

# Threshold factorization of the Drell-Yan quark-gluon channel and two-loop soft function at NLP

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LBNL

based on a publication **in preparation** with  
Alessandro Broggio and Leonardo Vernazza



## Motivations and focus

$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Recently calculated to N<sup>3</sup>LO

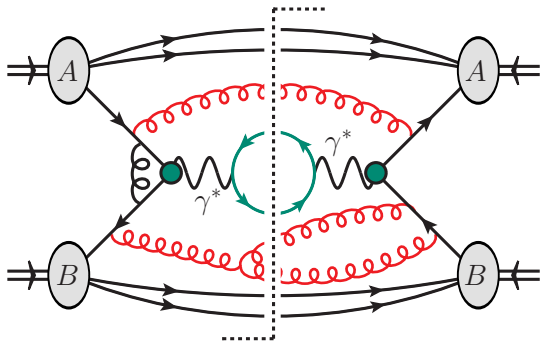
[C. Duhr, F. Dulat,  
B. Mistlberger, 2001.07717]

Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \rightarrow 1$$

Define power counting  
parameter  $\lambda$ :

$$\lambda = \sqrt{1-z}$$



Schematic form for production cross-sections near threshold,  $z \rightarrow 1$  ( $Q^2 \lambda^2 \gg \Lambda_{\text{QCD}}^2$ ):

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[ c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left( c_{nm} \left[ \frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \right]$$

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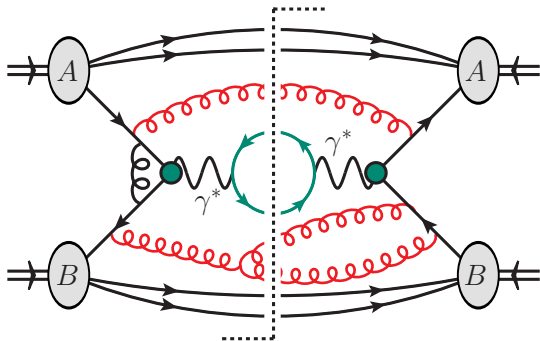
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Naturally, the idea is to factorize the physics appearing at different scales:

$$\sigma \sim H \otimes J \otimes \dots \otimes J \otimes S$$

and solve RG equations for each object to sum the large logarithms.

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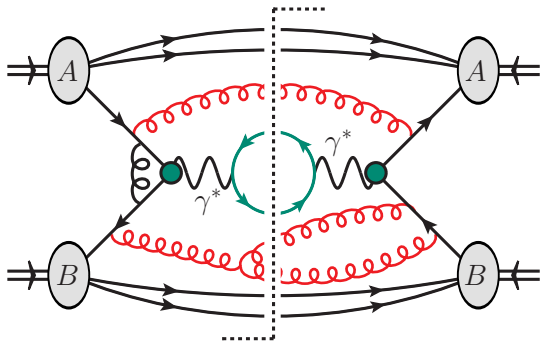
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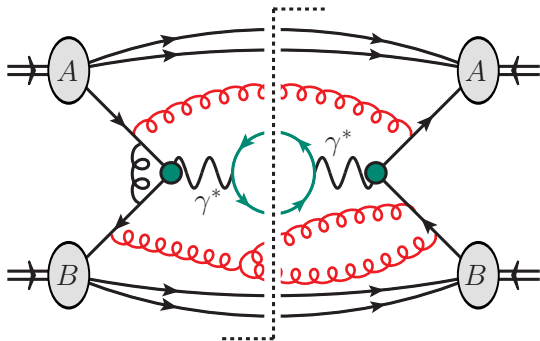
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## Plenty recent NLP studies

Subleading power resummed thrust spectrum for  $H \rightarrow gg$  (LL)

[I. Moulton, I. Stewart, G. Vita, H. Zhu, 1804.04665, 1910.14038]

Drell-Yan and Higgs production at threshold (LL)

[M. Beneke, A. Broggio, M. Garry, SJ, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, M. Garry, SJ, R. Szafron, L. Vernazza, J. Wang, 1910.12685]

Resummation of rapidity logarithms: the EE correlator in N=4 SYM (LL)

[I. Moulton, G. Vita, K. Yan, 1912.02188]

Factorization at Subleading Power and Endpoint Divergences in SCET (LL, NLL)

[Z. L. Liu, M. Neubert, 1912.08818 ]

[Z. L. Liu, B. Mecej, M. Neubert, X. Wang, 2009.04456, 2009.06779]

[Z. L. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

DIS  $x \rightarrow 1$  [M. Beneke, M. Garry, SJ, R. Szafron, L. Vernazza, J. Wang, 2008.04943]

Soft-collinear gravity beyond the leading power

[M. Beneke, P. Hager, R. Szafron, 2112.04983]  $\rightarrow$  see talk by P. Hager

Factorization at subleading power in Deep Inelastic Scattering in the  $x \rightarrow 1$  limit

[M. Luke, J. Roy, A. Spourdakis, 2210.02529]  $\rightarrow$  see talk by M. Luke

## Plenty recent NLP studies

Power-enhanced QED corrections to  $B_q \rightarrow \mu^+ \mu^-$  (LL)

[M. Beneke, C. Bobeth, R. Szafron, 1908.07011]

Violation of KSZ theorem in SCET

[M. Beneke, M. Garry, R. Szafron, J.Wang, 1907.05463]

Subleading Power Factorization with Radiative Functions

[I. Moulton, I. Stewart, G. Vita, 1905.07411]

Drell-Yan  $q_T$  Resummation of Fiducial Power Corrections at  $N^3LL$

[M. Ebert, J. Michel, I. Stewart, F. Tackmann, 2006.11382]

Resummation of double logarithms in loop-induced processes with EFT

[J. Wang, 1912.09920]

