Dilatations, Boosts, and the Phase of the S-Matrix

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Based on work with Ira Rothstein

SCET 2023 (Berkeley)

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Talk Overview

1. Anomalous dimensions of form factors from unitarity cuts [Caron-Huot, Wilhelm, 2016]

2. Explore using the lens of SCET

3. Two Examples: Sudakov Form Factor, TMD Soft Function

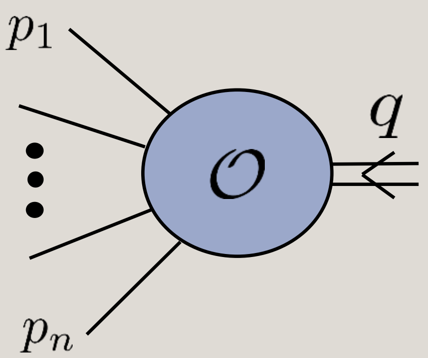
4. Apply to $2 \to 2$ forward scattering amplitude and Glauber operators

Form Factors and Dilatations

- Form Factor: $F_{\mathcal{O}}(p_1, ..., p_n) = \langle p_1, ..., p_n | \, \mathcal{O}(q) \, |0\rangle$

1. On-Shell: $p_i^2 = 0$

2. Time-Like: $s_{ij} = (p_i + p_j)^2 > 0$



- s_{ij} enter through RG logs:

$$\log\left(\frac{-s_{ij}}{\mu^2}\right) = \log\left(\frac{|s_{ij}|}{\mu^2}\right) - i\pi$$

μ-dependence $\Leftrightarrow \mathrm{Im}\left[F_{\mathcal{O}}\right]$

Form Factors and Dilatations

- Dilatation Operator $$DF_{\mathcal{O}} = \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu} F_{\mathcal{O}}$$

$$e^{i\pi D} F_{\mathcal{O}}(p_1, ..., p_n) = F_{\mathcal{O}}(-p_1, ..., -p_n) = F^*_{\mathcal{O}}(p_1, ..., p_n)$$

- Optical Theorem for form factors: $$\mathcal{O}^\dagger = S^\dagger \mathcal{O} S^\dagger$$

$$\boxed{e^{i\pi D} F_{\mathcal{O}} = S^\dagger F_{\mathcal{O}}}$$

"The dilatation operator is minus the phase of the S-matrix, divided by π."

Form Factors and Dilatations

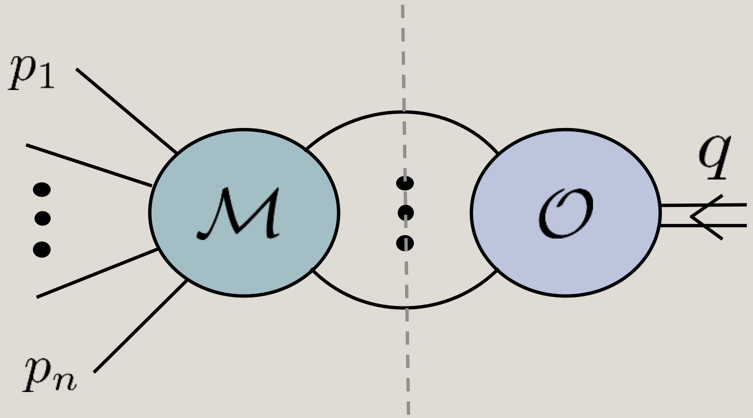
- Identify dilatations with RGE

$$D\left[\log\frac{-s_{ij}}{\mu^2}\right] = -\mu\frac{\partial}{\partial\mu}\left[\log\frac{-s_{ij}}{\mu^2}\right]$$

$$\Rightarrow DF_{\mathcal{O}} = \left(\gamma_{\mathrm{UV}} - \gamma_{\mathrm{IR}} + \beta(g^2)\frac{\partial}{\partial g}\right)F_{\mathcal{O}}$$

$$e^{i\pi D}F_{\mathcal{O}} = S^\dagger F_{\mathcal{O}}$$

$$\left(e^{i\pi\gamma_\mu} - 1\right)F_{\mathcal{O}} = \sum_{\mathrm{cuts}}$$



Form Factors and Dilatations

Several limitations/shortcomings of this formula:

1. Breaks down for massive particles: $\log \frac{M^2}{\mu^2}$ undetected by dilatations
2. Generally need to subtract the IR anomalous dimension – accomplished by subtracting cuts of the stress-tensor:

$$\gamma_{\rm UV} = -\frac{2}{\pi} {\rm Im} \log \frac{F_{\mathcal{O}}}{\langle p_1, \ldots | \, T^{\mu\nu} \, |0\rangle} + \beta\text{-function terms}$$

3. Only applies in hard scattering-like kinematics

How do we extend to more general cases?

Form Factors in SCET

- Factorization: $\hat{F} = \mathcal{H} \otimes \underbrace{(J_{n_1} \otimes ... \otimes J_{n_k}) \otimes \mathcal{S}}_{F_{\rm SCET}}$

- 3 Sources of $\mathrm{Im}[\hat{F}]$
  1. Hard logs in $\mathcal{H}$: $\sim \log \frac{-Q^2}{\mu^2}$ — $Q^2 > 0$: Hard scale
  2. Ultrasoft logs (SCET$_{\rm I}$): $\sim \log \frac{-Q^2 \mu^2}{p_i^2 p_j^2}$ — $p_i^2 << Q^2$ $\quad p_i^2 < 0$
  3. Rapidity Logs in $F_{\rm SCET}$: $\sim \log \frac{-Q^2}{m_{\rm IR}^2}$ — $m_{\rm IR}^2 << Q^2$: IR scale

- Can no longer identify $D \not\simeq -\mu \frac{\partial}{\partial \mu}$

Form Factors in SCET

μ - RGE

$\updownarrow$

Dilatations

$$D \log \frac{-Q^2}{\mu^2} = -\mu \frac{\partial}{\partial \mu} \log \frac{-Q^2}{\mu^2}$$

$\Leftrightarrow$

ν - RGE

$\updownarrow$

Boosts

$$K_z \log \frac{-Q^2}{\nu^2} = -\nu \frac{\partial}{\partial \nu} \log \frac{-Q^2}{\nu^2}$$

Just replace dilatations with boosts!

$$K_z = \sum_n n^\mu \bar{n}^\nu M^n_{\mu\nu}$$

$$M^n_{\mu\nu} = \sum_{i \in n} \left(p_{i,\mu} \frac{\partial}{\partial p_i^\nu} - p_{i,\nu} \frac{\partial}{\partial p_i^\mu} \right)$$

Complex Boosts

- Boosts by $i\pi$ take incoming states to outgoing states

$$e^{i\pi K_z}(n\cdot p, \bar{n}\cdot p, \vec{p}_\perp) = (-n\cdot p, -\bar{n}\cdot p, \vec{p}_\perp)$$

- Boosting form factors:

$$e^{-i\pi K_z} F_{\text{SCET}}(\{p_i\}) = F_{\text{SCET}}(\{-p_i\}) = F^*_{\text{SCET}}(\{p_i\})$$

- Soft function is boost invariant:

$$e^{-i\pi K_z} F_{\text{SCET}} = \left(e^{-i\pi K_z} J_{n_1}\right) \otimes \ldots \otimes \left(e^{-i\pi K_z} J_{n_k}\right) \otimes \mathcal{S}$$

$$= e^{i\pi(\gamma_\nu^{n_1} + \ldots + \gamma_\nu^{n_k})} \left(J_{n_1} \otimes \ldots \otimes J_{n_k} \otimes \mathcal{S}\right)$$

