$

A special property of the LL series and F_L channel: All the IR divergences at LL for F_L are generated by BFKL equation and can be absorbed into the same $\bar{\Gamma}_{gg}$:

$$\bar{\mathcal{F}}_g^{(0)}(n, \boldsymbol{q}_\perp) = \frac{1}{\pi \boldsymbol{k}_\perp^2} \times \gamma_{gg} \times \tilde{R}(n, \boldsymbol{k}_\perp, \epsilon) \times \bar{\Gamma}_{gg}\,.$$

LL small-x resummation by Catani and Hautmann

Notice how BFKL kernel acts on $\delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp)$:

$$K \otimes_\perp \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) \sim \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon}$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon} \sim \frac{1}{\epsilon}\,\frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-2\epsilon}$$

$$\vdots$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\ell\epsilon} \sim \frac{1}{\ell\epsilon}\,\frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-(\ell+1)\epsilon}$$

This generates a series solution for $\bar{\mathcal{F}}_g^{(0)}$ (= resummation):

$$\bar{\mathcal{F}}_g^{(0)} \sim \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) + \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\sum_{\ell=1}^{\infty} c_\ell(\epsilon)\left(\frac{\bar{\alpha}_s}{n}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon}\right)^{\ell}, \qquad c_\ell(\epsilon) = \frac{1}{\ell!}\left(-\frac{1}{\epsilon}\right)^{\ell}\left(1+\mathcal{O}(\epsilon^2)\right)$$

In the case of F_L channel, these are all the IR divergences at LL, so we know $\gamma_{gg}(n)$ and $\bar{\Gamma}_{gg}$. Rest goes into $\bar{H}_L^{(g)}$ coefficient function.

LL small-x resummation by Catani and Hautmann

Notice how BFKL kernel acts on $\delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp)$:

$$K \otimes_\perp \delta^{(2-2\epsilon)}(\boldsymbol{q}_\perp) \sim \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon}$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\epsilon} \sim \frac{1}{\epsilon}\,\frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-2\epsilon}$$

$$\vdots$$

$$K \otimes_\perp \frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-\ell\epsilon} \sim \frac{1}{\ell\epsilon}\,\frac{1}{\boldsymbol{k}_\perp^{2-2\epsilon}}\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right)^{-(\ell+1)\epsilon}$$

> Resummation of F_2 and γ_{qg} is not straightforward in this framework, because F_2 involves IR divergences NOT generated by BFKL evolution alone!. They introduced a new *quark's Green's function* to capture this non-BFKL divergence. This approach has not been extended beyond LL.

Goal of this work: provide a new framework for higher order resummation using a factorization derived in SCET with Glauber operators of Rothstein and Stewart [RS16].

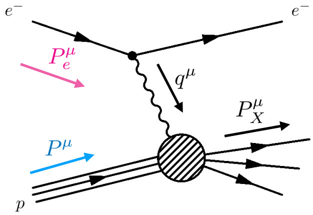
See also Ciafaloni et al. [Cia+04], Altarelli, Ball, and Forte [ABF06], and Thorne [Tho01] and references therein.

Outline

EFT modes and power counting

Center of mass light cone coordinates:

$$P^\mu = \frac{\sqrt{s}}{2} n^\mu \,, \qquad P_e^\mu = \frac{\sqrt{s}}{2} \bar{n}_\mu$$



Power counting parameters:

$$\lambda \sim x_b \qquad \text{and} \qquad \lambda' \sim \frac{\Lambda_{\rm QCD}}{Q} \,.$$

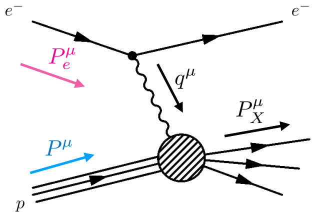
Two possible scenarios based on the scaling of the invariant mass of hadronic state:

	Hard scattering		***Forward scattering***
$\frac{\boldsymbol{P_X^2}}{\boldsymbol{s}} = \frac{(q+P)^2}{s} = \frac{Q^2}{s}\frac{(1-x_b)}{x_b}$	$\sim \lambda^0$	or	$\sim \lambda$
$\boldsymbol{q^\mu} = -\frac{Q^2}{\sqrt{s}}\frac{n^\mu}{2} + \frac{Q^2}{x_b\sqrt{s}}\frac{\bar{n}^\mu}{2} + q_\perp^\mu$	$\sim \sqrt{s}(1, \lambda, \sqrt{\lambda})$	or	$\sim \sqrt{s}(\lambda, \lambda^2, \lambda)$
	(collinear to e^-)		($\bar{n}s$ Glauber)

EFT modes and power counting

Center of mass light cone coordinates:

$$P^\mu = \frac{\sqrt{s}}{2} n^\mu \,, \qquad P_e^\mu = \frac{\sqrt{s}}{2} \bar{n}_\mu$$



