Pure Quark and Gluon Observables

Xiaojun Yao
InQubator for Quantum Simulation
University of Washington

Collaborator: Iain W. Stewart
JHEP 09 (2022) 120

The XXth annual workshop on Soft-Collinear Effective Theory (SCET 2023)
Lawrence Berkeley National Laboratory

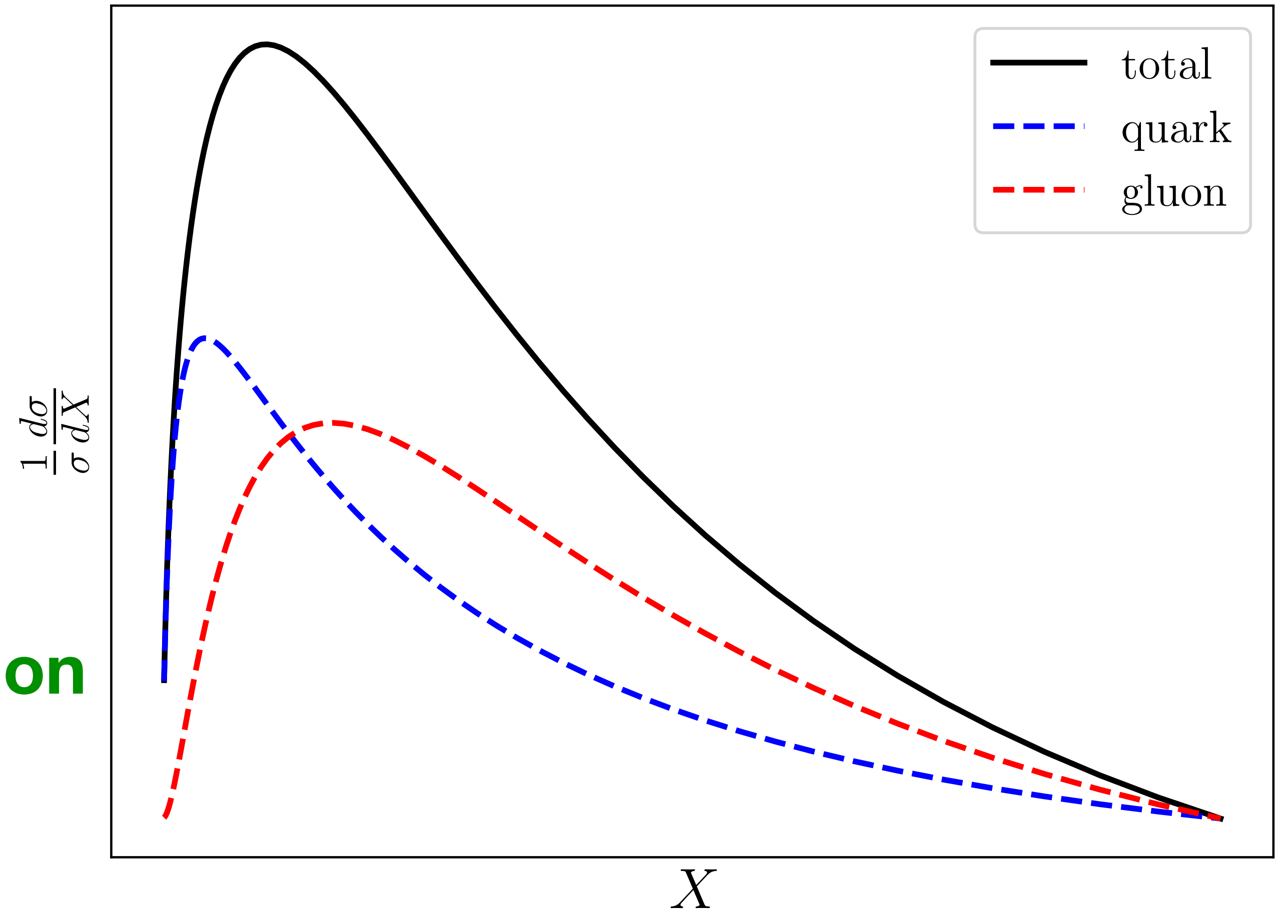
March 28, 2023

Disentangling Quark- and Gluon-Initiated Jets

- **Jet observables contain quark & gluon contributions**

$$X = f_q X_q + f_g X_g$$

Only know total distribution from measurements, want to know individual distribution



- **Motivations of quark/gluon discrimination**
  - Better understand QCD jets
  - Improve probes of QGP in heavy ion collisions
  - Constrain parton shower generators
  - Increase sensitivity in BSM physics searches
- **Can we extract quark & gluon fractions (and distributions)?**

