Threshold factorization of the Drell-Yan quark-gluon channel and two-loop soft function at NLP

Sebastian Jaskiewicz

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LBNL

based on a publication in preparation with
Alessandro Broggio and Leonardo Vernazza

Durham University

Motivations and focus

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

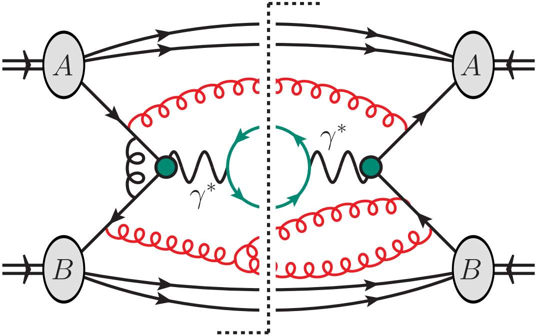
Recently calculated to N^{3}LO
[C. Duhr, F.Dulat,
B. Mistlberger, 2001.07717]

Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \to 1$$

Define power counting parameter λ:

$$\lambda = \sqrt{1-z}$$



Schematic form for production cross-sections near threshold, $z \to 1$ $(Q^2\lambda^2 \gg \Lambda^2_{\mathrm{QCD}})$:

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \right]$$

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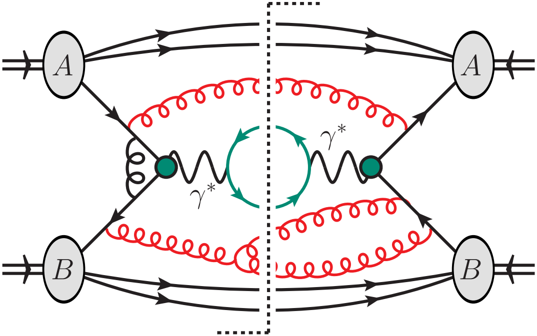

Threshold limit:

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Naturally, the idea is to factorize the physics appearing at different scales:

$$\sigma \sim H \otimes J \otimes \ldots \otimes J \otimes S$$

and solve RG equations for each object to sum the large logarithms.

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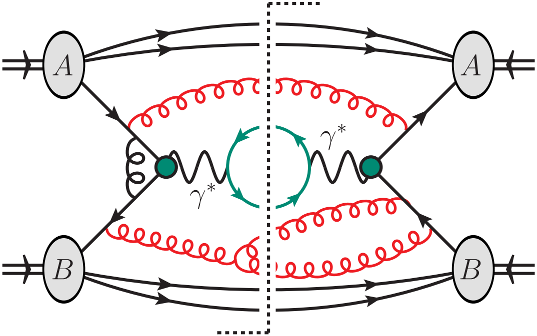

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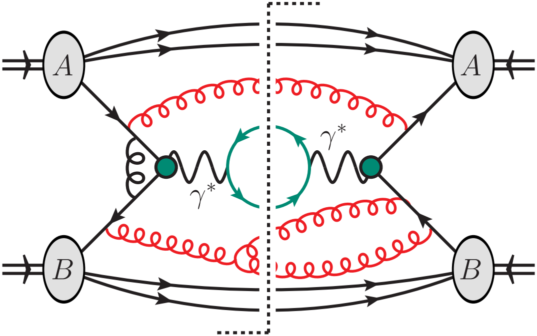

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Plenty recent NLP studies

Subleading power resummed thrust spectrum for $H \to gg$ (LL)

[I. Moult, I. Stewart, G. Vita, H. Zhu, 1804.04665, 1910.14038]

Drell-Yan and Higgs production at threshold (LL)

[M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

[M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1910.12685]

Resummation of rapidity logarithms: the EE correlator in N=4 SYM (LL)

[I. Moult, G. Vita, K. Yan, 1912.02188]

Factorization at Subleading Power and Endpoint Divergences in SCET (LL, NLL)

[Z. L. Liu, M.Neubert, 1912.08818]

[Z. L. Liu, B. Mecaj, M.Neubert, X.Wang, 2009.04456, 2009.06779]

[Z. L. Liu, M.Neubert, M. Schnubel, X. Wang, 2112.00018]

DIS $x \to 1$ [M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 2008.04943]

Soft-collinear gravity beyond the leading power

[M. Beneke, P. Hager, R. Szafron, 2112.04983] →see talk by P. Hager

Factorization at subleading power in Deep Inelastic Scattering in the $x \to 1$ limit

[M. Luke, J. Roy, A. Spourdalakis, 2210.02529] →see talk by M. Luke

Plenty recent NLP studies

Power-enhanced QED corrections to $B_q \to \mu^+\mu^-$ (LL)

[M. Beneke, C. Bobeth , R. Szafron, 1908.07011]

Violation of KSZ theorem in SCET

[M. Beneke, M. Garny, R. Szafron, J.Wang, 1907.05463]

Subleading Power Factorization with Radiative Functions

[I. Moult, I. Stewart, G. Vita, 1905.07411]

Drell-Yan q_T Resummation of Fiducial Power Corrections at N^{3}LL

[M. Ebert, J. Michel, I. Stewart, F. Tackmann, 2006.11382]

Resummation of double logarithms in loop-induced processes with EFT

[J. Wang, 1912.09920]

Refactorisation in subleading $\bar{B} \to X_s\gamma$

[T. Hurth, R. Szafron, 2301.01739] →see talk by R. Szafron

Structure-Dependent QED Effects in Exclusive B Decays at Subleading Power

[C. Cornella, M. König, M. Neubert, 2212.14430] →see talks by C. Cornella and M. Neubert

Muon-electron backward scattering: endpoint singularities in SCET

[G. Bell, P. Böer, T. Feldmann, 2205.06021]→see talk by P. Böer

Next-to-leading power endpoint factorization and resummation for off-diagonal "gluon" thrust

[M. Beneke, M. Garny, **SJ**, J. Strohm, R. Szafron, L. Vernazza, J.Wang, 2205.04479] →see talk by J. Strohm last year

Brief introduction to NLP SCET

In this talk we employ position-space SCET [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358]