Power counting parameters:

$$\lambda \sim x_b \quad \text{and} \quad \lambda' \sim \frac{\Lambda_{\text{QCD}}}{Q} \,.$$

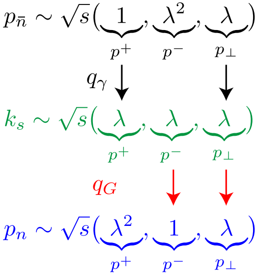
Collinear modes in the proton:	$p_c^\mu \sim \sqrt{s}\left(\frac{\Lambda_{\text{QCD}}^2}{s}, 1, \Lambda_{\text{QCD}}\right) \sim \sqrt{s}\left((\lambda\lambda')^2, 1, \lambda\lambda'\right)$
Small-x_b resummation requires collinear modes with higher virtuality $p_n^2 \sim Q^2$:	$p_n^\mu \sim \sqrt{s}(\lambda^2, 1, \lambda)$

We do not enforce $\lambda' \ll 1$ until later.

EFT modes and power counting

Forward scattering

P_X^2/s	$\sim \lambda$
q^μ	$\sim \sqrt{s}(\lambda, \lambda^2, \lambda)$
p_n^μ	$\sim \sqrt{s}(\lambda^2, 1, \lambda)$



The photon cannot interact directly with collinear mode without knocking it offshell. The leading terms start at $\mathcal{O}(\alpha_s^2)$ due to intermediate soft sector:

$$p_s = (p_s^+, p_s^-, p_{s\perp}) \sim \sqrt{s}(\lambda, \lambda, \lambda)\,.$$

Need additional Glauber modes for soft-collinear interaction:

$$q_G^\mu = q'^\mu \sim \sqrt{s}(\lambda^2, \lambda, \lambda)\,.$$

Consistent with $P_X^2/s \sim \lambda$ since now we have either soft or collinear particles in the final state:

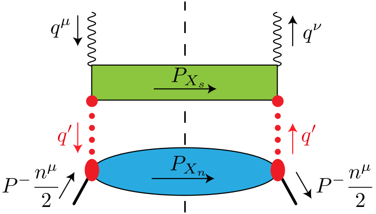
$$P_X^2 \sim (p_n + p_s)^2 \sim p_n^- p_s^+ \sim s\lambda\,.$$

Outline

Small-x factorization formula

We include *two insertions* of the ns Glauber action:

$$S_G^{ns} = 8\pi\alpha_s \sum_{i,j,A} \int \mathrm{d}^d y \int \mathrm{d}^d x \int \frac{\mathrm{d}^d q'}{(2\pi)^d} \frac{e^{\mathrm{i}(x-y)\cdot q'}}{\boldsymbol{q}_\perp'^2} \mathcal{O}_n^{iA}(x)\mathcal{O}_s^{j_n A}(y) \,.$$



Factorization formula at NLL:

$$W^{\alpha\beta}(q,P) = \int \mathrm{d}^{d-2} q'_\perp \; S^{\alpha\beta}\Big(q, q'_\perp, \frac{\nu}{x_b P^-}, \epsilon\Big) \; C\Big(q'_\perp, P, \frac{\nu}{P^-}, \epsilon\Big) + \dots .$$

The collinear and soft functions are defined as

$$C \equiv \frac{1}{\pi\nu}\frac{1}{\boldsymbol{q}_\perp'^2} \sum_{i,j,A} \int \frac{\mathrm{d}q'^+}{2\pi} \int \mathrm{d}^d x \, e^{\mathrm{i}\frac{x^- q'^+}{2} + \mathrm{i} x_\perp \cdot q'_\perp} \langle P | \mathcal{O}_n^{i\,A}(x) \mathcal{O}_n^{j\,A}(0) | P \rangle_\nu \,,$$

$$S^{\alpha\beta} \equiv \frac{\nu}{\boldsymbol{q}_\perp'^2} \frac{(2\pi\iota\mu^2)^{4-d}(8\pi\alpha_s(\mu^2))^2}{16\pi^2 (N_c^2-1)} \sum_{i,j,A} \int \frac{\mathrm{d}q'^-}{4\pi} \int \mathrm{d}^d z \, e^{\mathrm{i} z\cdot q} \int \mathrm{d}^d y_L \mathrm{d}^d y_R$$

$$\times e^{-\mathrm{i}\frac{q'^-(y_L^+ - y_R^+)}{2} - \mathrm{i} q'_\perp \cdot (y_{L\perp} - y_{R\perp})} \langle 0 | \bar{T}\{J^\alpha(z) \mathcal{O}_s^{i_n\,A}(y_L)\} T\{J^\beta(0)\mathcal{O}_s^{j_n\,A}(y_R)\} | 0 \rangle_\nu \,.$$