Some Strategies of Disentangling

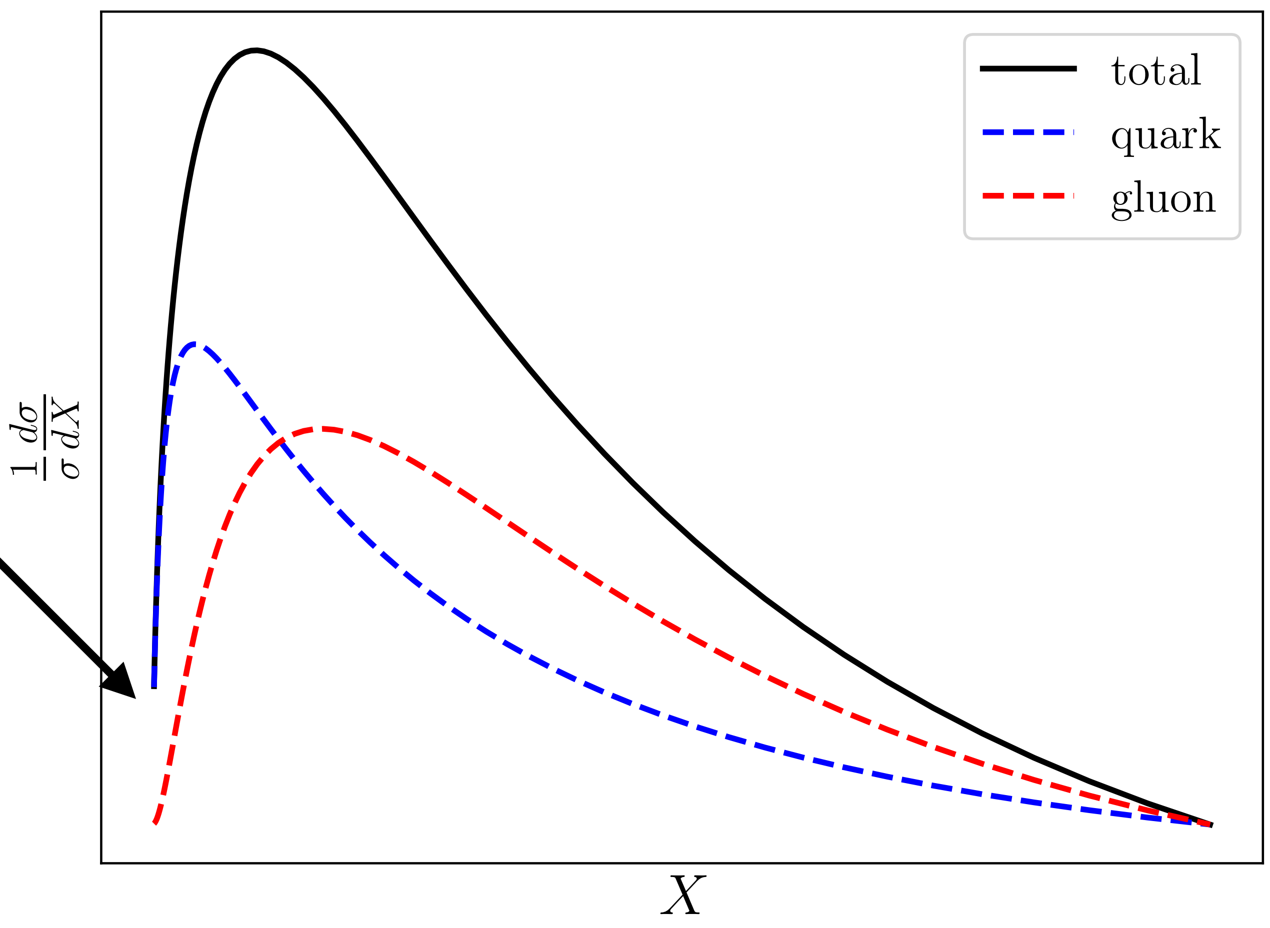
- **Strategy 1: different Sudakov factors near tail on left**

$$d\sigma_q/dX \sim \exp(-\# C_F \log^2)$$

$$d\sigma_g/dX \sim \exp(-\# C_A \log^2)$$

Tail region large experimental uncertainty

Not 100% efficiency



$$C_j = \begin{cases} C_F = \frac{N_c^2-1}{2N_c} & \text{quark} \\ C_A = N_c & \text{gluon} \end{cases}$$

Some Strategies of Disentangling

E.M.Metodiev, J.Thaler, 1802.00008

P.T.Komiske, E.M.Metodiev, J.Thaler, 1809.01140

- **Strategy 2: jet topics**

Consider two jet samples (A = Z+jet, B = dijets)

$$p_A(x) = f_q^A p_q(x) + f_g^A p_g(x)$$
$$p_B(x) = f_q^B p_q(x) + f_g^B p_g(x)$$

From experimental data, calculate topic fractions

$$p_{T1}(x) = \frac{p_A(x) - \kappa(A|B)p_B(x)}{1-\kappa(A|B)} \longrightarrow p_q(x)$$