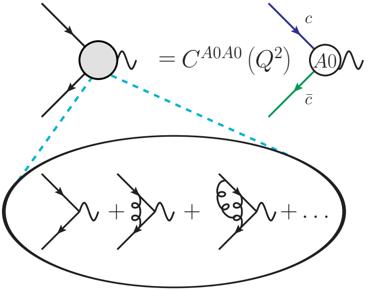
$$\psi(x) \to \underbrace{\psi_1(x) + \cdots + \psi_N(x)}_{N \text{ collinear fermion fields}} + q(x) \qquad \mathcal{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter $\lambda = \sqrt{1-z}$:

$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots \qquad \text{E.g.} \quad \mathcal{L}_c^{(1)} = \bar{\chi}_c \, i x_\perp^\mu \left[i n_- \partial \mathcal{B}_\mu^+ \right] \frac{\not{n}_+}{2} \chi_c$$

We keep the *collinear*, *anti-collinear*, and *soft* degrees of freedom. The *hard* modes are integrated out:

$$\bar{\psi}\gamma^\mu\psi = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k} \; C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, \ldots, t_{i_{n_i}})$$



SCET introduction: Lagrangian

In this talk we employ position-space SCET [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358]

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Separate collinear sectors interact only through soft gluon interactions. Focusing on the LP term:

$$\mathcal{L}_c^{(0)} = \bar{\xi}_c \left(i n_- D_c + g\, n_- A_s(x_-) + i \not{D}_{\perp c} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi_c + \mathcal{L}_{c,\text{YM}}^{(0)}$$

with $i n_- D_c = i n_- \partial + g\, n_- A_c(x)$, $x_-^\mu = (n_+ x) n_-^\mu / 2$.

The decoupling transformation, $\xi_c \to Y_+ \xi_c^{(0)}$ and $A_c^\mu \to Y_+ A_c^{(0)\mu} Y_+^\dagger$, separates the soft and collinear sectors at LP [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

$$\bar{\xi}_c \left(i n_- D_c + g_s n_- A_s \right) \frac{\not{n}_+}{2} \xi_c = \bar{\xi}_c^{(0)} \, i n_- D_c^{(0)} \frac{\not{n}_+}{2} \xi_c^{(0)}$$

where

$$Y_\pm(x) = \mathbf{P} \exp \left[i g_s \int_{-\infty}^{0} ds \, n_\mp A_s(x + s n_\mp) \right]$$

Brief introduction to NLP SCET

Generic N-jet operator has the form:

[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

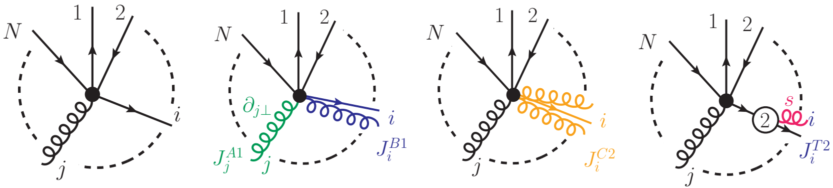
$$J = \int \prod_{i=1}^{N} \prod_{k_i=1}^{n_i} dt_{ik_i}\; C(\{t_{ik_i}\}) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, ..., t_{i_{n_i}})$$

where the Js are constructed by multiplying collinear gauge invariant building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad\qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

by acting on these with derivatives $i\partial^\mu_{\perp i} \sim \lambda$, and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

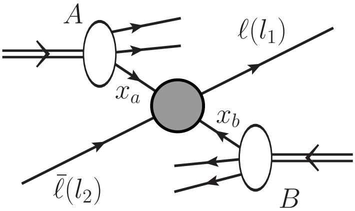
Generic leading power N-jet operator:



The Drell-Yan Process

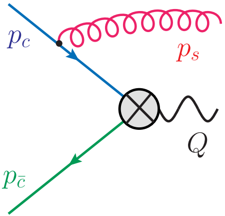
$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)\,] + X(p_X)$$

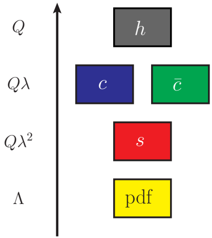
$$z = \frac{Q^2}{\hat{s}} \to 1 \qquad \lambda = \sqrt{(1-z)}$$



$$\begin{aligned} p_c &= (n_+p_c, n_-p_c, p_{c\perp}) \sim Q(1,\lambda^2,\lambda) \\ p_{\bar{c}} &= (n_+p_{\bar{c}}, n_-p_{\bar{c}}, p_{\bar{c}\perp}) \sim Q(\lambda^2,1,\lambda) \\ p_s &= (n_+p_s, n_-p_s, p_{s\perp}) \sim Q(\lambda^2,\lambda^2,\lambda^2) \end{aligned}$$

$$Q^2\lambda^2 = Q^2(1-z) \gg \Lambda^2_{\rm QCD} \qquad p_{c-\rm PDF} \sim (Q, \Lambda^2/Q, \Lambda)$$





Drell Yan: Factorization of the partonic cross-section

First let us compare leading power and *next*-to-leading power cross-sections schematically:

$$\frac{d\sigma_{\mathrm{DY}}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\mathrm{LP}}(z) \; + \; \hat{\sigma}_{ab}^{\mathrm{NLP}}(z) \; + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$

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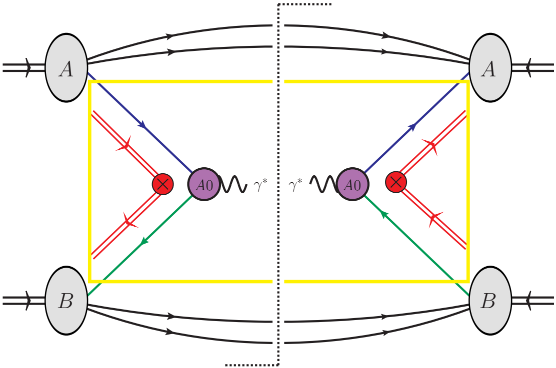

$$\hat{\sigma}^{\,\mathrm{LP}}(z) = Q \, H(Q^2) \, S_{\mathrm{DY}}(\Omega)$$

[G. Sterman 1987] [S. Catani, L. Trentadue 1989] [G. P. Korchemsky G. Marchesini, 1993]

[S. Moch, A. Vogt, hep-ph/0508265] [T. Becher, M. Neubert, G. Xu, 0710.0680]