Refactorisation in subleading  $\bar{B} \rightarrow X_s \gamma$

[T. Hurth, R. Szafron, 2301.01739] →see talk by R. Szafron

Structure-Dependent QED Effects in Exclusive B Decays at Subleading Power

[C. Cornella, M. König, M. Neubert, 2212.14430] →see talks by C. Cornella and M. Neubert

Muon-electron backward scattering: endpoint singularities in SCET

[G. Bell, P. Böer, T. Feldmann, 2205.06021] →see talk by P. Böer

Next-to-leading power endpoint factorization and resummation for off-diagonal “gluon” thrust

[M. Beneke, M. Garry, SJ, J. Strohm, R. Szafron, L. Vernazza, J.Wang, 2205.04479] →see talk by J. Strohm last year

# Brief introduction to NLP SCET

In this talk we employ position-space SCET [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358]

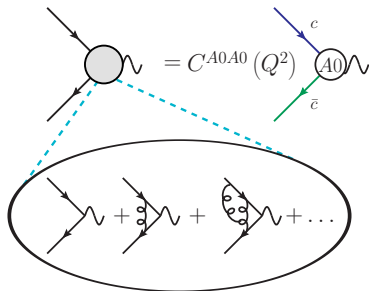
$$\psi(x) \rightarrow \underbrace{\psi_1(x) + \dots + \psi_N(x)}_{N \text{ collinear fermion fields}} + q(x) \quad \mathcal{L}_{\text{SCET}} = \sum_{i=1}^N \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the **small parameter**  $\lambda = \sqrt{1-z}$ :

$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \dots \quad \text{E.g.} \quad \mathcal{L}_c^{(1)} = \bar{\chi}_c i x_{\perp}^{\mu} [i n_{-} \partial \mathcal{B}_{\mu}^{+}] \frac{\not{n}_{+}}{2} \chi_c$$

We keep the *collinear*, *anti-collinear*, and *soft* degrees of freedom. The *hard* modes are integrated out:

$$\bar{\psi} \gamma^{\mu} \psi = \int \prod_{i=1}^N \prod_{k=1}^{n_i} dt_{i_k} C(\{t_{i_k}\}) \prod_{i=1}^N J_i(t_{i_1}, \dots, t_{i_{n_i}})$$





# SCET introduction: Lagrangian

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Separate collinear sectors interact only through **soft gluon interactions**. Focusing on the LP term:

$$\mathcal{L}_c^{(0)} = \bar{\xi}_c \left( i n_{-} D_c + g n_{-} A_s(x_{-}) + i \not{D}_{\perp c} \frac{1}{i n_{+} D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_{+}}{2} \xi_c + \mathcal{L}_{c, \text{YM}}^{(0)}$$

with  $i n_{-} D_c = i n_{-} \partial + g n_{-} A_c(x)$ ,  $x_{\pm}^{\mu} = (n_{\pm} x) n_{\pm}^{\mu} / 2$ .

The **decoupling transformation**,  $\xi_c \rightarrow Y_{+} \xi_c^{(0)}$  and  $A_c^{\mu} \rightarrow Y_{+} A_c^{(0)\mu} Y_{+}^{\dagger}$ , separates the soft and collinear sectors at LP [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

$$\bar{\xi}_c (i n_{-} D_c + g_s n_{-} A_s) \frac{\not{n}_{+}}{2} \xi_c = \bar{\xi}_c^{(0)} i n_{-} D_c^{(0)} \frac{\not{n}_{+}}{2} \xi_c^{(0)}$$

where

$$Y_{\pm}(x) = \mathbf{P} \exp \left[ i g_s \int_{-\infty}^0 ds n_{\mp} A_s(x + s n_{\mp}) \right]$$

# Brief introduction to NLP SCET

Generic N-jet operator has the form:

[M. Beneke, M. Garry, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

$$J = \int \prod_{i=1}^N \prod_{k_i=1}^{n_i} dt_{ik_i} C(\{t_{ik_i}\}) \prod_{i=1}^N J_i(t_{i1}, t_{i2}, \dots, t_{in_i})$$

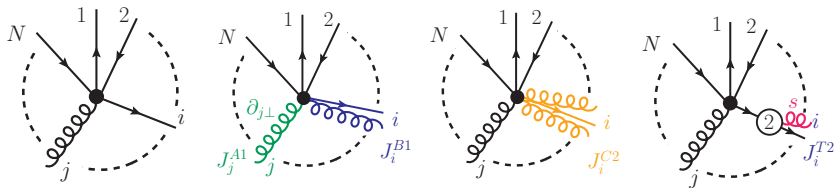
where the  $J$ s are constructed by multiplying collinear gauge invariant building blocks in the same direction ( up to  $\mathcal{O}(\lambda^2)$  )

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i$$

$$\mathcal{A}_{i\perp}^\mu(t_i n_{i+}) \equiv W_i^\dagger [iD_{\perp i}^\mu W_i]$$

by acting on these with derivatives  $i\partial_{\perp i}^\mu \sim \lambda$ , and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

Generic leading power  $N$ -jet operator:



# The Drell-Yan Process

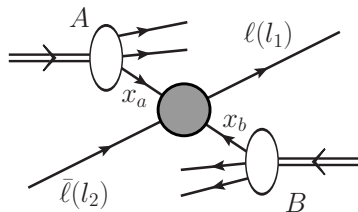
$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

$$z = \frac{Q^2}{\hat{s}} \rightarrow 1 \quad \lambda = \sqrt{1-z}$$

$$p_c = (n_+ p_c, n_- p_c, p_{c\perp}) \sim Q(1, \lambda^2, \lambda)$$

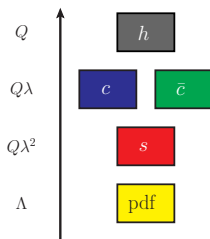
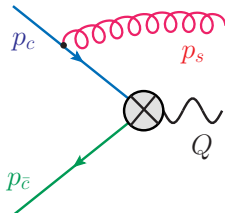
$$p_{\bar{c}} = (n_+ p_{\bar{c}}, n_- p_{\bar{c}}, p_{\bar{c}\perp}) \sim Q(\lambda^2, 1, \lambda)$$