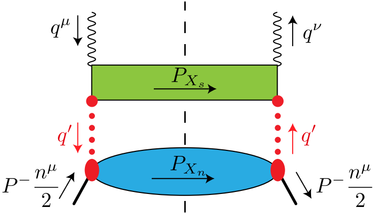
Small-x factorization formula

We include *two insertions* of the ns Glauber action:

$$S_G^{ns} = 8\pi\alpha_s \sum_{i,j,A} \int \mathrm{d}^d y \int \mathrm{d}^d x \int \frac{\mathrm{d}^d q'}{(2\pi)^d} \frac{e^{\mathrm{i}(x-y)\cdot q'}}{\boldsymbol{q}_\perp'^2} \mathcal{O}_n^{iA}(x)\mathcal{O}_s^{j_n A}(y)\,.$$

Factorization formula at NLL:

$$W^{\alpha\beta}(q,P) = \int \mathrm{d}^{d-2} q'_\perp \; S^{\alpha\beta}\Big(q, q'_\perp, \frac{\nu}{x_b P^-}, \epsilon\Big) \; C\Big(q'_\perp, P, \frac{\nu}{P^-}, \epsilon\Big) + \dots .$$



Here small-x_b logs are resummed via *rapidity evolution* for $\nu_S \sim x_b P^-$ and $\nu_C \sim P^-$

$$\frac{\nu_S}{\nu_C} = x_b$$

IR divergences

The convolution itself generates IR divergences as nothing prevents $\boldsymbol{q}'_\perp$ from entering the IR region:

$$\frac{1}{x_b} F_a(q,P) = \int_0^\infty \mathrm{d}^{d-2} q'_\perp \, S_a\left(q, q'_\perp, \frac{\nu}{x_b P^-}, \epsilon\right) C\left(q'_\perp, \frac{\nu}{P^-}, \epsilon\right), \qquad [S^{\mu\nu}] = 4-d\,, \qquad [C] = -2\,.$$

To see this explicitly, let us note that the SCET$_{\text{II}}$ collinear function has the all-orders expansion:

$$C(q'_\perp, \epsilon) = \frac{1}{\boldsymbol{q}'^2_\perp} \sum_{\ell=0}^\infty C^{(\ell)}(\alpha_s, \epsilon) \left(\frac{\boldsymbol{q}'^2_\perp}{\mu^2}\right)^{-\ell\epsilon}.$$

Alternative form of the factorization formula:

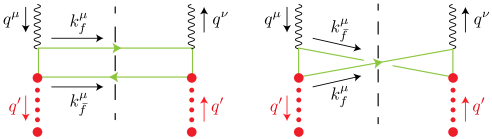
$$\frac{1}{x_b} F_a = \sum_{\ell=0}^\infty \left(\frac{\boldsymbol{q}^2_\perp}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a(\gamma = -\ell\epsilon)\,.$$

The γ-transform of the soft function:

$$\tilde{S}_a(\gamma) \sim \int \frac{\mathrm{d}^{2-2\epsilon} q'_\perp}{\boldsymbol{q}'^2_\perp} \left(\frac{\boldsymbol{q}'^2_\perp}{\mu^2}\right)^\gamma S_a(q_\perp, q'_\perp, \epsilon)\,.$$

IR divergences

Leading order soft function calculated from



is IR finite for $\gamma \neq 0$:

$$\tilde{S}_2^{\rm LO}(\gamma) = \alpha_s^2 n_f T_F \left(\frac{\nu}{x_b P^-}\right)\left(\frac{\pi^2\left(-3\gamma^2+3\gamma+2\right)\csc^2(\pi\gamma)}{8\Gamma(\frac{5}{2}-\gamma)\Gamma(\frac{3}{2}+\gamma)}\right) + \mathcal{O}(\epsilon)\,,$$

$$\tilde{S}_L^{\rm LO}(\gamma) = \alpha_s^2 n_f T_F \left(\frac{\nu}{x_b P^-}\right)\left(\frac{\pi^2\left(-\gamma+1\right)\csc^2(\pi\gamma)}{4\Gamma(\frac{5}{2}-\gamma)\Gamma(\frac{3}{2}+\gamma)}\right) + \mathcal{O}(\epsilon)\,.$$

But notice the LO collinear function:

$$C_\kappa^{\rm LO}(q'_\perp) = \frac{P^-}{\nu}\frac{c_\kappa}{\pi \boldsymbol{q}'^2_\perp}\,, \qquad c_\kappa = C_F, C_A \qquad \text{(bad boundary condition!)}$$

IR divergences

In the convolution the collinear function forces us to set $\gamma = -\ell\epsilon$,

$$\frac{1}{x_b} F_a = \sum_{\ell=0}^{\infty} \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a\left(\gamma = -\ell\epsilon \right).$$

which implies

$$\lim_{\epsilon\to 0} \tilde{S}_2^{\text{LO}}(-\ell\epsilon) = \frac{2\alpha_s^2 n_f T_F}{3\pi} \frac{1}{(\ell+1)(\ell+2)} \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + \mathcal{O}(\epsilon^0)\right),$$

$$\lim_{\epsilon\to 0} \tilde{S}_L^{\text{LO}}(-\ell\epsilon) = \frac{2\alpha_s^2 n_f T_F}{3\pi} \frac{1}{(\ell+1)} \left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right).$$

The $\tilde{S}_a$ soft function will contribute to the PDF despite being a vacuum matrix element.