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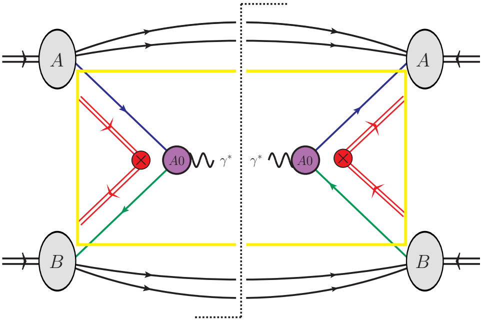

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{i\Omega\, x^0/2}\, \frac{1}{N_c} {\rm Tr}\, \langle 0|\bar{\mathbf{T}} \left[Y_+^\dagger(x) Y_-(x)\right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0)\right] |0\rangle$$

$$Y_\pm(x) \quad = \quad \mathbf{P}\exp\left[ig_s \int_{-\infty}^0 ds\, n_\mp A_s(x+sn_\mp)\right].$$

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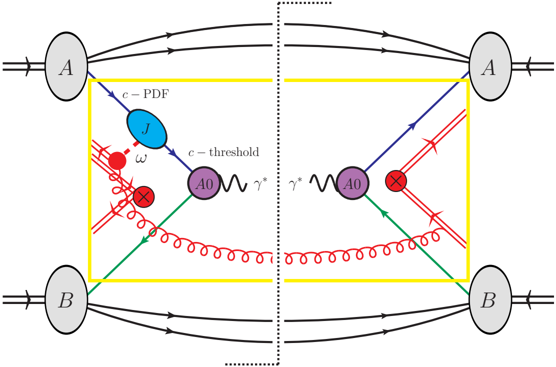


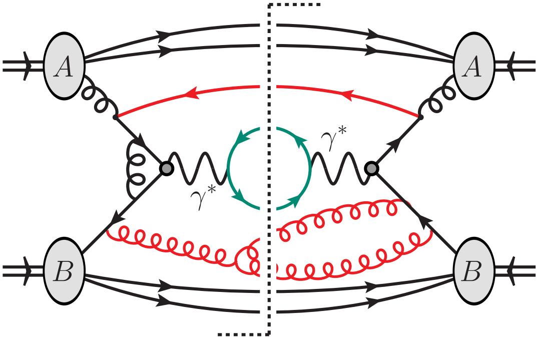
$$\hat{\sigma}_{q\bar{q}}^{\rm NLP}(z) = \sum_{\rm terms} \left[C \otimes J \otimes \bar{J} \right]^2 \otimes S$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

The off-diagonal DY process

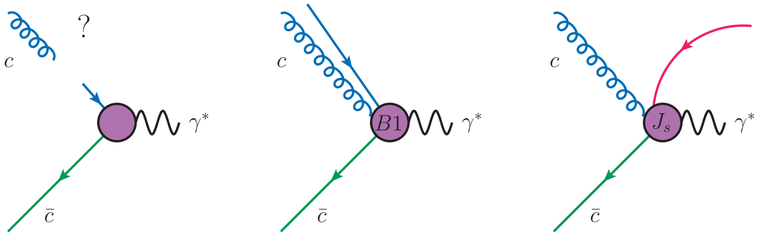
Now we consider initial state gluon from a proton A, which must be converted into a collinear quark via an emission of a soft antiquark.



In the EFT picture

This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\rho\psi(0) = \sum_{m_1,m_2} \int \{dt_k\}\,\{d\bar{t}_{\bar{k}}\}\,\widetilde{C}^{\,m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)\,J_s(0)\,J_\rho^{m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)$$



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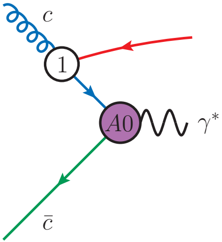
This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, \underbrace{\bar{\chi}_{\bar{c}}(\bar{t}n_-)\,\gamma_{\perp\mu}\,\chi_c(tn_+)}_{J^{A0}_\mu(t,\bar{t})}$$



Beyond LP, the decoupling transformation does not remove soft-collinear interactions in the Lagrangian.

$$\left(J^{T2}_{A0,\xi q}(t)\right)^\mu = i\int d^dz\, \mathbf{T}\left[\chi_c\,(tn_+)\;\mathcal{L}^{(1)}_{\xi q}(z)\right] \qquad \mathcal{L}^{(1)}_{\xi q} = \bar{q}_+ \not{\mathcal{A}}_{c\perp}\chi_c + \text{h.c.}$$

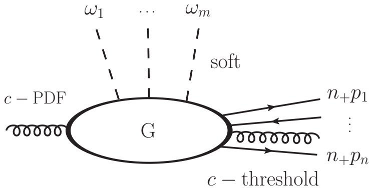
Collinear functions at NLP

PDF collinear modes can be radiated into the final state:
$p_c \sim Q(1,\lambda^2,\lambda)$ and $p_{c-\mathrm{PDF}} \sim (Q,\Lambda^2/Q,\Lambda)$

$$i\int d^dz\, \mathbf{T}\left[\xi_c(tn_+)\mathcal{L}^{(1)}_{\xi q}(z)\right] = 2\pi\int du\int dz_-\, \tilde{G}_{\xi q}\left(t,u;z_-\right)\mathcal{A}^{\mathrm{PDF}}_{c\perp}(un_+)\,\mathfrak{s}_{\xi q}(z_-)$$

Only one soft strucutre present at NLP

$$\mathfrak{s}_{\xi q}(z_-) = \frac{g_s}{in_-\partial_z}q^+(z_-)$$



Collinear functions at NLP

Definition is analogous to the collinear functions in the $q\bar{q}$ channel.

$$i\int d^4z\,\mathbf{T}\Big[\chi_{c,\gamma f}\,(tn_+)\;\mathcal{L}^{(2)}(z)\Big]$$
$$= 2\pi\sum_i\int du\int\frac{d(n_+z)}{2}\,\tilde{J}_{i;\gamma\beta,\mu,fbd}\left(t,u;\frac{n_+z}{2}\right)\chi^{\rm PDF}_{c,\beta b}(un_+)\,\mathfrak{s}_{i;\mu,d}(z_-)$$

$$\mathfrak{s}_i(z_-)\in\left\{\frac{i\partial^\mu_\perp}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\,,\;\frac{i\partial_{[\mu_\perp}}{in_-\partial}\mathcal{B}^+_{\nu_\perp]}(z_-)\,,\;\frac{1}{(in_-\partial)}\left[\mathcal{B}^+_{\mu_\perp}(z_-),\mathcal{B}^+_{\nu_\perp}(z_-)\right],\ldots\right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