$$p_s = (n_+ p_s, n_- p_s, p_{s\perp}) \sim Q(\lambda^2, \lambda^2, \lambda^2)$$



$$Q^2 \lambda^2 = Q^2(1-z) \gg \Lambda_{\text{QCD}}^2$$

$$p_{c\text{-PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$$



## Drell Yan: Factorization of the partonic cross-section

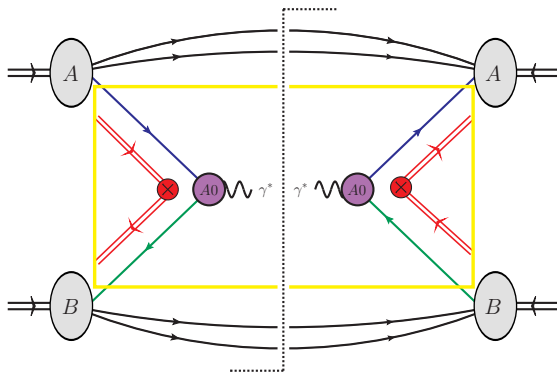
First let us compare **leading power** and **next-to-leading power** cross-sections schematically:

$$\frac{d\sigma_{\text{DY}}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left( \hat{\sigma}_{ab}^{\text{LP}}(z) + \hat{\sigma}_{ab}^{\text{NLP}}(z) + \dots \right) + \mathcal{O}(\Lambda/Q)$$

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$$\hat{\sigma}^{\text{LP}}(z) = Q H(Q^2) S_{\text{DY}}(\Omega)$$

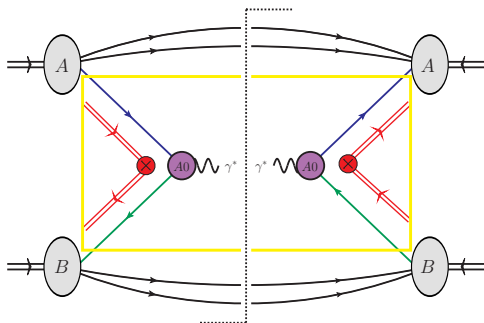
[G. Sterman 1987] [S. Catani, L. Trentadue 1989] [G. P. Korchemsky G. Marchesini, 1993]

[S. Moch, A. Vogt, hep-ph/0508265] [T. Becher, M. Neubert, G. Xu, 0710.0680]

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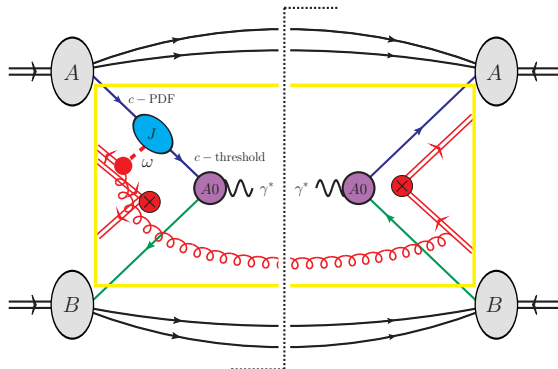
$$S_{\text{DY}}(\Omega) = \int \frac{dx^0}{4\pi} e^{i\Omega x^0/2} \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[ Y_+^\dagger(x) Y_-(x) \right] \mathbf{T} \left[ Y_-^\dagger(0) Y_+(0) \right] | 0 \rangle$$

$$Y_\pm(x) = \mathbf{P} \exp \left[ ig_s \int_{-\infty}^0 ds n_\mp A_s(x + sn_\mp) \right].$$

# Drell Yan: Factorization of the partonic cross-section

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$$\hat{\sigma}_{q\bar{q}}^{NLP}(z) = \sum_{\text{terms}} [C \otimes J \otimes \bar{J}]^2 \otimes S$$

[M. Beneke, A. Broggio, M. Garry, SJ, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, SJ, L. Vernazza, 1912.01585]

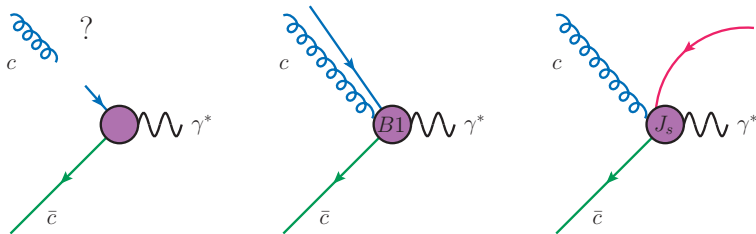




## In the EFT picture

This is inherently a subleading power channel.

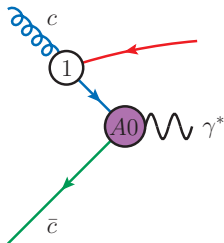
$$\bar{\psi}\gamma_\rho\psi(0) = \sum_{m_1, m_2} \int \{dt_k\} \{d\bar{t}_{\bar{k}}\} \tilde{C}^{m_1, m_2}(\{t_k\}, \{\bar{t}_{\bar{k}}\}) J_s(0) J_\rho^{m_1, m_2}(\{t_k\}, \{\bar{t}_{\bar{k}}\})$$



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This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\mu\psi = \int dt d\bar{t} \tilde{C}^{A0}(t, \bar{t}) \underbrace{\bar{\chi}_{\bar{c}}(\bar{t}n_-) \gamma_{\perp\mu} \chi_c(tn_+)}_{J_\mu^{A0}(t, \bar{t})}$$



Beyond LP, the **decoupling transformation** does not remove soft-collinear interactions in the Lagrangian.