IR divergences

In the convolution the collinear function forces us to set $\gamma = -\ell\epsilon$,

$$\frac{1}{x_b} F_a = \sum_{\ell=0}^{\infty} \left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-(\ell+2)\epsilon} C^{(\ell)} \times \tilde{S}_a\left(\gamma = -\ell\epsilon\right).$$

We find that for $\gamma \neq 0$, $\tilde{S}_L$ and $\tilde{S}_2$ are proportional to the off-shell cross section that appear in [CH94]:

$$\tilde{S}_2(\gamma, \epsilon = 0) = \left(\frac{\nu}{x_b P^-}\right) \alpha_s \frac{h_2(\gamma)}{\gamma^2}, \qquad (1)$$

$$\tilde{S}_L(\gamma, \epsilon = 0) = \left(\frac{\nu}{x_b P^-}\right) \alpha_s \frac{h_L(\gamma)}{\gamma}.$$

This is not the right limit for us and the full ϵ dependence is needed to perform small-x_b resummation.

> In [CH94] these IR divergences were separately captured in the gluon and quark Green's functions for F_2 and F_L. For us they are automatically accounted for via the same soft function $\tilde{S}_a$.

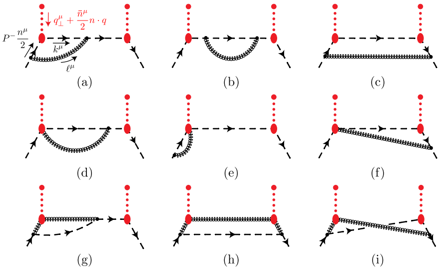
Collinear function at NLO

We computed the collinear function at NLO

$$C_q^{\text{NLO}} = \bar{\alpha}_s C_q^{\text{LO}} \times (-2\pi)\, I_\epsilon\big[\boldsymbol{q}_\perp'^2\big] \left(\frac{1}{\eta} + \ln\left(\frac{\nu}{P^-}\right) + \frac{3}{4} \right),$$

$$C_g^{\text{NLO}} = \bar{\alpha}_s C_g^{\text{LO}} \times (-2\pi)\, I_\epsilon\big[\boldsymbol{q}_\perp'^2\big]$$

$$\times \left(\frac{1}{\eta} + \ln\left(\frac{\nu}{P^-}\right) + \frac{11}{12} - \frac{n_f T_R}{4 C_A}\left(1 - \frac{1}{3(1-\epsilon)}\right)\right),$$



$$(2\pi) I_\epsilon\big[\boldsymbol{r}_\perp^2\big] = -\frac{1}{\epsilon} + \ln\left(\frac{\boldsymbol{r}_\perp^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\,, \qquad \bar{\alpha}_s \equiv \frac{\alpha_s C_A}{\pi}$$

We see that the one-loop contribution is IR divergent and exhibits a rapidity divergence.

Process independence and the BFKL equation

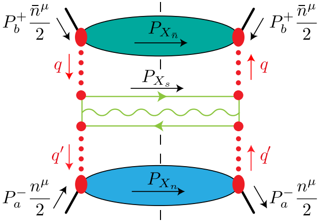
Rothstein and Stewart [RS16] showed that for $pp \to pp$ forward scattering

$$\sigma^{pp\to pp} \sim C_n \otimes S^{pp} \otimes C_{\bar{n}}$$

and S^{pp} satisfies the BFKL equation:

$$\nu \frac{\mathrm{d}}{\mathrm{d}\nu} S^{pp} \sim +2\bar{\alpha}_s \iota^\epsilon K \otimes_\perp S^{pp}$$

The collinear function is process independent and is expected to satisfy the BFKL equation from RG consistency:



Drell-Yan

$$\nu \frac{d}{d\nu} C = -C - \bar{\alpha}_s \iota^\epsilon K \otimes_\perp C\,.$$

The predicted rapidity logarithm agrees with our NLO result:

$$C_{\kappa\,\mathrm{LL}} = \frac{\nu}{P^-}\frac{c_\kappa}{\pi \boldsymbol{q}_\perp'^2}\left(1 - \bar{\alpha}_s(2\pi) I_\epsilon\big[\boldsymbol{q}_\perp'^2\big] \ln\Big(\frac{\nu}{P^-}\Big)\right) + \mathcal{O}(\alpha_s^2)\,.$$