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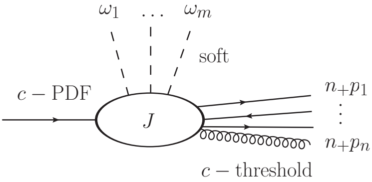


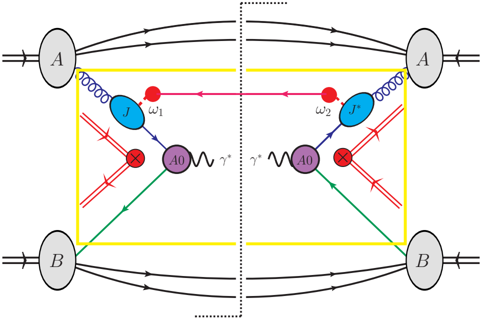
NLP factorization formula for Drell-Yan: $g\bar{q}$

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \qquad \Delta_{g\bar{q},\mathrm{NLP}}(z) = \Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z)$$

where [**SJ**, PhD thesis][A. Broggio, **SJ**, L. Vernazza, to appear]

$$\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z) = 8H(Q^2) \int d\omega\, d\omega'\, G^*_{\xi q}(x_a n_+ p_A; \omega')\, G_{\xi q}(x_a n_+ p_A; \omega)\, S(\Omega, \omega, \omega')$$



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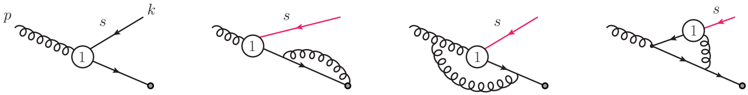


$$G_{\xi q} = -\frac{1}{2} + \frac{i\alpha_s}{4\pi}\left[\frac{(n_+p)(n_-k)}{\mu^2}\right]^{-\epsilon}\Big(C_F - C_A\Big)\Big\{2 - 4\epsilon - \epsilon^2\Big\}\frac{1}{\epsilon^2}\frac{e^{\epsilon\gamma_E}\,\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{\Gamma[2-2\epsilon]}$$

Calculation of this object up to $\mathcal{O}(\alpha_s^2)$ performed in [Z.L. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

NLP factorization formula for Drell-Yan: $g\bar{q}$

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)}\frac{\hat{\sigma}(z)}{z} \qquad \Delta_{g\bar{q},\text{NLP}}(z) = \Delta^{dyn}_{g\bar{q},\text{NLP}}(z)$$

where

[A. Broggio, **SJ**, L. Vernazza, to appear]

$$\Delta^{dyn}_{g\bar{q},\text{NLP}}(z) = 8H(Q^2)\int d\omega\, d\omega'\, G^*_{\xi q}(x_a n_+ p_A;\omega')\, G_{\xi q}(x_a n_+ p_A;\omega)\; S(\Omega,\omega,\omega')$$

In this talk, we focus on $\mathcal{O}(\alpha_s^2)$ calculation of:

$$S(\Omega,\omega,\omega') = \int \frac{dx^0}{4\pi}\int\frac{dz_-}{2\pi}\int\frac{dz'_-}{2\pi}e^{-iz_-\omega}e^{+iz'_-\omega'}e^{+i\frac{1}{2}x^0\Omega}$$

$$\times\frac{1}{C_F C_A}\langle 0|\bar{\mathbf{T}}\Big(\frac{g_s}{in_-\partial_{z'}}\bar{q}_{+\sigma}(x^0+z'_-)\,\mathbf{T}^D\,\big\{Y_+^\dagger(x^0)Y_-(x^0)\big\}\Big)$$

$$\times\frac{\not{n}_{-\sigma\beta}}{4}\mathbf{T}\Big(\big\{Y_-^\dagger(0)Y_+(0)\big\}\,\mathbf{T}^D\,\frac{g_s}{in_-\partial_z}q_{+\beta}(z_-)\Big)|0\rangle$$

Generalized soft functions

We make use of the soft building blocks

$$\mathcal{B}^{\mu}_{\pm} = Y^{\dagger}_{\pm}\,[i\,D^{\mu}_{s}\,Y_{\pm}]\,, \quad q^{\pm} = Y^{\dagger}_{\pm}\,q_{s}$$

The only relevant soft function is

$$\mathcal{S}(\Omega,\omega,\omega') = \sum_{X}\int\frac{dz_-}{2\pi}\int\frac{dz'_-}{2\pi}e^{-iz_-\omega}e^{+iz'_-\omega'}$$
$$\times\frac{1}{C_F C_A}\langle 0|\bar{\mathbf{T}}\Big(\frac{g_s}{in_-\partial_{z'}}\,\bar{q}_{+\sigma k}(z'_-)\,\mathbf{T}^{D}\,\{Y^{\dagger}_{+}(0)Y_{-}(0)\}\Big)|X_s\rangle$$
$$\times\frac{\not{n}_{-\sigma\beta}}{4}\langle X_s|\mathbf{T}\Big(\{Y^{\dagger}_{-}(0)Y_{+}(0)\}\,\mathbf{T}^{D}\,\frac{g_s}{in_-\partial_z}\,q_{+\beta}(z_-)\Big)|0\rangle\delta(\Omega-2E_X)$$

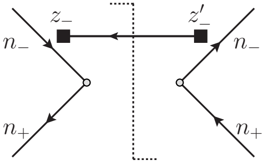
Not functions of Wilson lines only!

$$\langle q_b(k)|\frac{g_s}{in_-\partial_z}q_{+\beta j}(z_-)|0\rangle = \delta_{bj}\frac{g_s}{in_-\partial_z}v_{s\beta}(k)e^{iz_-k} = -\delta_{bj}\frac{g_s}{n_-k}v_{s\beta}(k)e^{iz_-k}$$

$\rightarrow\quad \delta(\omega-n_-k_1),\ \delta(\omega'-n_-k_1-n_-k_2)$ constraints in the integrand.

Soft function at first order

We compute directly from the operatorial definition. Wilson lines can be set to unity.