If mutual irreducibility true

$$p_{T2}(x) = \frac{p_B(x) - \kappa(B|A)p_A(x)}{1-\kappa(B|A)} \longrightarrow p_g(x)$$

$$\boxed{\hat{\kappa}(q|g) = \hat{\kappa}(g|q) = 0}$$

Mutual irreducibility usually not true

For soft drop mass distribution, $\kappa_{LL}(q\,|\,g) = C_F/C_A$

$$\kappa(A|B) = \min_x \frac{p_A(x)}{p_B(x)}$$

- **Key is to find observables with mutual irreducibility**

Goal and Path Forward

- Find observables whose gluon (or quark) contribution vanishes in a range of phase space

$$\hat{\kappa}(q|g) = \hat{\kappa}(g|q) = 0$$

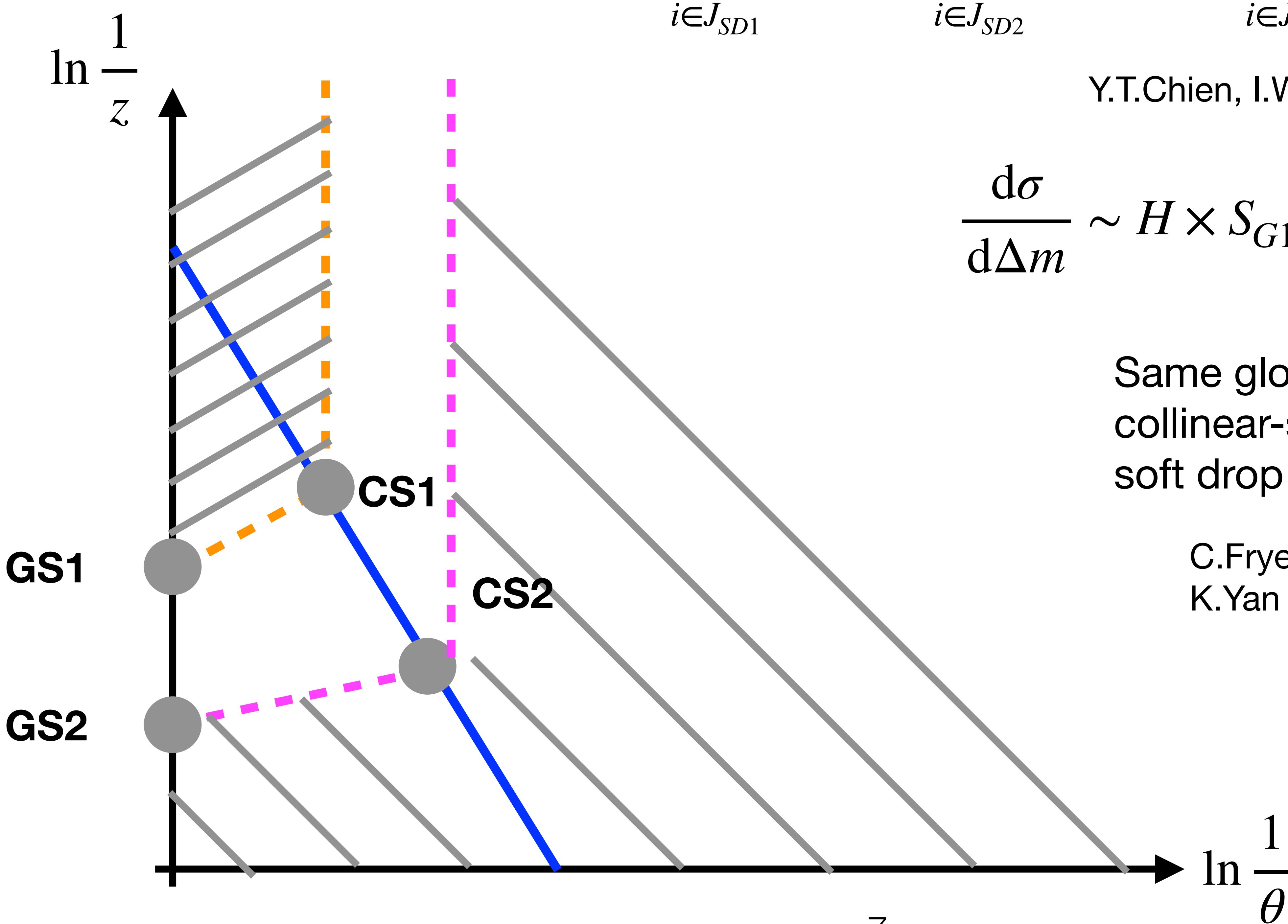
- Collinear drop grooming: cumulative jet mass in perturbative and nonperturbative regimes

I. Cumulative Jet Mass in Collinear Drop

Jet Mass in Collinear Drop

- **Jet mass in CD: CD defined from two SD's, second one more aggressive**

$$\Delta m^2 = m^2_{J_{SD1}} - m^2_{J_{SD2}} = \Big(\sum_{i\in J_{SD1}} p_i^\mu\Big)^2 - \Big(\sum_{i\in J_{SD2}} p_i^\mu\Big)^2 = Q\Big(\sum_{i\in J_{SD1}} p_i^+ - \sum_{i\in J_{SD2}} p_i^+\Big)$$