Leading log small-x_b resummation

Setting $\nu = \nu_S$ trivializes rapidity logs in the soft function:

$$\frac{1}{x_b} F_a^\kappa(x_b, Q^2) = \int \mathrm{d}^{d-2} q'_\perp S_a\big(1, q_\perp, q'_\perp, \epsilon\big) C_\kappa\big(x_b, q'_\perp, \epsilon\big)$$

Mellin space :

$$\bar{C}_\kappa\big(n, q'_\perp, \epsilon\big) = \frac{c_\kappa}{n\pi \boldsymbol{q}'^2_\perp} + \frac{\bar{\alpha}_s \iota^\epsilon}{n} K \otimes_\perp \bar{C}_\kappa\big(n, q'_\perp, \epsilon\big) \quad , \quad c_\kappa = C_F, C_A \,, \quad \bar{\alpha}_s = \frac{\alpha_s C_A}{\pi}$$

Solve for $\bar{C}_\kappa$ as a power series as before:

$$\bar{C}_{\kappa,\mathrm{LL}}(n, q'_\perp, \epsilon) = \frac{1}{n} \frac{c_\kappa}{\pi \boldsymbol{q}'^2_\perp} \sum_{\ell=0}^{\infty} c_{\ell+1}(\epsilon) \left(\frac{\bar{\alpha}_s}{n} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{\boldsymbol{q}'^2_\perp}{\mu^2}\right)^{-\epsilon} \right)^\ell , \qquad c_\ell(\epsilon) = \frac{1}{\ell!} \left(\frac{-1}{\epsilon}\right)^\ell \Big(1 + \mathcal{O}(\epsilon^2)\Big)$$

Now include the soft contribution to arrive at small-x_b resummed structure functions:

$$\bar{F}^\kappa_{a,\mathrm{LL}}(n, Q^2) = \frac{c_\kappa}{n\pi} \left(\frac{\boldsymbol{q}^2_\perp}{\mu^2}\right)^{-2\epsilon} \sum_{\ell=0}^{\infty} d_{a,\ell+1}(\epsilon) \left(\frac{\bar{\alpha}_s}{n} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{\boldsymbol{q}^2_\perp}{\mu^2}\right)^{-\epsilon} \right)^\ell \quad , \quad d_{a,\ell+1}(\epsilon) \equiv c_{\ell+1}(\epsilon) \tilde{S}_a(1, -\ell\epsilon, \alpha_s, \epsilon)$$

Outline

Small-x vs. twist expansion

Here we are dealing with two different power expansions simultaneously:

$$\lambda \sim x_b \qquad \text{and} \qquad \lambda' \sim \frac{\Lambda_{\text{QCD}}}{Q} \,.$$

Key subtleties:

- Small-x_b and twist expansions *do not commute*.
- Both expansions have terms that are leading power in one but subleading in the other.

Consider the fixed order series: Leading twist-2 contributions at $\mathcal{O}(\alpha_s^0)$ and $\mathcal{O}(\alpha_s)$ are actually power suppressed in x_b-expansion. For example,

$$H_L^{(g)}(x) \sim \alpha_s\, x(1-x) + \mathcal{O}(\alpha_s^2) \qquad \Leftrightarrow \qquad \bar{H}_L^{(g)}(x) \sim \alpha_s\Big(\frac{1}{n+2} - \frac{1}{n+3}\Big) + \mathcal{O}(\alpha_s^2)$$

Thus in connecting with the twist expansion we will have to include power suppressed pieces. (See an illustration in the backup.)

BFKL Resummation of F_L

We set $\mu^2 = Q^2$ and start with formula involving unknown pieces (HP = higher power)

$$\bar{F}^g_{L,\text{HP}} + \bar{F}^g_{L,\text{LL}}(n) = \bar{H}^{(g)}_L\Big(n, \frac{Q^2}{\mu^2} = 1, \alpha_s\Big)\bar{\Gamma}_{gg}(\alpha_s, n) \;.$$

Parameterize the the terms we want to determine for LL results as

$$\bar{H}^{(g)}_L = \frac{\alpha_s}{\pi}\sum_{k=0}^{\infty}\epsilon^k h^{(0,k)}_{L,g} + \frac{\alpha_s}{\pi}\sum_{\ell=1}^{\infty}\Big(\frac{\alpha_s}{\pi n}\Big)^{\ell}\sum_{k=0}^{\infty}\epsilon^k h^{(\ell,k)}_{L,g}\,,$$

$$\gamma_{gg} = \sum_{\ell=1}^{\infty}\gamma_{gg,\ell-1}\Big(\frac{\bar{\alpha}_s}{\pi}\Big)^{\ell}\,,$$

$$\bar{F}^g_{L,\text{HP}} = \frac{\alpha_s}{\pi}\sum_{k=-1}^{\infty}\epsilon^k f^{(k)}_{L,g}\,.$$

We have truncated the higher power pieces to $\mathcal{O}(\alpha_s)$ which is sufficient for LL resummation in small-x_b.