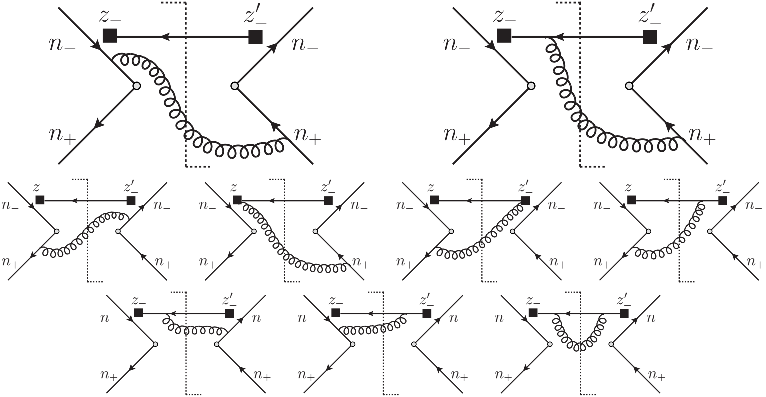


$$S^{(1)}_{g\bar{q}}(\Omega,\omega,\omega') \quad = \quad \frac{\alpha_s}{4\pi}\,\mu^{2\epsilon}\,\frac{1}{\omega^{1+\epsilon}}\,\frac{1}{(\Omega-\omega)^{\epsilon}}\,\theta(\Omega-\omega)\,\theta(\omega)\,\delta(\omega-\omega')\,\frac{e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]}$$

Inserting this result alongside the tree-level collinear functions inside our factorization formula, and performing ω, ω' convolution integrals gives the correct $\mathcal{O}(\alpha_s)$ result [R. Hamberg, W. van Neerven, T. Matsuura, 1991].

Some sample two-loop diagrams: double real

$$\mathcal{S}(\Omega,\omega,\omega') = \sum_{s,\lambda}\int \frac{d^dk_1}{(2\pi)^{d-1}}\delta^+(k_1^2)\int\frac{d^dk_2}{(2\pi)^{d-1}}\delta^+(k_2^2)\delta(\omega-n_-k_1-n_-k_2)\delta(\omega'-n_-k_1)$$
$$\times\frac{1}{C_FC_A}\langle 0|\bar{\mathbf{T}}\Big(\frac{g_s}{in_-\partial_{z'}}\,\bar{q}_{+\sigma k}(z'_-)\;\mathbf{T}^D\,\{Y_+^\dagger(0)Y_-(0)\}\Big)|X_s\rangle$$
$$\times\frac{\not{n}_{-\sigma\beta}}{4}\langle X_s|\mathbf{T}\Big(\{Y_-^\dagger(0)Y_+(0)\}\,\mathbf{T}^D\,\frac{g_s}{in_-\partial_z}\,q_{+\beta}(z_-)\Big)|0\rangle\delta(\Omega-2k_1^0-2k_2^0)$$

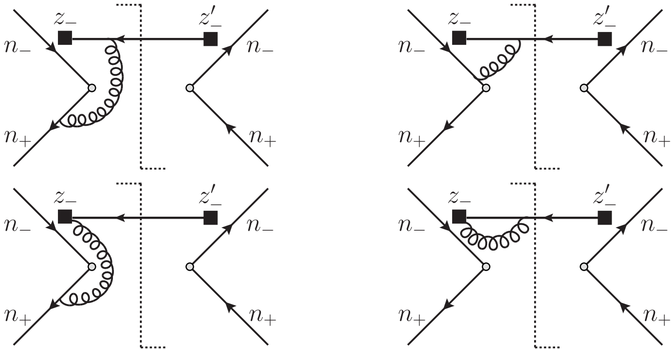


Some sample two-loop diagrams: virtual real

$$\mathcal{S}(\Omega,\omega,\omega') = \sum_{s,\lambda}\int \frac{d^dk_1}{(2\pi)^{d-1}}\delta^+(k_1^2)\int \frac{d^dk_2}{(2\pi)^{d-1}}\delta^+(k_2^2)\delta(\omega - n_-k_1 - n_-k_2)\delta(\omega' - n_-k_1)$$

$$\times \frac{1}{C_F C_A}\langle 0|\bar{\mathbf{T}}\Big(\frac{g_s}{in_-\partial_{z'}}\, \bar{q}_{+\sigma k}(z'_-)\, \mathbf{T}^D\,\{Y_+^\dagger(0)Y_-(0)\}\Big)|X_s\rangle$$

$$\times \frac{\not{n}_{-\sigma\beta}}{4}\langle X_s|\mathbf{T}\Big(\{Y_-^\dagger(0)Y_+(0)\}\,\mathbf{T}^D\,\frac{g_s}{in_-\partial_z}\, q_{+\beta}(z_-)\Big)|0\rangle\delta(\Omega - 2k_1^0 - 2k_2^0)$$



The calculation

Methods developed for calculations of two-loop soft functions at *leading power.*

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglia, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = k_1^2 \ , \quad P_4 = k_2^2 \ , \quad P_5 = \big(\omega - n_- k_1\big)$$

$$P_6 = \big(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\big) \ , \quad P_7 = \big(\omega' - n_- k_1 - n_- k_2\big)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E} \mu^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i}\left[\frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+}\right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

The calculation

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- The reduction is implemented in `LiteRed`
- 4 + 1 topologies are needed to reduce the soft functions
- We have 6 + 2 Master Integrals (MIs)
  - 4 MIs implementing the $\delta(\omega - n_- k_1)\delta(\omega' - n_- k_1 - n_- k_2)$ and $\omega \leftrightarrow \omega'$ constraint: $\hat{I}_1$ – $\hat{I}_4$
  - 1 MI with $\delta(\omega - n_- k_1)\delta(\omega - \omega')$: $\hat{I}_5$
  - 1 MI with $\delta(\omega - n_- k_1 - n_- k_2)\,\delta(\omega - \omega')$: $\hat{I}_6$