Y.T.Chien, I.W.Stewart, arXiv:1907.11107

$$\frac{d\sigma}{d\Delta m} \sim H \times S_{G1} \times S_{G2} \times S_C \otimes D_C$$

Same global soft and collinear-soft functions as in soft drop jet mass at NLL

C.Frye,A.J.Larkoski,M.D.Schwartz, K.Yan arXiv:1603.09338

Differential Jet Mass Distribution

- **Differential cross section**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Delta m^2} = \sum_{j=q,g} N_j^{\mathrm{CD}}(p_T, \eta_J, R, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu)\, P_j^{\mathrm{CD}}(\Delta m^2, Q, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu)$$

Determine normalization
Independent of Δm
Dependent on jet kinematics and R

Determine the shape
Dependent on Δm

- **Normalization**

$$N_j^{\mathrm{CD}}(p_T, \eta_J, R, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu) = H_j(p_T, \eta_J, R) \otimes_\Omega S_{G_j}(Q_{\mathrm{gs}1}, R, \beta_1, \mu) \otimes_\Omega S_{\overline{G}_j}(Q_{\mathrm{gs}2}, R, \beta_2, \mu)$$

Hard function only depends
on jet kinematics and R

Global soft function depends
on CD parameters

$$H_j(p_T, \eta_J, R) = \sum_{a,b} f_a \otimes f_b \otimes H_{abj}(p_T, \eta_J, R)$$

Differential Jet Mass Distribution

- **Differential cross section**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Delta m^2} = \sum_{j=q,g} N_j^{\mathrm{CD}}(p_T, \eta_J, R, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu)\, P_j^{\mathrm{CD}}(\Delta m^2, Q, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu)$$

Determine normalization
Independent of Δm
Dependent on jet kinematics and R

Determine the shape
Dependent on Δm

- **Normalization**

$$N_j^{\mathrm{CD}}(p_T, \eta_J, R, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu) = H_j(p_T, \eta_J, R) \otimes_\Omega S_{G_j}(Q_{\mathrm{gs}1}, R, \beta_1, \mu) \otimes_\Omega S_{\overline{G}_j}(Q_{\mathrm{gs}2}, R, \beta_2, \mu)$$