BFKL Resummation of F_L

We set $\mu^2 = Q^2$ and start with formula involving unknown pieces (HP = higher power)

$$\bar{F}^g_{L,\text{HP}} + \bar{F}^g_{L,\text{LL}}(n) = \bar{H}^{(g)}_L\Big(n, \frac{Q^2}{\mu^2} = 1, \alpha_s\Big)\bar{\Gamma}_{gg}(\alpha_s, n) \;.$$

By sequentially comparing the coefficients of $(\alpha_s/\epsilon)^\ell$, $\alpha_s(\alpha_s/\epsilon)^\ell, \ldots$ terms we find

$$\gamma_{gg} = \frac{\bar{\alpha}_s}{n} + 2\zeta_3\Big(\frac{\bar{\alpha}_s}{n}\Big)^4 + \ldots\,,$$

$$\bar{H}^{(g)}_L = \frac{2\alpha_s n_f T_F}{3\pi}\bigg(1 - \frac{1}{3}\frac{\bar{\alpha}_s}{n} + \Big(\frac{34}{9} - \zeta_2\Big)\Big(\frac{\bar{\alpha}_s}{n}\Big)^2 + \Big(-\frac{40}{27} + \frac{\pi^2}{18} + \frac{8}{3}\zeta_3\Big)\Big(\frac{\bar{\alpha}_s}{n}\Big)^3 + \ldots\bigg)\,,$$

$$\bar{F}^g_{L,\text{HP}} = \frac{2\alpha_s n_f T_F}{3\pi}\bigg(1 + 3\epsilon + \Big(6 - \frac{1}{2}\zeta_2\Big)\epsilon^2 + \Big(12 - \frac{\pi^2}{4} - \frac{7}{3}\zeta_3\Big)\epsilon^3 + \ldots\bigg)\,.$$

Series agree with LL results in Catani and Hautmann [CH94]. Interestingly, we simultaneously determine the LL results for γ_{gg} and $\bar{H}^{(g)}_L$. Also we determine the unknown power suppressed pieces in the structure function and the coefficient function self-consistently!

Resummation of F_2

For F_2, we write

$$\bar{F}^g_{2,\mathrm{HP}} + \bar{F}^g_{2,\mathrm{LL}}(n) = 2n_f\, \bar{\Gamma}_{qg} + \bar{H}^{(g)}_2\, \bar{\Gamma}_{gg}$$

Following the same steps as before, we find

$$\gamma_{qg} = \frac{\alpha_s T_F}{3\pi}\left(1 + \frac{5}{3}\frac{\bar{\alpha}_s}{n} + \frac{14}{9}\left(\frac{\bar{\alpha}_s}{n}\right)^2 + \left(\frac{82}{81} + 2\zeta_3\right)\left(\frac{\bar{\alpha}_s}{n}\right)^3 + \dots\right),$$

$$\bar{H}^{(g)}_2 = \frac{\alpha_s n_f T_F}{3\pi}\left(1 + \left(\frac{43}{9} - 2\zeta_2\right)\frac{\bar{\alpha}_s}{n} + \left(\frac{1234}{81} - \frac{13}{3}\zeta_2 + \frac{4}{3}\zeta_3\right)\left(\frac{\bar{\alpha}_s}{n}\right)^3 + \dots\right),$$

$$\bar{F}^g_{2,\mathrm{HP}} = \frac{\alpha_s n_f T_F}{3\pi}\left(-\frac{2}{\epsilon} + 1 + (1+\zeta_2)\epsilon + \left(1 - \frac{1}{2}\zeta_2 + \frac{14}{3}\zeta_3\right)\epsilon^2 + \dots\right).$$

The IR pole in $\bar{F}^g_{2,\mathrm{HP}}$ does not result from BFKL evolution. This required [CH94] to introduce a new auxiliary object, *the quark Green's function* (see backup). For us it results straightforwardly from our soft function $\tilde{S}_2$.

Comparison with previous work

> ***Objects in factorization:***

[CH94] Made use of off-shell cross sections which can only be guaranteed to be gauge invariant at leading order.

here Employed individually gauge invariant (to all orders) collinear and soft functions.

> ***Resummation of F_L vs. F_2:***

[CH94] Needed to define a separate quark Green's function for F_2

here Resummation of both F_2 and F_L follow from the same soft function.

> ***Manifest power counting***

[CH94] Included $\mathcal{O}(\alpha_s)$ higher power pieces from the beginning.

here The resummed structure function $\bar{F}^{\kappa}_{a,\mathrm{LL}}$ is manifestly leading power. We could self-consistently determine the power suppressed pieces by demanding consistency with twist factorization.