Reduced expressions

Double real

$$
\begin{aligned}
&S^{(2)2r0v}_{g\bar{q}}(\Omega,\omega,\omega') \\
&\quad = \frac{\alpha_s^2}{(4\pi)^2} T_F \Bigg[C_F \left(\frac{(4-\epsilon)(1-\epsilon)(1-2\epsilon)}{\epsilon^2\omega^2(\Omega-\omega)}\, \hat{I}_6 + \frac{4(2-3\epsilon)(1-3\epsilon)}{\epsilon^2\omega(\Omega-\omega)^2}\, \hat{I}_5 \right) \delta(\omega-\omega') \\
&\qquad + (C_A - 2C_F) \left(\frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\, \omega\omega'(\Omega-\omega)(\omega-\omega')}\, \hat{I}_3 - \frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\, \omega\omega'(\Omega-\omega')(\omega-\omega')}\, \hat{I}_1 \right. \\
&\qquad \left. + \frac{(\Omega-\omega)(\omega+\omega')}{2\omega\omega'} \hat{I}_4 + \frac{(\Omega-\omega')(\omega+\omega')}{2\omega\omega'} \hat{I}_2 \right) \Bigg]
\end{aligned}
$$

where for example

$$
\hat{I}_1(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(0,0,1,1,1,1,1), \quad \hat{I}_2(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(1,1,1,1,1,1,1)
$$

and family $\mathcal{A}$ propagators are

$$
\begin{aligned}
&P_1 = (k_1+k_2)^2, \quad P_2 = n_+k_2, \quad P_3 = k_1^2, \quad P_4 = k_2^2, \\
&P_5 = \big(\Omega - n_-k_1 - n_-k_2 - n_+k_1 - n_+k_2\big), \quad P_6 = \big(\omega - n_-k_1\big), \\
&P_7 = \big(\omega' - n_-k_1 - n_-k_2\big)
\end{aligned}
$$

The integrals are calculated using the differential equations method.

Double real result

This contribution is given by [A. Broggio, **SJ**, L. Vernazza, to appear]

$$
\begin{aligned}
S^{(2)2r0v}_{g\bar{q}}(\Omega,\omega,\omega') \quad = \quad & \frac{\alpha_s^2 T_F}{(4\pi)^2}\Bigg\{ C_F \frac{e^{2\epsilon\gamma_E}\Gamma[1-\epsilon]}{\epsilon^2}\frac{1}{\omega}\Bigg[\frac{4}{\Gamma[1-3\epsilon]}\left(\frac{\mu^4}{\omega(\Omega-\omega)^3}\right)^{\epsilon} \\
& + \frac{(4-\epsilon)\Gamma[2-\epsilon]}{(1-2\epsilon)\Gamma[1-2\epsilon]^2}\left(\frac{\mu^4}{\omega^2(\Omega-\omega)^2}\right)^{\epsilon}\Bigg]\delta(\omega-\omega')\theta(\Omega-\omega)\theta(\omega) \\
& + (C_A-2C_F)\frac{2e^{2\epsilon\gamma_E}}{\epsilon\Gamma[1-2\epsilon]}\frac{\omega+\omega'}{\omega\omega'(\omega'-\omega)}\left(\frac{\mu^4}{\omega(\omega'-\omega)(\Omega-\omega')^2}\right)^{\epsilon} \\
& \times\left[{}_2F_1\left(1,-\epsilon,1-\epsilon,\frac{\omega}{\omega-\omega'}\right)-1\right]\theta(\omega)\theta(\omega')\theta(\omega'-\omega)\theta(\Omega-\omega')\Bigg\}
\end{aligned}
$$

Integrating over ω,ω' gives

$$
\begin{aligned}
\Delta^{(2)}_{g\bar{q}}(z)|_{\mathrm{NLP,s,2r}} \quad = \quad & -\left(\frac{\alpha_s}{4\pi}\right)^2 T_F\left(\frac{\mu^2}{\Omega^2}\right)^{2\epsilon}\frac{e^{2\epsilon\gamma_E}\Gamma[1-\epsilon]^2}{\epsilon^3(1-2\epsilon)\Gamma[1-4\epsilon]}\Bigg\{C_F\Big[12-(21-\epsilon)\epsilon\Big] \\
& -(C_A-2C_F)\Bigg[\frac{1}{\epsilon}-9+14\epsilon-\frac{{}_2F_1\left[2,2,3-2\epsilon,1\right]}{1-\epsilon} \\
& +4(1-2\epsilon)\,{}_3F_2\left[\{1,1,-\epsilon\},\{1-\epsilon,-2\epsilon\},1\right]\Bigg]\Bigg\}
\end{aligned}
$$

Reduced expressions

Virtual-real

$$S^{(2)1r1v}_{g\bar{q}}(\Omega,\omega,\omega') = \mathrm{Re}\Big\{ -i\,\frac{\alpha_s^2\,T_F}{(4\pi)^2}(2C_F - C_A)\Big[\frac{1}{\omega\omega'}\hat{J}_1 + \frac{(\omega+\omega')(\Omega-\omega')}{\omega\omega'}\hat{J}_2\Big]\Big\}$$

The master integrals are

$$\hat{J}_1(\Omega,\omega,\omega') \equiv \hat{J}_{\mathcal{A}}(0,1,1,1,1,1,1), \quad \hat{J}_2(\Omega,\omega,\omega') \equiv \hat{J}_{\mathcal{A}}(1,1,1,1,1,1,1)$$

in terms of the relevant propagators

$$P_1 = k^2, \quad P_2 = (k+k_1)^2, \quad P_3 = n_+k, \quad P_4 = k_1^2,$$
$$P_5 = \big(\Omega - n_-k_1 - n_+k_1\big), \quad P_6 = \big(\omega - n_-k - n_-k_1\big), \quad P_7 = \big(\omega' - n_-k_1\big)$$

The propagators $\{P_4, P_5, P_6, P_7\}$ are cut.

Virtual real result

This contribution is given by [A. Broggio, **SJ**, L. Vernazza, to appear]

$$
\begin{aligned}
S_{g\bar{q}}^{(2)1r1v}(\Omega,\omega,\omega') \quad &= \quad \frac{\alpha_s^2\,T_F}{(4\pi)^2}(2C_F - C_A)\frac{e^{2\epsilon\gamma_E}\,\Gamma[1+\epsilon]}{\epsilon\,\Gamma[1-\epsilon]} \\
&\cdot \mathrm{Re}\Bigg\{\left[\frac{\omega'-\omega}{\omega(\omega')^2}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,-\frac{\omega}{\omega'}\right) - \frac{1}{\omega\omega'}\right]\left(\frac{\mu^4}{\omega\omega'(\Omega-\omega')^2}\right)^{\epsilon}\theta(\omega) \\
&+ \frac{2(\omega+\omega')}{\omega\omega'(\omega'-\omega)}\left(\frac{\mu^4}{(\omega'-\omega)^2(\Omega-\omega')^2}\right)^{\epsilon}\frac{\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]}\theta(\omega'-\omega)\Bigg\}\theta(\omega')\theta(\Omega-\omega')
\end{aligned}
$$