$$\left(J_{A0, \xi q}^{T2}(t)\right)^\mu = i \int d^d z \mathbf{T} \left[ \chi_c(tn_+) \mathcal{L}_{\xi q}^{(1)}(z) \right] \quad \mathcal{L}_{\xi q}^{(1)} = \bar{q}_+ \mathcal{A}_{c\perp} \chi_c + \text{h.c.}$$

## Collinear functions at NLP

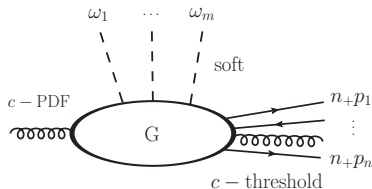
PDF collinear modes can be radiated into the final state:

$$p_c \sim Q(1, \lambda^2, \lambda) \text{ and } p_{c\text{-PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$$

$$i \int d^d z \mathbf{T} \left[ \xi_c(tn_+) \mathcal{L}_{\xi_q}^{(1)}(z) \right] = 2\pi \int du \int dz_- \tilde{G}_{\xi_q}(t, u; z_-) \mathcal{A}_{c\perp}^{\text{PDF}}(un_+) \mathbf{s}_{\xi_q}(z_-)$$

Only one soft structure present at NLP

$$\mathbf{s}_{\xi_q}(z_-) = \frac{g_s}{in_- \partial_z} q^+(z_-)$$



## Collinear functions at NLP

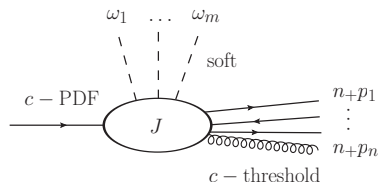
Definition is analogous to the collinear functions in the  $q\bar{q}$  channel.

$$\begin{aligned}
 & i \int d^4 z \mathbf{T} \left[ \chi_{c,\gamma f}(tn_+) \mathcal{L}^{(2)}(z) \right] \\
 &= 2\pi \sum_i \int du \int \frac{d(n+z)}{2} \tilde{J}_{i;\gamma\beta,\mu,fb d} \left( t, u; \frac{n+z}{2} \right) \chi_{c,\beta b}^{\text{PDF}}(un_+) \mathbf{s}_{i;\mu,d}(z_-)
 \end{aligned}$$

$$\mathbf{s}_i(z_-) \in \left\{ \frac{i\partial_{\perp}^{\mu}}{in_{-}\partial} \mathcal{B}_{\mu\perp}^{+}(z_-), \frac{i\partial_{[\mu\perp}}{in_{-}\partial} \mathcal{B}_{\nu\perp]}^{+}(z_-), \frac{1}{(in_{-}\partial)} [\mathcal{B}_{\mu\perp}^{+}(z_-), \mathcal{B}_{\nu\perp}^{+}(z_-)], \dots \right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]



# NLP factorization formula for Drell-Yan: $g\bar{q}$

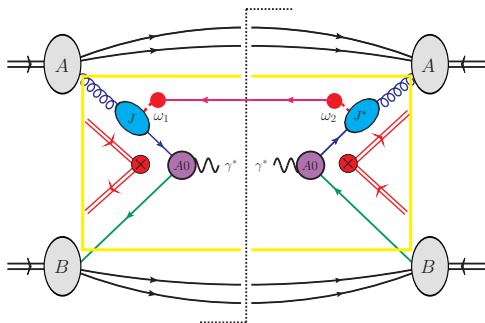
The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \quad \Delta_{g\bar{q},\text{NLP}}(z) = \Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z)$$

where

[SJ, PhD thesis][A. Broggio, SJ, L. Vernazza, to appear]

$$\Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z) = 8H(Q^2) \int d\omega d\omega' G_{\xi q}^*(x_a n + p_A; \omega') G_{\xi q}(x_a n + p_A; \omega) S(\Omega, \omega, \omega')$$



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$$G_{\xi q} = -\frac{1}{2} + \frac{i\alpha_s}{4\pi} \left[ \frac{(n+p)(n-k)}{\mu^2} \right]^{-\epsilon} (C_F - C_A) \left\{ 2 - 4\epsilon - \epsilon^2 \right\} \frac{1}{\epsilon^2} \frac{e^{\epsilon\gamma_E} \Gamma[1+\epsilon] \Gamma[1-\epsilon]^2}{\Gamma[2-2\epsilon]}$$

Calculation of this object up to  $\mathcal{O}(\alpha_s^2)$  performed in [Z.L. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

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$$\Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z) = 8H(Q^2) \int d\omega d\omega' G_{\xi q}^*(x_a n_+ p_A; \omega') G_{\xi q}(x_a n_+ p_A; \omega) S(\Omega, \omega, \omega')$$

In this talk, we focus on  $\mathcal{O}(\alpha_s^2)$  calculation of:

$$\begin{aligned} S(\Omega, \omega, \omega') &= \int \frac{dx^0}{4\pi} \int \frac{dz_-}{2\pi} \int \frac{dz'_-}{2\pi} e^{-iz_- \omega} e^{+iz'_- \omega'} e^{+i\frac{1}{2}x^0 \Omega} \\ &\times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left( \frac{g_s}{in_- \partial_{z'}} \bar{q}_{+\sigma}(x^0 + z'_-) \mathbf{T}^D \{ Y_+^\dagger(x^0) Y_-(x^0) \} \right) \\ &\times \frac{\not{n}_{-\sigma\beta}}{4} \mathbf{T} \left( \{ Y_-^\dagger(0) Y_+(0) \} \mathbf{T}^D \frac{g_s}{in_- \partial_z} q_{+\beta}(z_-) \right) | 0 \rangle \end{aligned}$$

## Generalized soft functions

We make use of the soft building blocks

$$\mathcal{B}_{\pm}^{\mu} = Y_{\pm}^{\dagger} [i D_s^{\mu} Y_{\pm}], \quad q_{\pm}^{\pm} = Y_{\pm}^{\dagger} q_s$$