> ***NLO computation***

[CC99] Calculated *impact factor* analogous to our collinear function, but required a careful subtraction of Green's function pieces, inducing factorization scheme dependencies.

here Computation of factorized functions in our formalism follow straightforwardly from operator definitions. No process or factorization scheme dependence.

Conclusion

- We have shown how to construct from the SCET framework with Glauber interactions
  - small-x_b factorization to NLL,
  - and resummation done explicitly to LL.
- Factorization involves a universal collinear function. Such universality is not obvious in the traditional approach.
- Advantages of the EFT approach:
  - Factorization functions gauge invariant to all orders.
  - No separate Green's functions needed to be calculated.
  - Off-shell cross sections replaced by one soft function $S^{\alpha\beta}$ for all DIS channels.
  - Manifest power counting.
  - No factorization or scheme dependencies.
  - Universal, process independent, collinear-function.
- This work provides a new framework for extending resummed calculations for coefficient functions and anomalous dimensions to higher logarithmic orders.

Thank you!

Contact

DESY. Deutsches
Elektronen-Synchrotron

www.desy.de

Aditya Pathak
0000–0001–8149–2817
Theory Group
aditya.pathak@desy.de
+49–40–8998–4589

Outline

Backup

Resummation of γ_{gg} by Catani and Hautmann [CH94]

LL resummation by Catani and Hautmann [CH94]:

$$\bar{F}_L^{(g)}\left(n, \frac{Q^2}{\mu^2}\right) = h_L\big(\gamma_{gg}(\alpha_s, n)\big) R(\alpha_s, n) \left(\frac{Q^2}{\mu^2}\right)^{\gamma_{gg}(\alpha_s, n)} \bar{\Gamma}_{gg}(\alpha_s, n, \epsilon)\,.$$

Interaction with photons in terms of IR finite, off-shell cross section:

$$h_L(\gamma) = \gamma \int_0^\infty \frac{\mathrm{d}\boldsymbol{k}_\perp^2}{\boldsymbol{k}_\perp^2} \left(\frac{\boldsymbol{k}_\perp^2}{Q^2}\right)^\gamma \hat{\sigma}_L^g\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}, \alpha_s, \epsilon = 0\right).$$

Resummation of $(\alpha_s/n)^\ell$ poles in γ_{gg} via a separate calculation of gluon Green's function:

$$\mathcal{F}_g^{(0)}(n, \boldsymbol{k}_\perp, \alpha_s, \mu, \epsilon) = \frac{\gamma_{gg}(n, \alpha_s)}{\pi \boldsymbol{k}_\perp^2} \tilde{R}(n, \boldsymbol{k}_\perp, \mu_F, \alpha_s; \mu, \epsilon) \; \bar{\Gamma}_{gg}(\alpha_s, n, \epsilon,) \;.$$

Special property of LL BFKL: all the LL IR divergences in F_L are generated by BFKL evolution, and captured in the gluon Green's function. R describes scheme dependence. Consistency with DGLAP allows a closed form determination of γ_{gg} and a differential equation or a series solution for R.

Resummation of γ_{qg} by Catani and Hautmann [CH94]

For F_2 structure function, they showed

$$\gamma_{gg}(N,\alpha_s)\bar{H}_2^{(g)}(n,Q^2/\mu^2=1,\alpha_s)+2n_f\gamma_{qg}(\alpha_s,n)=h_2(\gamma)R(n,\alpha_s)\,,$$

where

$$h_2(\gamma)=\gamma\int_0^\infty\frac{\mathrm{d}\boldsymbol{k}_\perp^2}{\boldsymbol{k}_\perp^2}\left(\frac{\boldsymbol{k}_\perp^2}{Q^2}\right)^\gamma\frac{\partial}{\partial\ln Q^2}\hat{\sigma}_2^g\left(\frac{\boldsymbol{k}_\perp^2}{Q^2},\alpha_s,\epsilon=0\right).$$

Notice that they needed to take $\ln Q^2$ derivative as $\hat{\sigma}_2^g$ is not collinear safe. The structure of IR divergences in γ_{qg} gets polluted by $1/\epsilon$ divergence in $\hat{\sigma}_2^g$, so define a new *quark Green function*:

$$G_{qg}^{(0)}(n,\alpha_s,\epsilon)=\int\mathrm{d}^{d-2}\boldsymbol{k}_\perp\,\hat{K}_{qg}\left(\frac{\boldsymbol{k}_\perp^2}{Q^2},\alpha_s,\mu,\epsilon\right)\mathcal{F}_g^{(0)}(n,\boldsymbol{k}_\perp,\alpha_s,\mu,\epsilon)\,.$$

K_{qg} includes the $1/\epsilon$ pole associated with $\hat{\sigma}_2^g$ (same as what we saw in $\bar{F}_{2,\mathrm{HP}}^g$ above). Consistency with DGLAP resummation then enables determination of γ_{qg} anomalous dimension using $G_{qg}^{(0)}$, although not in a closed form as in γ_{gg}.