Integrating over ω,ω'

$$
\begin{aligned}
\Delta_{g\bar{q}}^{(2)}(z)|_{\mathrm{NLP,s,1r1v}} \quad &= \quad -\left(\frac{\alpha_s}{4\pi}\right)^2 T_F\left(C_A - 2C_F\right)\left(\frac{\mu^2}{\Omega^2}\right)^{2\epsilon} \\
&\times \frac{2\mathrm{Re}[e^{-i\epsilon\pi}]e^{2\epsilon\gamma_E}\,\Gamma[1-2\epsilon]\Gamma[1-\epsilon]^2\Gamma[1+\epsilon]^2}{\epsilon^3\Gamma[1-4\epsilon]}
\end{aligned}
$$

Fixed-order checks

$$\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z) = 8H(Q^2)\int d\omega\, d\omega'\, G^*_{\xi q}(x_a n_+ p_A;\omega_2)\, G_{\xi q}(x_a n_+ p_A;\omega) S(\Omega,\omega,\omega')$$

We have all the ingredients up to two-loops. Different contributing regions are

- Hard: 1-loop hard and NLO soft functions
- Collinear: 1-loop collinear and NLO soft functions
- Soft: NNLO soft functions, all else tree-level

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

$$\begin{aligned}\Delta^{(2)}_{g\bar{q}}(z) \quad = \quad & C_F\left(\frac{13}{3}\ln^3(1-z) + \ln(1-z)(3-12\zeta_2) + \zeta_3 - 2\right) \\ & +C_A\left(\frac{35}{3}\ln^3(1-z) - 6\ln^2(1-z) + \ln(1-z)(-21-8\zeta_2) + 6\zeta_2 + 38\zeta_3 - 5\right)\end{aligned}$$

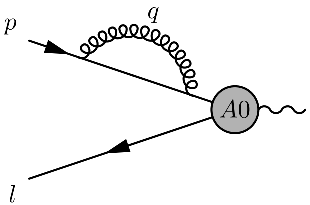
Outlook and Summary

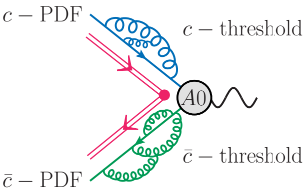
- Endpoint divergences present already at LL, similar to off-diagonal channels of other processes
- Framework developed to the point where refactorization ideas can be applied.
- Our derivation and validation of the factorization formula provides the groundwork for further such studies.
- We provide higher perturbative order data for the objects appearing in the factorization formula to facilitate further investigations into the open conceptual issues.

Thank you

Auxiliary slides

Absence of collinear functions at LP





The purely threshold-collinear loops are scaleless and vanish.

At leading power, we can apply the decoupling transformation $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$ which removed the soft-collinear interactions from the Lagrangian.

[C. Bauer, D. Pirjol, and I. Stewart, hep-ph/0109045]

The threshold-collinear modes can be identified with c-PDF modes, $\chi_c \to \chi_c^{\mathrm{PDF}}$

$$\chi_c(tn_+) = \int du\, \tilde{J}(t,u)\, \chi_c^{\mathrm{PDF}}(un_+) \qquad \tilde{J}(t,u) = \delta(t-u)$$

Collinear functions at NLP

Beyond LP, the decoupling transformation does not remove soft-collinear interactions in the Lagrangian.

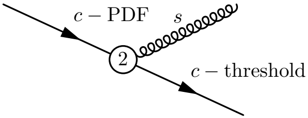
Consider an example of subleading SCET Lagrangian:

[M. Beneke and Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2}\bar{\chi}_c^{(0)} z_\perp^\mu z_\perp^\rho \left[i\partial_\rho i n_- \partial \mathcal{B}_\mu^+(z_-) \right] \frac{\not{n}_+}{2} \chi_c^{(0)} \qquad \text{where} \qquad \mathcal{B}_\pm^\mu = Y_\pm^\dagger \left[i D_s^\mu Y_\pm \right]$$

The subleading power Lagrangian terms enter the basis through time-ordered product operators

$$\left(J_{A0,2\xi}^{T2}(t) \right)^\mu = i \int d^4 z \, \mathbf{T} \left[J_{A0}^\mu(t) \, \mathcal{L}_{2\xi}^{(2)}(z) \right]$$



Collinear functions at NLP

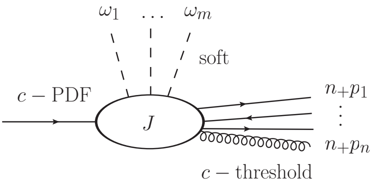
PDF collinear modes can be radiated into the final state:
$p_c \sim Q(1,\lambda^2,\lambda)$ and $p_{c-\text{PDF}} \sim (Q,\Lambda^2/Q,\Lambda)$

$$i\int d^4z\, \mathbf{T}\Big[\chi_{c,\gamma f}(tn_+)\,\mathcal{L}^{(2)}(z)\Big]$$
$$= 2\pi\sum_i \int du \int \frac{d(n_+z)}{2}\,\tilde{J}_{i;\gamma\beta,\mu,fbd}\left(t,u;\frac{n_+z}{2}\right)\chi^{\text{PDF}}_{c,\beta b}(un_+)\,\mathfrak{s}_{i;\mu,d}(z_-)$$

$$\mathfrak{s}_i(z_-) \in \left\{\frac{i\partial^\mu_\perp}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\,,\ \frac{i\partial_{[\mu_\perp}}{in_-\partial}\mathcal{B}^+_{\nu_\perp]}(z_-)\,,\ \frac{1}{(in_-\partial)}\left[\mathcal{B}^+_{\mu_\perp}(z_-),\mathcal{B}^+_{\nu_\perp}(z_-)\right],\ldots\right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]



$$
\begin{aligned}
S_{g\bar{q}}^{(2)1r1v}(\Omega,\omega,\omega') \quad = \quad & \mathrm{Re}\Big\{ i\, g_s^4\, T_F(2C_F - C_A) \int [dk_1] \int [dk] \\
& \cdot (2\pi)\delta(k_1^2)\theta(k_1^0)\, \delta(\Omega - 2k_1^0)\, \delta(\omega' - n_-k_1)\delta(\omega - n_-k - n_-k_1) \\
& \cdot \frac{(k+k_1)^2 - k^2 + n_-k_1(n_+k + n_+k_1) + n_+k_1(n_-k + n_-k_1)}{k^2(k+k_1)^2(n_+k)(-n_-k - n_-k_1)(-n_-k_1)} \Big\}
\end{aligned}
$$