The only relevant soft function is

$$\begin{aligned} \mathcal{S}(\Omega, \omega, \omega') &= \sum_X \int \frac{dz_-}{2\pi} \int \frac{dz'_-}{2\pi} e^{-iz_- \omega} e^{+iz'_- \omega'} \\ &\times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left( \frac{g_s}{in_- \partial_{z'}} \bar{q}_{+\sigma k}(z'_-) \mathbf{T}^D \{Y_+^{\dagger}(0) Y_-(0)\} \right) | X_s \rangle \\ &\times \frac{\not{q}_{-\sigma\beta}}{4} \langle X_s | \mathbf{T} \left( \{Y_-^{\dagger}(0) Y_+(0)\} \mathbf{T}^D \frac{g_s}{in_- \partial_z} q_{+\beta}(z_-) \right) | 0 \rangle \delta(\Omega - 2E_X) \end{aligned}$$

Not functions of Wilson lines only!

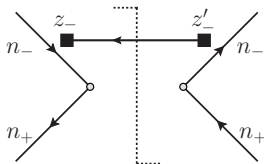
$$\langle q_b(k) | \frac{g_s}{in_- \partial_z} q_{+\beta j}(z_-) | 0 \rangle = \delta_{bj} \frac{g_s}{in_- \partial_z} v_{s\beta}(k) e^{iz_- k} = -\delta_{bj} \frac{g_s}{n_- k} v_{s\beta}(k) e^{iz_- k}$$

→  $\delta(\omega - n_- k_1), \delta(\omega' - n_- k_1 - n_- k_2)$  constraints in the integrand.



## Soft function at first order

We compute directly from the operatorial definition. Wilson lines can be set to unity.

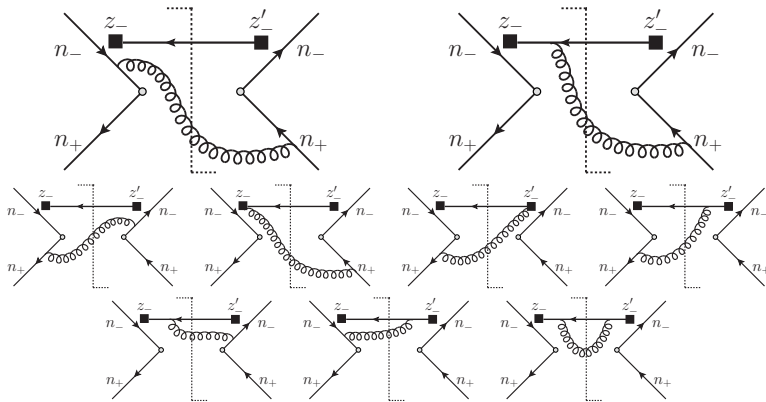


$$S_{g\bar{q}}^{(1)}(\Omega, \omega, \omega') = \frac{\alpha_s}{4\pi} \mu^{2\epsilon} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^\epsilon} \theta(\Omega - \omega) \theta(\omega) \delta(\omega - \omega') \frac{e^{\epsilon\gamma_E}}{\Gamma[1 - \epsilon]}$$

Inserting this result alongside the tree-level collinear functions inside our factorization formula, and performing  $\omega, \omega'$  convolution integrals gives the correct  $\mathcal{O}(\alpha_s)$  result [R. Hamberg, W. van Neerven, T. Matsuura, 1991].

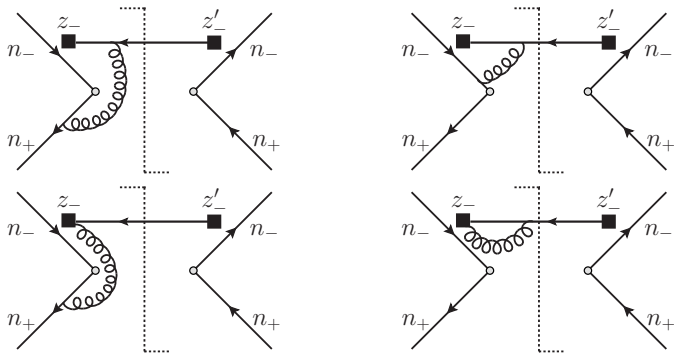
## Some sample two-loop diagrams: double real

$$\begin{aligned}
 \mathcal{S}(\Omega, \omega, \omega') &= \sum_{s, \lambda} \int \frac{d^d k_1}{(2\pi)^{d-1}} \delta^+(k_1^2) \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_2^2) \delta(\omega - n_- k_1 - n_- k_2) \delta(\omega' - n_- k_1) \\
 &\times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left( \frac{g_s}{i n_- \partial_{z'}} \bar{q}_{+\sigma k}(z'_-) \mathbf{T}^D \{ Y_+^\dagger(0) Y_-(0) \} \right) | X_s \rangle \\
 &\times \frac{\not{p}_{-\sigma\beta}}{4} \langle X_s | \mathbf{T} \left( \{ Y_-^\dagger(0) Y_+(0) \} \mathbf{T}^D \frac{g_s}{i n_- \partial_z} q_{+\beta}(z-) \right) | 0 \rangle \delta(\Omega - 2k_1^0 - 2k_2^0)
 \end{aligned}$$



## Some sample two-loop diagrams: virtual real

$$\begin{aligned}
 \mathcal{S}(\Omega, \omega, \omega') &= \sum_{s, \lambda} \int \frac{d^d k_1}{(2\pi)^{d-1}} \delta^+(k_1^2) \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_2^2) \delta(\omega - n_- k_1 - n_- k_2) \delta(\omega' - n_- k_1) \\
 &\quad \times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left( \frac{g_s}{i n_- \partial_{z'}} \bar{q}_{+\sigma k}(z'_-) \mathbf{T}^D \{ Y_+^\dagger(0) Y_-(0) \} \right) | X_s \rangle \\
 &\quad \times \frac{\not{q}_{-\sigma\beta}}{4} \langle X_s | \mathbf{T} \left( \{ Y_-^\dagger(0) Y_+(0) \} \mathbf{T}^D \frac{g_s}{i n_- \partial_z} q_{+\beta}(z_-) \right) | 0 \rangle \delta(\Omega - 2k_1^0 - 2k_2^0)
 \end{aligned}$$



## The calculation

Methods developed for calculations of two-loop soft functions at *leading power*.