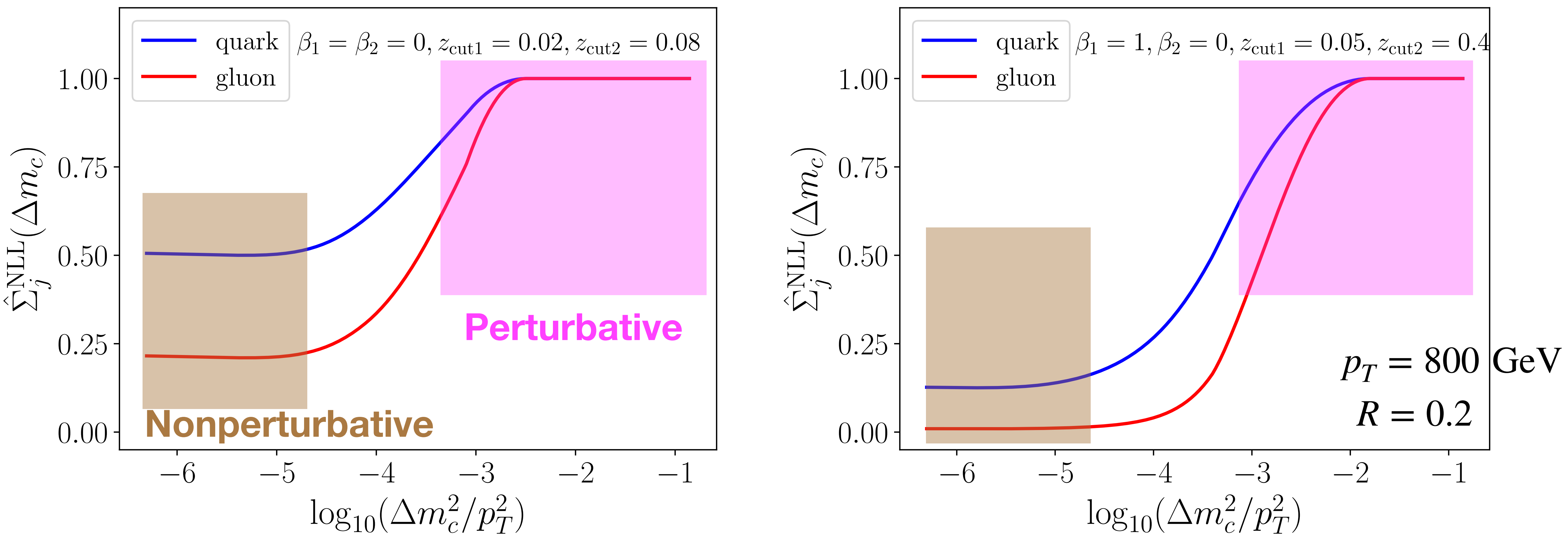
- **Shape**

$$\hat{P}_j^{\mathrm{CD}}(\Delta m^2, Q, \tilde{z}_{\mathrm{cut}\,i}, \beta_i, \mu) = Q_{\mathrm{cut}1}^{\frac{1}{1+\beta_1}} Q_{\mathrm{cut}2}^{\frac{1}{1+\beta_2}} \int \mathrm{d}\ell_1^+ \mathrm{d}\ell_2^+ \delta\big(\Delta m^2 - Q\ell_1^+ - Q\ell_2^+\big) \times \hat{S}_{C_j}\big(\ell_1^+ Q_{\mathrm{cut}1}^{\frac{1}{1+\beta_1}}, \beta_1, \mu\big) \hat{D}_{C_j}\big(\ell_2^+ Q_{\mathrm{cut}2}^{\frac{1}{1+\beta_2}}, \beta_2, \mu\big)$$

Convolution of two collinear-soft functions

Cumulative Jet Mass in CD: Perturbative

$$\hat{\Sigma}(\Delta m_c^2) = \frac{1}{\hat{\sigma}} \int_0^{\Delta m_c^2} \mathrm{d}\Delta m^2 \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\Delta m^2} = \sum_{j=q,g} f_j \hat{\Sigma}_j \qquad \hat{\Sigma}_j(\Delta m_c \to p_T) = 1$$



Finite constant as $\Delta m_c \to 0$ in CD; In SD, go to zero

A significant fraction of events have the two SD jet masses equal, the constant gives this fraction

The constant depends on quark/gluon, CD parameters —> exploited later

Nonperturbative Correction via Shape Function

- **Convoluting perturbative CS function with nonperturbative shape function**

$$S_{C_j}\left(\ell_1^+ Q_{\mathrm{cut}1}^{\frac{1}{1+\beta_1}}, \beta_1, \mu\right) = \int_0^{+\infty} \mathrm{d}k_1\, \hat{S}_{C_j}\left(\ell_1^+ Q_{\mathrm{cut}1}^{\frac{1}{1+\beta_1}} - k_1^{\frac{2+\beta_1}{1+\beta_1}}, \beta_1, \mu\right) F_1^j(k_1, \beta_1)$$

Independent of $z_{\mathrm{cut}1}$

A.H.Hoang, S.Mantry, A.Pathak, I.W.Stewart, arXiv:1906.11843

- **In Laplace space:**

$$\widetilde{S}_{C_j}\left(yQQ_{\mathrm{cut}1}^{\frac{-1}{1+\beta_1}}, \beta_1, \mu\right) = \widetilde{\hat{S}}_{C_j}\left(yQQ_{\mathrm{cut}1}^{\frac{-1}{1+\beta_1}}, \beta_1, \mu\right) \widetilde{F}_1^j(yQQ_{\mathrm{cut}1}^{\frac{-1}{1+\beta_1}}, \beta_1)$$

Same procedure can be done for the other collinear-soft function

Can define shape function in MSbar scheme at $\mu = \Lambda_{\mathrm{CS}}$

$$\widetilde{F}_i^{(\overline{\mathrm{MS}})j}\left(yQQ_{\mathrm{cut}\,i}^{\frac{-1}{1+\beta_i}}, \beta_i, \Lambda_{\mathrm{cs}}\right) = \widetilde{\hat{S}}_{C_j}\left(yQQ_{\mathrm{cut}\,i}^{\frac{-1}{1+\beta_i}}, \beta_i, \Lambda_{\mathrm{cs}}\right) \widetilde{F}_i^j\left(yQQ_{\mathrm{cut}\,i}^{\frac{-1}{1+\beta_i}}, \beta_i\right)$$

Λ_{CS} dependence canceled between perturbative evolution and shape function

Cumulative Jet Mass in Nonperturbative Regime

$$\Sigma^{\mathrm{NLL}}(\Delta m_c) = \sum_{j=q,g} f_j \, \hat{\Sigma}_j^{\mathrm{NLL}} \, \mathcal{F}_j(\Delta m_c)$$

$$\hat{\Sigma}_j^{\mathrm{NLL}} = \exp\left[\frac{2C_j}{1+\beta_1}K(\mu_{\mathrm{gs1}},\mu) - \frac{2C_j}{1+\beta_2}K(\mu_{\mathrm{gs2}},\mu) + \frac{2C_j(\beta_1-\beta_2)}{(1+\beta_1)(1+\beta_2)}K(\Lambda_{\mathrm{cs}},\mu)\right]$$

$$\times \left(\frac{\mu_{\mathrm{gs1}}}{Q_{\mathrm{gs1}}}\right)^{\frac{2C_j}{1+\beta_1}\omega(\mu_{\mathrm{gs1}},\mu)} \left(\frac{\mu_{\mathrm{gs2}}}{Q_{\mathrm{gs2}}}\right)^{\frac{-2C_j}{1+\beta_2}\omega(\mu_{\mathrm{gs2}},\mu)} \left(\frac{Q_{\mathrm{cut1}}^{\frac{1}{1+\beta_1}}}{\Lambda_{\mathrm{cs}}^{\frac{1}{1+\beta_1}}} \frac{\Lambda_{\mathrm{cs}}^{\frac{1}{1+\beta_2}}}{Q_{\mathrm{cut2}}^{\frac{1}{1+\beta_2}}}\right)^{2C_j\omega(\Lambda_{\mathrm{cs}},\mu)}$$