Full Result

- Simplify using soft-collinear consistency

$$\gamma_\nu^s + \sum_n \gamma_\nu^n = 0$$

- $Z_{\mathcal{O}}$ can be complex - optical theorem only holds for bare operator
  - Optical theorem for renormalized form factor:

$$\mathcal{O}^{\dagger R}_{\text{SCET}} = S^\dagger \mathcal{O}^R_{\text{SCET}} S^\dagger - 2i\text{Im}[Z_{\mathcal{O}}^{-1}]\mathcal{O}^B_{\text{SCET}}$$

$$e^{-i\pi\gamma_\nu^s} F^R_{\text{SCET}} = S^\dagger F^R_{\text{SCET}} - 2i\text{Im}[Z_{\mathcal{O}}^{-1}] F^B_{\text{SCET}}$$

The rapidity anomalous dimension is (almost) the phase of the S-matrix (in SCET_{II}).

At one loop:

Tree form-factors

$$S = 1 + i\mathcal{M}$$

$$\gamma_\nu^{s(1)} F^{(0)}_{\rm SCET} = \frac{1}{\pi}\left[\mathcal{M}^\dagger F^{R(1)}_{\rm SCET} + 2{\rm Im}[Z_{\mathcal{O}}^{-1}]^{(1)} F^{(0)}_{\rm SCET}\right]$$

1-loop rapidity anomalous dimension

Cuts of one-loop form factor

Pure divergence: Renormalizes $1/\varepsilon_{\rm UV}$ terms in the cut

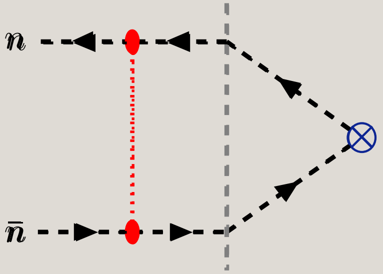
- We can compute ℓ-loop anomalous dimensions from cuts of ℓ- loop of form-factors

Example 1: Sudakov Form Factor

- Time-like form factor $Q^2 > 0:\quad \mathcal{O} = \bar{\xi}_n W_n S_n^\dagger \Gamma S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n}}$

- Only one diagram at one loop:

$$\gamma_\nu^{s(1)} = \frac{1}{\pi}$$



$$+ \frac{2}{\pi}\mathrm{Im}[Z_{\mathcal{O}}^{-1}]$$

$$= \frac{C_F \alpha_s}{\pi}\Gamma(\varepsilon)\left(\frac{\mu^2}{M^2}\right)^{\varepsilon} - \frac{C_F \alpha_s}{\pi}\frac{1}{\varepsilon}$$

Example 1: Sudakov Form Factor

- Two loops:

$$e^{-i\pi\gamma_\nu^s} F^R_{\mathrm{SCET}} = S^\dagger F^R_{\mathrm{SCET}} - 2i\mathrm{Im}[Z_{\mathcal{O}}^{-1}] F^B_{\mathrm{SCET}}$$

Iterative terms

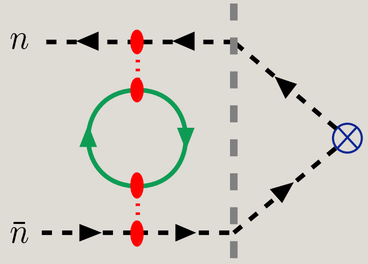
$$\gamma_\nu^{s(2)} F^{(0)}_{\mathrm{SCET}} + \gamma_\nu^{s(1)} F^{R(1)}_{\mathrm{SCET}} - \frac{i\pi}{2!}(\gamma_\nu^{s(1)})^2 F^{(0)}_{\mathrm{SCET}} = \frac{1}{\pi}\left(\mathcal{M}^\dagger F^R_{\mathrm{SCET}}\right)^{(2)} + \frac{2}{\pi}\mathrm{Im}[Z_{\mathcal{O}}^{-1}]^{(2)} F^{(0)}_{\mathrm{SCET}} + \frac{2}{\pi}\mathrm{Im}[Z_{\mathcal{O}}^{-1}]^{(1)} F^{(1)}_{\mathrm{SCET}}$$