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglia, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = k_1^2, \quad P_4 = k_2^2, \quad P_5 = (\omega - n_- k_1)$$

$$P_6 = (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), \quad P_7 = (\omega' - n_- k_1 - n_- k_2)$$

$$\hat{I}_{\mathcal{I}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^7 \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i} \left[ \frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+} \right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

## The calculation

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$$P_6 = (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), \quad P_7 = (\omega' - n_- k_1 - n_- k_2)$$

$$\hat{I}_{\mathcal{S}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^7 \frac{1}{P_i^{\alpha_i}}$$

- ▶ The reduction is implemented in **LiteRed**
- ▶ 4 + 1 topologies are needed to reduce the soft functions
- ▶ We have 6 + 2 Master Integrals (MIs)
  - ▶ 4 MIs implementing the  $\delta(\omega - n_- k_1) \delta(\omega' - n_- k_1 - n_- k_2)$  and  $\omega \leftrightarrow \omega'$  constraint:  $\hat{I}_1 - \hat{I}_4$
  - ▶ 1 MI with  $\delta(\omega - n_- k_1) \delta(\omega - \omega')$ :  $\hat{I}_5$
  - ▶ 1 MI with  $\delta(\omega - n_- k_1 - n_- k_2) \delta(\omega - \omega')$ :  $\hat{I}_6$

## Reduced expressions

Double real

$$\begin{aligned} S_{g\bar{q}}^{(2)2r0v}(\Omega, \omega, \omega') &= \frac{\alpha_s^2}{(4\pi)^2} T_F \left[ C_F \left( \frac{(4-\epsilon)(1-\epsilon)(1-2\epsilon)}{\epsilon^2 \omega^2 (\Omega-\omega)} \hat{I}_6 + \frac{4(2-3\epsilon)(1-3\epsilon)}{\epsilon^2 \omega (\Omega-\omega)^2} \hat{I}_5 \right) \delta(\omega-\omega') \right. \\ &\quad + (C_A - 2C_F) \left( \frac{(1-2\epsilon)(\omega+\omega')}{\epsilon \omega \omega' (\Omega-\omega)(\omega-\omega')} \hat{I}_3 - \frac{(1-2\epsilon)(\omega+\omega')}{\epsilon \omega \omega' (\Omega-\omega')(\omega-\omega')} \hat{I}_1 \right. \\ &\quad \left. \left. + \frac{(\Omega-\omega)(\omega+\omega')}{2\omega\omega'} \hat{I}_4 + \frac{(\Omega-\omega')(\omega+\omega')}{2\omega\omega'} \hat{I}_2 \right) \right] \end{aligned}$$

where for example

$$\hat{I}_1(\Omega, \omega, \omega') \equiv \hat{I}_{\mathcal{A}}(0, 0, 1, 1, 1, 1, 1), \quad \hat{I}_2(\Omega, \omega, \omega') \equiv \hat{I}_{\mathcal{A}}(1, 1, 1, 1, 1, 1, 1)$$

and family  $\mathcal{A}$  propagators are

$$\begin{aligned} P_1 &= (k_1 + k_2)^2, & P_2 &= n_+ k_2, & P_3 &= k_1^2, & P_4 &= k_2^2, \\ P_5 &= (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), & P_6 &= (\omega - n_- k_1), \\ P_7 &= (\omega' - n_- k_1 - n_- k_2) \end{aligned}$$

The integrals are calculated using the differential equations method.

## Reduced expressions

Virtual-real

$$S_{g\bar{q}}^{(2)1r1v}(\Omega, \omega, \omega') = \text{Re} \left\{ -i \frac{\alpha_s^2 T_F}{(4\pi)^2} (2C_F - C_A) \left[ \frac{1}{\omega\omega'} \hat{J}_1 + \frac{(\omega + \omega')(\Omega - \omega')}{\omega\omega'} \hat{J}_2 \right] \right\}$$

The master integrals are

$$\hat{J}_1(\Omega, \omega, \omega') \equiv \hat{J}_{\mathcal{A}}(0, 1, 1, 1, 1, 1, 1), \quad \hat{J}_2(\Omega, \omega, \omega') \equiv \hat{J}_{\mathcal{A}}(1, 1, 1, 1, 1, 1, 1)$$

in terms of the relevant propagators

$$P_1 = k^2, \quad P_2 = (k + k_1)^2, \quad P_3 = n_+ k, \quad P_4 = k_1^2, \\ P_5 = (\Omega - n_- k_1 - n_+ k_1), \quad P_6 = (\omega - n_- k - n_- k_1), \quad P_7 = (\omega' - n_- k_1)$$

The propagators  $\{P_4, P_5, P_6, P_7\}$  are cut.