We choose $\Lambda_{\mathrm{CS}} \sim 2\ \mathrm{GeV}$

Sudakov factors with opposite signs

$$\mathcal{F}_j(\Delta m_c) = \int \mathrm{d}k_1 \mathrm{d}k_2 \, F_1^j(k_1,\beta_1) F_2^j(k_2,\beta_2) \, \Theta\left(\Delta m_c^2 - Q Q_{\mathrm{cut1}}^{\frac{-1}{1+\beta_1}} k_1^{\frac{2+\beta_1}{1+\beta_1}} - Q Q_{\mathrm{cut2}}^{\frac{-1}{1+\beta_2}} k_2^{\frac{2+\beta_2}{1+\beta_2}}\right)$$

II. Construction of Pure Quark and Gluon Observables

Construction of Pure Quark/Gluon Observables

- **General strategy**

Take cumulative jet mass in CD $\Sigma(\Delta m_c) = \sum_{j=q,g} f_j \hat{\Sigma}_j \mathcal{F}_j(\Delta m_c)$

Form linear combination from two sets of CD parameters: $z_{\text{cut}\,i}^{(a)}$, $z_{\text{cut}\,i}^{(b)}$, $i = 1{,}2$

$$\mathcal{Q} = \Sigma(\Delta m_c^{(b)}, z_{\text{cut}\,i}^{(b)}, \beta_i) - \xi_g \Sigma(\Delta m_c^{(a)}, z_{\text{cut}\,i}^{(a)}, \beta_i)$$

$$\mathcal{G} = \Sigma(\Delta m_c^{(b)}, z_{\text{cut}\,i}^{(b)}, \beta_i) - \xi_q \Sigma(\Delta m_c^{(a)}, z_{\text{cut}\,i}^{(a)}, \beta_i)$$

p_T, η, R omitted

E.g. find value of ξ_g such that gluon contribution to $\mathcal{Q}$ vanishes

$$\hat{\Sigma}_g^{(b)} \mathcal{F}_g^{(b)} - \xi_g \hat{\Sigma}_g^{(a)} \mathcal{F}_g^{(a)} = 0$$

Key: values of ξ_q, ξ_g independent of $\Delta m_c^{(a)}$ and $\Delta m_c^{(b)}$

Make the two shape functions same so ξ_i independent of NP physics

Rebinning Jet Masses

- Make arguments of shape function the same in (a) and (b)

$$(Q^{(a)}_{\text{cut1}})^{\frac{1}{1+\beta_1}}(\Delta m^{(a)}_c)^2 = (Q^{(b)}_{\text{cut1}})^{\frac{1}{1+\beta_1}}(\Delta m^{(b)}_c)^2$$

$$(Q^{(a)}_{\text{cut2}})^{\frac{1}{1+\beta_2}}(\Delta m^{(a)}_c)^2 = (Q^{(b)}_{\text{cut2}})^{\frac{1}{1+\beta_2}}(\Delta m^{(b)}_c)^2$$

- Solved by constraints

$$(\Delta m^{(b)}_c)^2 = (\Delta m^{(a)}_c)^2 \left(\frac{z^{(a)}_{\text{cut1}}}{z^{(b)}_{\text{cut1}}}\right)^{\frac{1}{1+\beta_1}} \qquad z^{(b)}_{\text{cut2}} = z^{(a)}_{\text{cut2}} \left(\frac{z^{(b)}_{\text{cut1}}}{z^{(a)}_{\text{cut1}}}\right)^{\frac{1+\beta_2}{1+\beta_1}}$$

- Shape functions become common factor $\mathscr{F}^{(a)}_j = \mathscr{F}^{(b)}_j \equiv \mathscr{F}_j$

$$\mathscr{Q} = \sum_{j=q,g} f_j \mathscr{F}_j \left(\hat{\Sigma}^{(b)}_j - \xi_g \hat{\Sigma}^{(a)}_j\right) \qquad \mathscr{G} = \sum_{j=q,g} f_j \mathscr{F}_j \left(\hat{\Sigma}^{(b)}_j - \xi_q \hat{\Sigma}^{(a)}_j\right)$$

$$\xi_g = \frac{\hat{\Sigma}^{(b)}_g}{\hat{\Sigma}^{(a)}_g} \qquad \xi_q = \frac{\hat{\Sigma}^{(b)}_q}{\hat{\Sigma}^{(a)}_q}$$