Problems at the Endpoint

Focus on one piece of the factorization formula

$$\int d\omega\, J_1^{(1)}(x_a n_+ p_A;\omega)\, S_1^{(1)}(\Omega,\omega)$$

with

$$J_1^{(1)}(x_a\, n_+ p_A;\, \omega) = \frac{\alpha_s}{4\pi}\frac{1}{(x_a\, n_+ p_A)}\left(\frac{(x_a\, n_+ p_A)\,\omega}{\mu^2}\right)^{-\epsilon}\frac{e^{\epsilon\,\gamma_E}\,\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]}$$
$$\times\left(C_F\left(-\frac{4}{\epsilon}+3+8\epsilon+\epsilon^2\right)-C_A\left(-5+8\epsilon+\epsilon^2\right)\right)$$

$$S_1\,(\Omega,\omega) = \frac{\alpha\, C_F}{2\pi}\frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]}\,\frac{1}{\omega^{1+\epsilon}}\,\frac{1}{(\Omega-\omega)^{\epsilon}}\theta(\omega)\theta(\Omega-\omega)\; +\; \mathcal{O}(\alpha^2)$$

As we have seen before, performing the $d\omega$ convolution integral in d-dimensions, and only *after* expanding in ϵ gives the correct results.

Problems at the Endpoint

$$\int_0^{\Omega} d\omega \, \underbrace{\left(n_+ p \, \omega\right)^{-\epsilon}}_{\text{collinear piece}} \, \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^{\epsilon}}}_{\text{soft piece}}$$

For resummation, we treat the two objects independently, and expand in ϵ prior to performing the final convolution. However, there is a problem! At two loops:

$$J_1^{(1)}(x_a n_+ p_A; \omega) \sim \alpha_s \log(\omega)$$

and

$$S_1\,(\Omega, \omega) \sim \alpha_s \, \delta(\omega) + \mathcal{O}(\alpha^2)$$

Hence, first expanding in ϵ and performing convolution after yields

$$\Delta^{dyn\,(2)}_{\mathrm{NLP-coll}}(z) = \frac{\alpha_s^2}{(4\pi)^2} \Bigg(C_F^2 \left(-\frac{32}{\epsilon^2} - \frac{8}{\epsilon} \left[5 - 8 \ln(1-z) - 4 \int d\omega \, \delta(\omega) \ln\left(\frac{\omega}{Q}\right) \right] \right) + C_A C_F \, \frac{40}{\epsilon} + \mathcal{O}(\epsilon^0) \Bigg)$$

A couple of issues arise. The convolution $d\omega$ integral is now divergent at the endpoint. This prohibits the application of standard RG methods.

DE method for MIs

Convenient to change to dimensionless variable $\omega \to r\,\Omega$

$$I'_1(r) = \frac{1}{\Omega^2}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_1(\Omega, r), \qquad I'_2(r) = \frac{1}{\Omega}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_2(\Omega, r),$$

$$I'_3(r) = \left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_3(\Omega, r), \qquad I'_4(r) = \Omega\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_4(\Omega, r),$$

$$I'_5(r) = \Omega^2\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_5(\Omega, r)$$

System of DEs can be put into canonical form [J. Henn, 1304.1806]

$$\frac{d\vec{I}(r)}{dr} = \epsilon\, A(r)\cdot\vec{I}(r)$$

$$I'_1(r) = \frac{2(1-r)^2}{2-9\epsilon+9\epsilon^2}I_1(r),$$

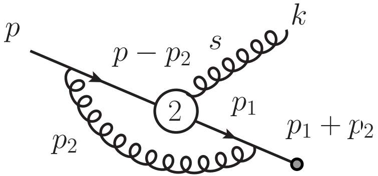
$$I'_3(r) = \frac{1}{\epsilon^2}I_3(r),$$

$$I'_4(r) = -\frac{1}{\epsilon^2(1-r)}I_4(r),$$

$$I'_5(r) = -\frac{1+r}{2\epsilon^2(1-r)r}I_4(r) + \frac{1}{\epsilon^2 r}I_5(r)$$

$$A(r) = \begin{bmatrix} -\frac{1}{r}+\frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

Collinear functions at one-loop



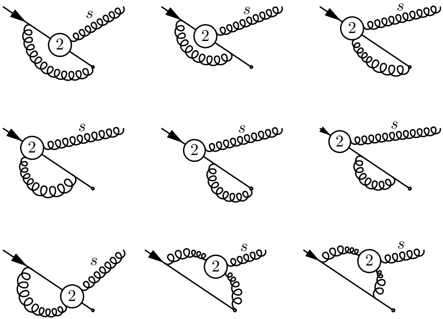
Use NLP Feynman rules to calculate these new objects up to $\mathcal{O}(\alpha_s)$. [M. Beneke, M. Garny, R. Szafron, J. Wang, 1808.04742]
Some non-standard features present!

$$X^\mu = \partial^\mu \left[(2\pi)^d \delta^{(d)} \left(\sum p_{\text{in}} - \sum p_{\text{out}}\right)\right]$$

$$\mathcal{O}(\lambda^2): \qquad i g_s \mathbf{T}^A A^{\rho\nu}(k,p,p') \frac{\not{n}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu})$$

$$A^{\rho\nu}(k,p,p') \equiv \frac{1}{2}\left[(n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left(\frac{\not{p}'_\perp}{n_+ p'} \gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p}\right)\right]$$

Collinear functions at one-loop



$$J_1\left(n_+q, n_+p;\, \omega\right) = -\frac{1}{n_+p}\delta(n_+q - n_+p) + 2\,\frac{\partial}{\partial n_+q}\delta(n_+q - n_+p)$$

$$+\frac{\alpha_s}{4\pi}\,\frac{1}{(n_+p)}\left(\frac{n_+p\,\omega}{\mu^2}\right)^{-\epsilon}\frac{e^{\epsilon\gamma_E}\,\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]}$$

$$\times\left(C_F\left(-\frac{4}{\epsilon}+3+8\epsilon+\epsilon^2\right) - C_A\left(-5+8\epsilon+\epsilon^2\right)\right)\delta(n_+q - n_+p)$$

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]