- Cut diagrams have one Glauber loop: two is purely real
- Imaginary terms are entirely iterative

Example 1: Sudakov Form Factor

- Two loops: switch to massive quarks

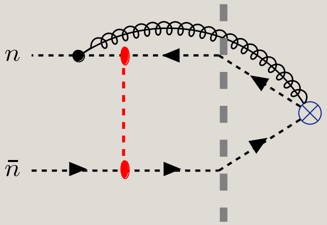
- Soft Loops:



$$= \gamma_\nu^{s(2)} F^{(0)}_{\mathrm{SCET}}$$

[Hoang, Pathak, Pietrulewicz, Stewart, 2015]

- Collinear Loops:



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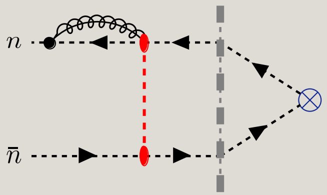
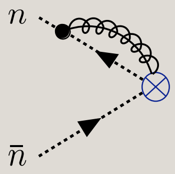


$$= \gamma_\nu^{s(1)} \Bigg(\quad \Bigg)$$



  – Diagrams with collinear loops are purely iterative and do not contribute to the anomalous dimension (at this order)

  – Many other collinear diagrams, all cancel via collapse rule or vanish identically

Example II: TMDs

- Observables are real functions – vanishing phase

- For some observables (e.g. TMDs), we can construct a form factor with the same rapidity anomalous dimension

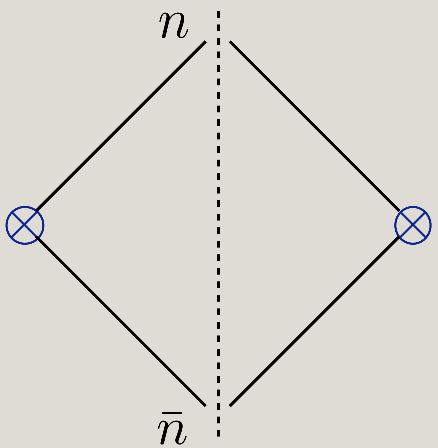
- TMDPDF: $\langle p_n| \, B^{A\mu}_{n\perp}(\vec{b}_\perp) B^{A\nu}_{n\perp}(0) \, |p_n\rangle \to \int dx^+ e^{iQx^+} \, \langle p_n, p'_n| \, B^{A\mu}_{n\perp}(x^+, \vec{b}_\perp) B^{A\nu}_{n\perp}(0) \, |0\rangle$

- Flips direction of soft Wilson lines to give a time-like soft function $\tilde{\mathcal{S}}_{\rm TMD}$

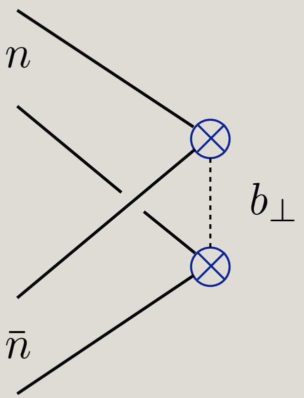
$$\tilde{\mathcal{S}}_{\rm TMD}(\nu^2) = \mathcal{S}_{\rm TMD}(-\nu^2 + i0) \;\; \Rightarrow \;\; \nu \frac{d}{d\nu} \tilde{\mathcal{S}}_{\rm TMD} = \nu \frac{d}{d\nu} \mathcal{S}_{\rm TMD}$$

Example II: TMD Soft Function

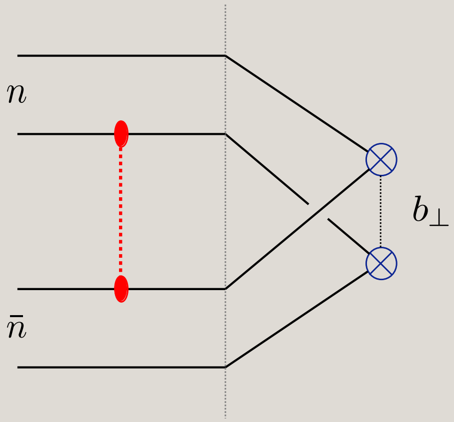
$\mathcal{S}_{\rm TMD}$:



$\tilde{\mathcal{S}}_{\rm TMD}$:



- One-loop calculation:



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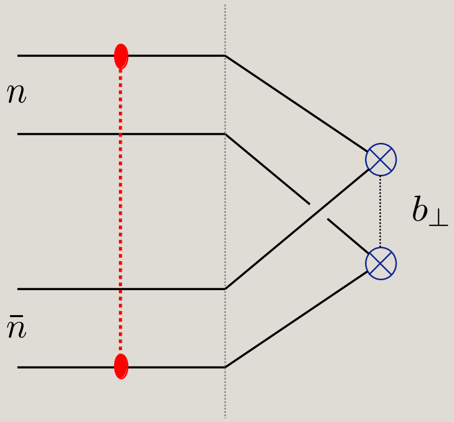
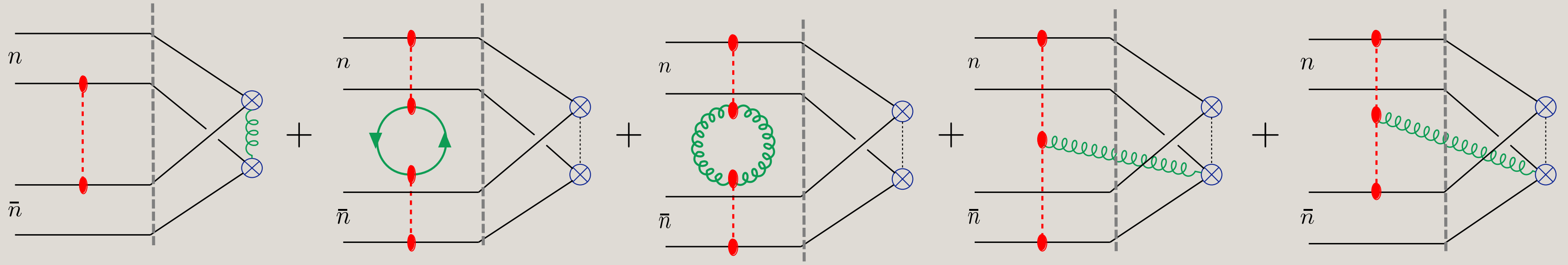


$$\sim \gamma_\nu^{s(1)}(\vec{b}_\perp^2)$$

Example II: TMD Soft Function

- 2-loops: collinear loops are purely iterative again!

- Only soft graphs contribute:



$$\sim \gamma_\nu^{s(2)}(\vec{b}_\perp^2) + \textit{iterative terms}$$

- Every integral reduces to one-loop bubble integrals

- Rapidity divergences cancel – compute γ_ν from rapidity finite integrals

Regge Trajectory from Complex Boosts

- Glaubers conserve large $n\cdot p\,,\bar{n}\cdot p:$ no way to write as a time-like form factor

- Instead use odd-signature amplitude:

$$\mathcal{M}^{(-)}_{2\to 2} = \frac{1}{2}\left(\mathcal{M}_{2\to 2}(s,t) - \mathcal{M}_{2\to 2}(u,t)\right)$$

[Caron-Huot, Gardi, Vernazza, 2016]

- Only depends on s, u through $\frac{1}{2}\left(\log\frac{s}{-t} + \log\frac{-s}{-t}\right) = \log\left|\frac{s}{t}\right| + \frac{i\pi}{2}$