Coefficients defined purely perturbatively

Works in perturbative and NP regimes

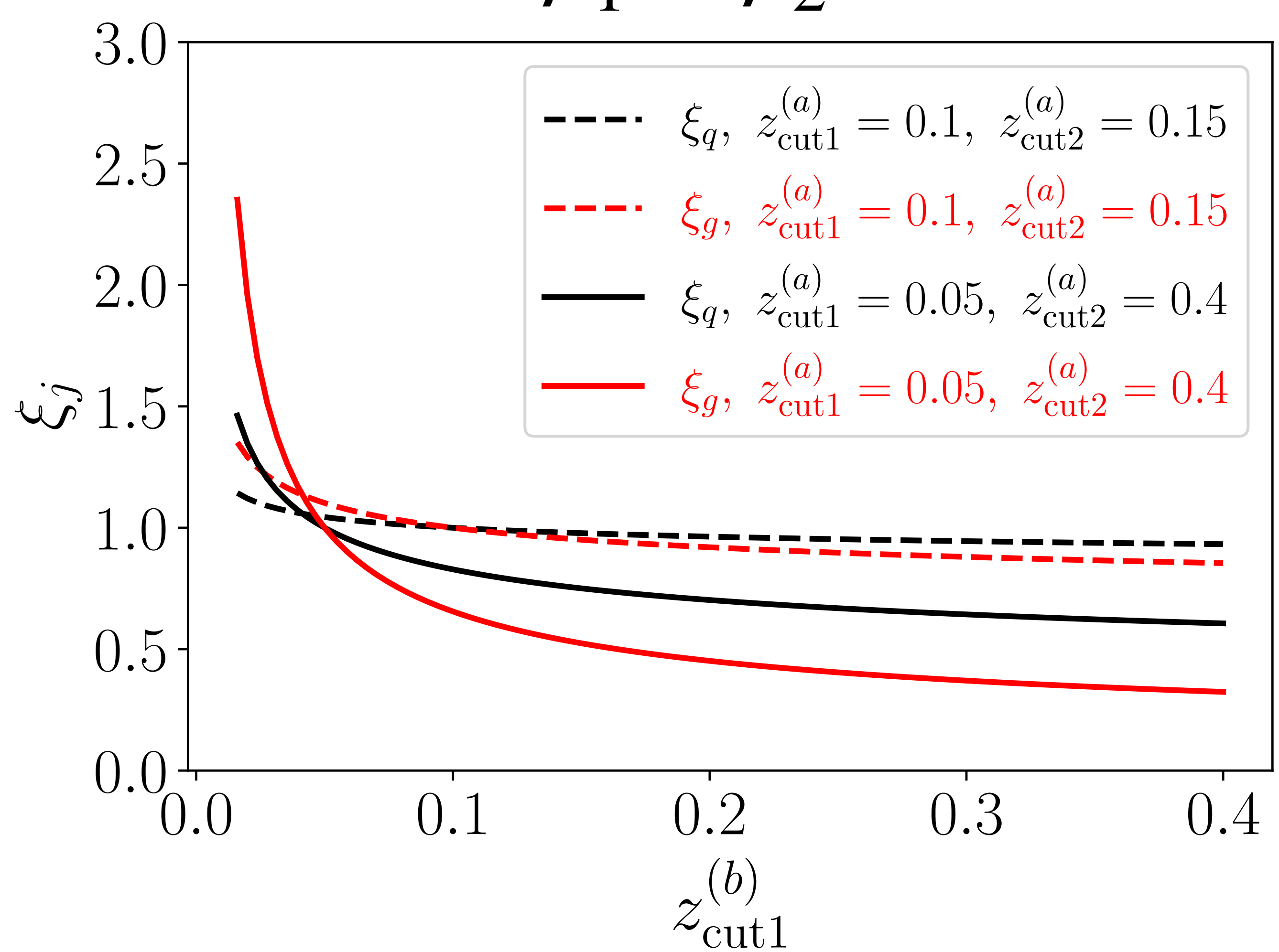
Perturbative Determination of ξ_j

- **At NLL**

$$\xi_j = \exp\left[\frac{2C_j}{1+\beta_1}K\left(\mu_{\rm gs1}^{(b)},\mu_{\rm gs1}^{(a)}\right) - \frac{2C_j}{1+\beta_2}K\left(\mu_{\rm gs2}^{(b)},\mu_{\rm gs2}^{(a)}\right) + \frac{2C_j}{1+\beta_1}\omega\left(\mu_{\rm gs1}^{(a)},\mu_{\rm gs2}^{(a)}\right)\ln\frac{z_{\rm cut\,1}^{(a)}}{z_{\rm cut\,1}^{(b)}}\right]$$

$$\times\left(\frac{\mu_{\rm gs1}^{(b)}}{Q_{\rm gs1}^{(b)}}\right)^{\frac{2C_j}{1+\beta_1}\omega\left(\mu_{\rm gs1}^{(b)},\mu_{\rm gs1}^{(a)}\right)}\left(\frac{\mu_{\rm gs2}^{(b)}}{Q_{\rm gs2}^{(b)}}\right)^{\frac{-2C_j}{1+\beta_2}\omega\left(\mu_{\rm gs2}^{(b)},\mu_{\rm gs2}^{(a)}\right)}$$