## Fixed-order checks

$$\Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z) = 8H(Q^2) \int d\omega d\omega' G_{\xi q}^*(x_a n_{+p_A}; \omega_2) G_{\xi q}(x_a n_{+p_A}; \omega) S(\Omega, \omega, \omega')$$

We have all the ingredients up to two-loops. Different contributing regions are

- ▶ **Hard:** 1-loop hard and NLO soft functions
- ▶ **Collinear:** 1-loop collinear and NLO soft functions
- ▶ **Soft:** NNLO soft functions, all else tree-level

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

$$\begin{aligned} \Delta_{g\bar{q}}^{(2)}(z) &= C_F \left( \frac{13}{3} \ln^3(1-z) + \ln(1-z)(3 - 12\zeta_2) + \zeta_3 - 2 \right) \\ &+ C_A \left( \frac{35}{3} \ln^3(1-z) - 6 \ln^2(1-z) + \ln(1-z)(-21 - 8\zeta_2) + 6\zeta_2 + 38\zeta_3 - 5 \right) \end{aligned}$$

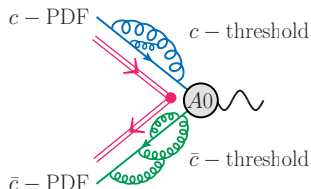
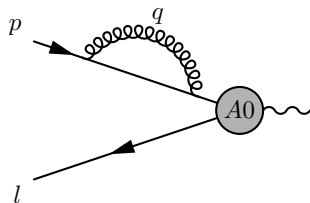
## Outlook and Summary

- ▶ Endpoint divergences present already at LL, similar to off-diagonal channels of other processes
- ▶ Framework developed to the point where refactorization ideas can be applied.
- ▶ Our derivation and validation of the factorization formula provides the groundwork for further such studies.
- ▶ We provide higher perturbative order data for the objects appearing in the factorization formula to facilitate further investigations into the open conceptual issues.

Thank you

# Auxiliary slides

## Absence of collinear functions at LP



The purely threshold-collinear loops are scaleless and vanish.

At **leading power**, we can apply the **decoupling transformation**  $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$  which removed the soft-collinear interactions from the Lagrangian.

[C. Bauer, D. Pirjol, and I. Stewart, hep-ph/0109045]

The threshold-collinear modes can be identified with  $c$ -PDF modes,  $\chi_c \rightarrow \chi_c^{\text{PDF}}$

$$\chi_c(tn_+) = \int du \tilde{J}(t, u) \chi_c^{\text{PDF}}(un_+) \quad \tilde{J}(t, u) = \delta(t - u)$$

## Collinear functions at NLP

Beyond LP, the **decoupling transformation** does not remove soft-collinear interactions in the Lagrangian.

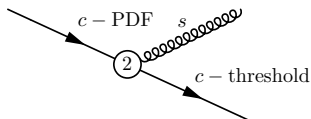
Consider an example of subleading SCET Lagrangian:

[M. Beneke and Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2} \bar{\chi}_c^{(0)} z_{\perp}^{\mu} z_{\perp}^{\rho} [i\partial_{\rho} i n_{-} \partial \mathcal{B}_{\mu}^{+}(z_{-})] \frac{\not{n}_{+}}{2} \chi_c^{(0)} \quad \text{where} \quad \mathcal{B}_{\pm}^{\mu} = Y_{\pm}^{\dagger} [iD_s^{\mu} Y_{\pm}]$$

The subleading power Lagrangian terms enter the basis through time-ordered product operators

$$\left( J_{A0,2\xi}^{T2}(t) \right)^{\mu} = i \int d^4 z \mathbf{T} \left[ J_{A0}^{\mu}(t) \mathcal{L}_{2\xi}^{(2)}(z) \right]$$



## Collinear functions at NLP

PDF collinear modes can be radiated into the final state:

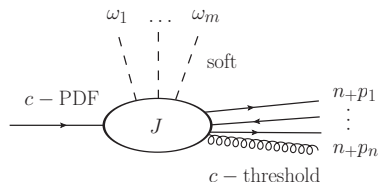
$$p_c \sim Q(1, \lambda^2, \lambda) \text{ and } p_{c\text{-PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$$

$$\begin{aligned} & i \int d^4 z \mathbf{T} \left[ \chi_{c, \gamma f}(tn_+) \mathcal{L}^{(2)}(z) \right] \\ &= 2\pi \sum_i \int du \int \frac{d(n+z)}{2} \tilde{J}_{i; \gamma \beta, \mu, fbd} \left( t, u; \frac{n+z}{2} \right) \chi_{c, \beta b}^{\text{PDF}}(un_+) \mathfrak{s}_{i; \mu, d}(z_-) \end{aligned}$$

$$\mathfrak{s}_i(z_-) \in \left\{ \frac{i\partial_{\perp}^{\mu}}{in_{-}\partial} \mathcal{B}_{\mu\perp}^{+}(z_-), \frac{i\partial_{[\mu\perp}}{in_{-}\partial} \mathcal{B}_{\nu\perp]}^{+}(z_-), \frac{1}{(in_{-}\partial)} [\mathcal{B}_{\mu\perp}^{+}(z_-), \mathcal{B}_{\nu\perp}^{+}(z_-)], \dots \right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]



$$\begin{aligned}
S_{g\bar{q}}^{(2)1r1v}(\Omega, \omega, \omega') &= \text{Re} \left\{ i g_s^4 T_F (2C_F - C_A) \int [dk_1] \int [dk] \right. \\
&\cdot (2\pi) \delta(k_1^2) \theta(k_1^0) \delta(\Omega - 2k_1^0) \delta(\omega' - n_- k_1) \delta(\omega - n_- k - n_- k_1) \\
&\cdot \left. \frac{(k + k_1)^2 - k^2 + n_- k_1 (n_+ k + n_+ k_1) + n_+ k_1 (n_- k + n_- k_1)}{k^2 (k + k_1)^2 (n_+ k) (-n_- k - n_- k_1) (-n_- k_1)} \right\}
\end{aligned}$$



## Problems at the Endpoint

Focus on one piece of the factorization formula

$$\int d\omega J_1^{(1)}(x_a n_{+p_A}; \omega) S_1^{(1)}(\Omega, \omega)$$

with

$$J_1^{(1)}(x_a n_{+p_A}; \omega) = \frac{\alpha_s}{4\pi} \frac{1}{(x_a n_{+p_A})} \left( \frac{(x_a n_{+p_A}) \omega}{\mu^2} \right)^{-\epsilon} \frac{e^{\epsilon \gamma_E} \Gamma[1 + \epsilon] \Gamma[1 - \epsilon]^2}{(-1 + \epsilon)(1 + \epsilon) \Gamma[2 - 2\epsilon]} \\ \times \left( C_F \left( -\frac{4}{\epsilon} + 3 + 8\epsilon + \epsilon^2 \right) - C_A (-5 + 8\epsilon + \epsilon^2) \right)$$

$$S_1(\Omega, \omega) = \frac{\alpha C_F}{2\pi} \frac{\mu^{2\epsilon} e^{\epsilon \gamma_E}}{\Gamma[1 - \epsilon]} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^\epsilon} \theta(\omega) \theta(\Omega - \omega) + \mathcal{O}(\alpha^2)$$

As we have seen before, performing the  $d\omega$  convolution integral in  $d$ -dimensions, and only *after* expanding in  $\epsilon$  gives the correct results.