How do IR poles exponentiate?

After resumming the leading $(\bar{\alpha}_s/n)^\ell$ terms:

$$\bar{F}^{\kappa}_{a,\mathrm{LL}}(n,Q^2) = \frac{c_\kappa}{n\pi}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-2\epsilon}\sum_{\ell=0}^{\infty} d_{a,\ell+1}(\epsilon)\left(\frac{\bar{\alpha}_s}{n}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\right)^{\ell}$$

In twist expansion the bare structure function (in dim-reg) factorizes as

$$\bar{F}^{\kappa}_{p}(n,Q^2) = \sum_{\kappa'} \bar{H}^{(\kappa')}_{p}\left(n,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right)\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2),n) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right).$$

In the fixed coupling approximation the partonic PDF is

$$\bar{\Gamma}_{\kappa'\kappa}(\alpha_s(\mu^2),n) = \mathrm{P}\exp\left(-\frac{1}{\epsilon}\int_0^{\alpha_s(\mu^2)}\frac{\mathrm{d}\alpha}{\alpha}\gamma^s(\alpha,n)\right)_{\kappa'\kappa}.$$

For parton $\kappa \to \kappa'$ it captures the infra-red divergences of the perturbative calculation.

How do IR poles exponentiate?

After resumming the leading $(\bar{\alpha}_s/n)^\ell$ terms:

$$\bar{F}^{\kappa}_{a,\mathrm{LL}}(n, Q^2) = \frac{c_\kappa}{n\pi}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-2\epsilon} \sum_{\ell=0}^{\infty} d_{a,\ell+1}(\epsilon)\left(\frac{\bar{\alpha}_s}{n}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\boldsymbol{q}_\perp^2}{\mu^2}\right)^{-\epsilon}\right)^\ell$$

Let us illustrate how the leading $(\alpha_s/\epsilon)^\ell$ IR poles exponentiate. The $d_{a,\ell}$ coefficients for $a = L$ behave as

$$\frac{1}{n}\left(\frac{\bar{\alpha}_s}{n}\right)^\ell d_{L,\ell+1}(\epsilon) = \frac{2\alpha_s n_f T_F}{3\pi}\left[\frac{1}{(\ell+1)!}\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)^{\ell+1} + \mathcal{O}(\epsilon^{-\ell})\right]$$

Thus,

$$\bar{F}^{g}_{L,\mathrm{LL}}(n) + \frac{2\alpha_s n_f T_F}{3\pi} = \frac{2\alpha_s n_f T_F}{3\pi}\left[\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)^\ell (1+\mathcal{O}(\epsilon))\right]$$

$$= \frac{2\alpha_s n_f T_F}{3\pi}\exp\left(-\frac{1}{\epsilon}\frac{\bar{\alpha}_s}{n}\right)\left(1+\mathcal{O}\left(\frac{\bar{\alpha}_s}{n}\right)\right) + \mathcal{O}\left(\frac{1}{\epsilon}\left(\frac{\bar{\alpha}_s}{n}\right)^2\right)$$

Necessary to add by hand the $\mathcal{O}(\alpha_s)$ term to factorize IR divergences.

References

[CH94] S. Catani and F. Hautmann. "High-energy factorization and small x deep inelastic scattering beyond leading order". In: Nucl. Phys. B427 (1994), pp. 475–524. DOI: 10.1016/0550-3213(94)90636-X. arXiv: hep-ph/9405388 [hep-ph].

[CC99] M. Ciafaloni and D. Colferai. "K factorization and impact factors at next-to-leading level". In: Nucl. Phys. B 538 (1999), pp. 187–214. arXiv: hep-ph/9806350.

[Tho01] Robert S. Thorne. "The Running coupling BFKL anomalous dimensions and splitting functions". In: Phys. Rev. D 64 (2001), p. 074005. arXiv: hep-ph/0103210.

[Cia+04] Marcello Ciafaloni et al. "The Gluon splitting function at moderately small x". In: Phys. Lett. B 587 (2004), pp. 87–94. arXiv: hep-ph/0311325.

[ABF06] Guido Altarelli, Richard D. Ball, and Stefano Forte. "Perturbatively stable resummed small x evolution kernels". In: Nucl. Phys. B 742 (2006), pp. 1–40. arXiv: hep-ph/0512237.

[RS16] Ira Z. Rothstein and Iain W. Stewart. "An Effective Field Theory for Forward Scattering and Factorization Violation". In: JHEP 08 (2016), p. 025. DOI: 10.1007/JHEP08(2016)025. arXiv: 1601.04695 [hep-ph].