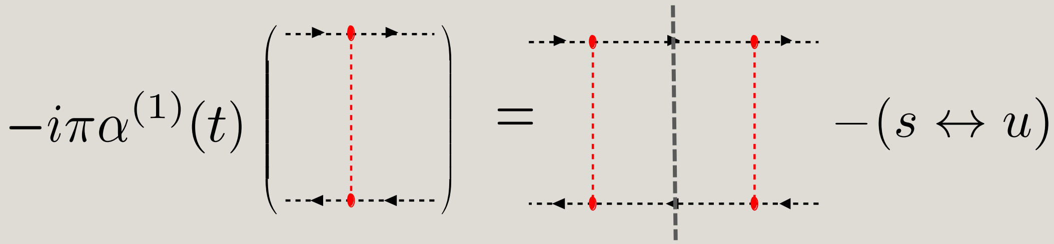
- Boost by $-\frac{i\pi}{2}$:

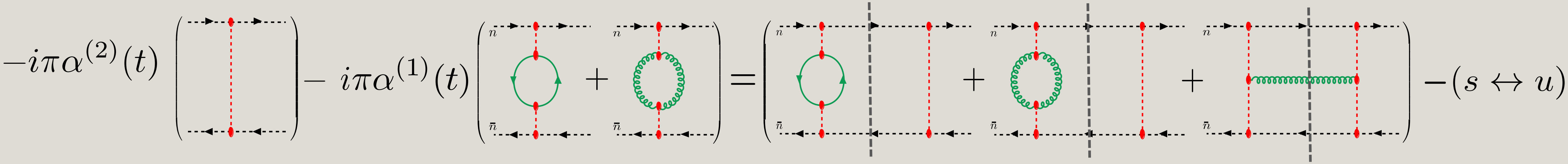
$$e^{-i\pi/2K_z}\mathcal{M}^{(-)} = \mathcal{M}^{(-)*} = S^\dagger\mathcal{M}^{(-)}$$

Glaubers and Complex Boosts

- Expand through 2 loops:

$$e^{i\pi/2K_z}\mathcal{M}^{(-)} = S^\dagger \mathcal{M}^{(-)}$$





Diagrammatically almost identical to results of [Moult, Raman, Ridgeway, Stewart, 2022]

Conclusion

- In form factors with rapidity logarithms, dilatations can be replaced with boosts in the S-matrix phase formula
- Provides an efficient method for extracting rapidity anomalous dimensions in terms of Glauber graphs
- In certain cases, we can extend beyond form factors

Dilatation Equation Derivation

- Conjugate Form Factor:

$$e^{i\pi D} F_{\mathcal{O}}(p_1, ..., p_n) = F_{\mathcal{O}}(-p_1, ..., -p_n) = F^*_{\mathcal{O}}(p_1, ..., p_n)$$

$$\langle -p_1, ..., p_n | \mathcal{O} |0\rangle = \langle 0 | \mathcal{O} |p_1, ..., p_n\rangle = \langle p_1, ..., p_n | \mathcal{O} |0\rangle^*$$

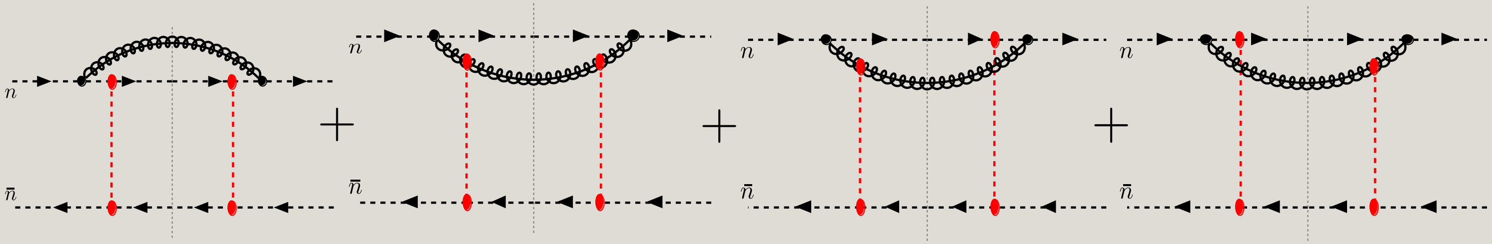
- Form Factor Optical Theorem:

$$i\mathcal{O} = i\delta S \qquad (S + i\delta S)(S + i\delta S)^\dagger = 1 \qquad S^\dagger \mathcal{O} - \mathcal{O}^\dagger S = 0$$

$$\left(S\, S^\dagger = 1\right)$$

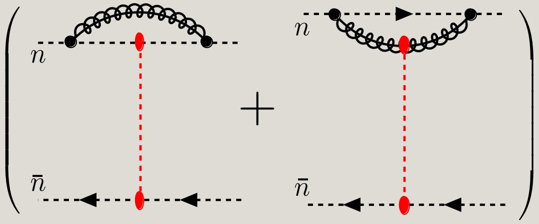
Glaubers and Complex Boosts

- Collinear diagrams:



$$- (s \leftrightarrow u)$$

$$= -\, i\pi\alpha^{(1)}(t)$$



- Collinear loops are entirely iterative terms

- Non-planar graphs cancelled by odd signature – misses pomeron

Glaubers and Complex Boosts: All Orders Formula

- $2 \to 2$ amplitude in terms of Glaubers:

$$\mathcal{M}_{2\to 2} = \sum_{i,j} J_{n(i)} \mathcal{S}_{(i,j)} J_{\bar{n}(j)}$$

- Rapidity RGE:

$$\nu \frac{d}{d\nu} J_{n(i)} = \gamma^{n}_{\nu(i,j)} J_{n(j)}$$

- Complex Boost:

$$e^{-i\pi/2K_z} \mathcal{M}^{(-)} = \sum_{i,j,k,l} J_{n(i)} \left(e^{i\pi/2\gamma^{n}_{\nu}}\right)_{(i,j)} \mathcal{S}_{(j,k)} \left(e^{i\pi/2\gamma^{\bar{n}}_{\nu}}\right)_{(k,l)} J_{\bar{n}(l)} - (s \leftrightarrow u)$$

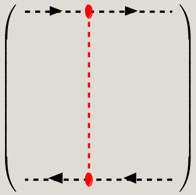
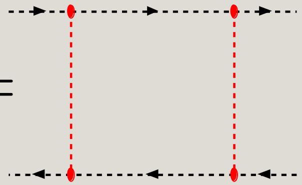
Glauber Anomalous Dimensions

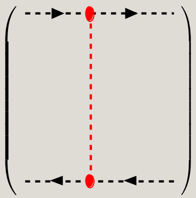
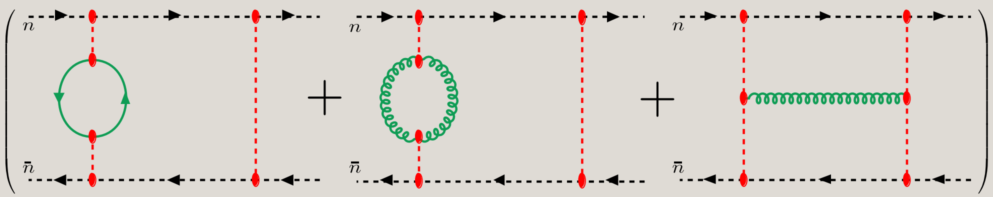
$$\alpha(t) = -2\frac{S_2}{S_1}$$

$$\alpha^{(1)}(t) = -2S_2^{(1)} \qquad \alpha^{(2)}(t) = -2\left(S_2^{(2)} - S_2^{(1)}S_1^{(1)}\right)$$

[Moult, Raman, Ridgway, Stewart, 2022]

- Look at these diagramically:

$$-\frac{i\pi}{2}\alpha^{(1)}(t)$$  $$=$$ 

$$-\frac{i\pi}{2}\alpha^{(2)}(t)$$  $$=$$  $$-\frac{i\pi}{2}\alpha^{(1)}(t)$$