## Problems at the Endpoint

$$\int_0^\Omega d\omega \underbrace{(n+p\omega)^{-\epsilon}}_{\text{collinear piece}} \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega-\omega)^\epsilon}}_{\text{soft piece}}$$

For resummation, we treat the two objects independently, and expand in  $\epsilon$  prior to performing the final convolution. However, there is a problem! At two loops:

$$J_1^{(1)}(x_a n + p_A; \omega) \sim \alpha_s \log(\omega)$$

and

$$S_1(\Omega, \omega) \sim \alpha_s \delta(\omega) + \mathcal{O}(\alpha^2)$$

Hence, first expanding in  $\epsilon$  and performing convolution after yields

$$\begin{aligned} \Delta_{\text{NLP-coll}}^{\text{dyn}(2)}(z) &= \frac{\alpha_s^2}{(4\pi)^2} \left( C_F^2 \left( -\frac{32}{\epsilon^2} - \frac{8}{\epsilon} \left[ 5 - 8 \ln(1-z) - 4 \int d\omega \delta(\omega) \ln\left(\frac{\omega}{Q}\right) \right] \right) \right. \\ &\quad \left. + C_A C_F \frac{40}{\epsilon} + \mathcal{O}(\epsilon^0) \right) \end{aligned}$$

A couple of issues arise. The convolution  $d\omega$  integral is now divergent at the endpoint. This prohibits the application of standard RG methods.

## DE method for MIs

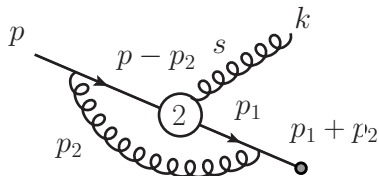
Convenient to change to dimensionless variable  $\omega \rightarrow r \Omega$

$$\begin{aligned} I'_1(r) &= \frac{1}{\Omega^2} \left( \frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_1(\Omega, r), & I'_2(r) &= \frac{1}{\Omega} \left( \frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_2(\Omega, r), \\ I'_3(r) &= \left( \frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_3(\Omega, r), & I'_4(r) &= \Omega \left( \frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_4(\Omega, r), \\ I'_5(r) &= \Omega^2 \left( \frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_5(\Omega, r) \end{aligned}$$

System of DEs can be put into canonical form [\[J. Henn, 1304.1806\]](#)

$$\frac{d\vec{I}(r)}{dr} = \epsilon A(r) \cdot \vec{I}(r)$$
$$\begin{aligned} I'_1(r) &= \frac{2(1-r)^2}{2-9\epsilon+9\epsilon^2} I_1(r), \\ I'_3(r) &= \frac{1}{\epsilon^2} I_3(r), \\ I'_4(r) &= -\frac{1}{\epsilon^2(1-r)} I_4(r), \\ I'_5(r) &= -\frac{1+r}{2\epsilon^2(1-r)r} I_4(r) + \frac{1}{\epsilon^2 r} I_5(r) \end{aligned} \quad A(r) = \begin{bmatrix} -\frac{1}{r} + \frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

## Collinear functions at one-loop



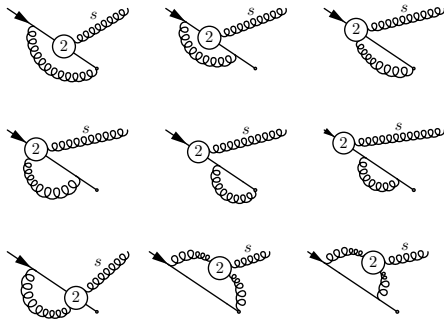
Use NLP Feynman rules to calculate these new objects up to  $\mathcal{O}(\alpha_s)$ . [M. Beneke, M. Garry, R. Szafron, J. Wang, 1808.04742] Some non-standard features present!

$$X^\mu = \partial^\mu \left[ (2\pi)^d \delta^{(d)} \left( \sum p_{\text{in}} - \sum p_{\text{out}} \right) \right]$$

$$\mathcal{O}(\lambda^2) : \quad ig_s \mathbf{T}^A A^{\rho\nu}(k, p, p') \frac{\not{p}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu})$$

$$A^{\rho\nu}(k, p, p') \equiv \frac{1}{2} \left[ (n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left( \frac{\not{p}'_\perp}{n_+ p'} \gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p} \right) \right]$$

## Collinear functions at one-loop



$$\begin{aligned}
 J_1(n+q, n+p; \omega) &= -\frac{1}{n+p} \delta(n+q - n+p) + 2 \frac{\partial}{\partial n+q} \delta(n+q - n+p) \\
 &+ \frac{\alpha_s}{4\pi} \frac{1}{(n+p)} \left( \frac{n+p\omega}{\mu^2} \right)^{-\epsilon} \frac{e^{\epsilon\gamma_E} \Gamma[1+\epsilon] \Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]} \\
 &\times \left( C_F \left( -\frac{4}{\epsilon} + 3 + 8\epsilon + \epsilon^2 \right) - C_A (-5 + 8\epsilon + \epsilon^2) \right) \delta(n+q - n+p)
 \end{aligned}$$

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]