$\beta_1 = \beta_2 = 0$



Stronger discrimination power requires largely separated ξ_j , and thus largely separated $z_{{\rm cut}\,i}$

Optimize Parameter Choice

- **Have constructed a class of pure quark and gluon observables, want to optimize parameters to maximize disentangling power**

  - GS scales perturbative: $Q_{\mathrm{gs}\,i}^{(a/b)} = p_T R\, z_{\mathrm{cut}\,i}^{(a/b)} (R/R_0)^{\beta_i} \gg \Lambda_{\mathrm{QCD}}$

  - Largely separated $z_{\mathrm{cut}1}^{(a)}$ and $z_{\mathrm{cut}2}^{(a)}$, largely separated $z_{\mathrm{cut}1}^{(a)}$ and $z_{\mathrm{cut}1}^{(b)}$

  - Remove contamination of external soft radiation (e.g. ISR) and underlying events (MPI): (1) small jet radius R; (2) $z_{\mathrm{cut}1}^{(a/b)} \gtrsim 0.15$ for $R \sim 1$

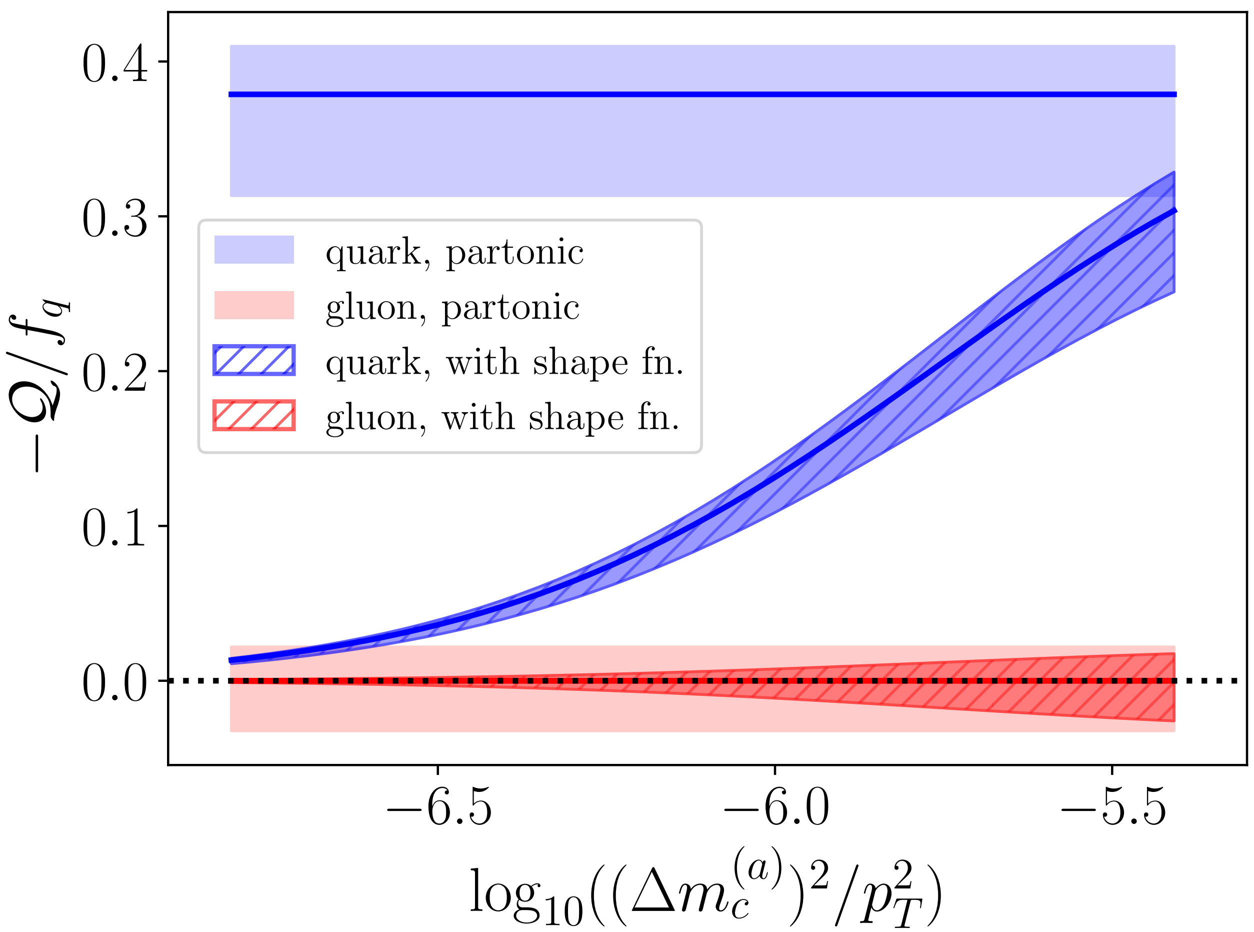
  - Factorization formula: $z_{\mathrm{cut}\,i} \ll 1$

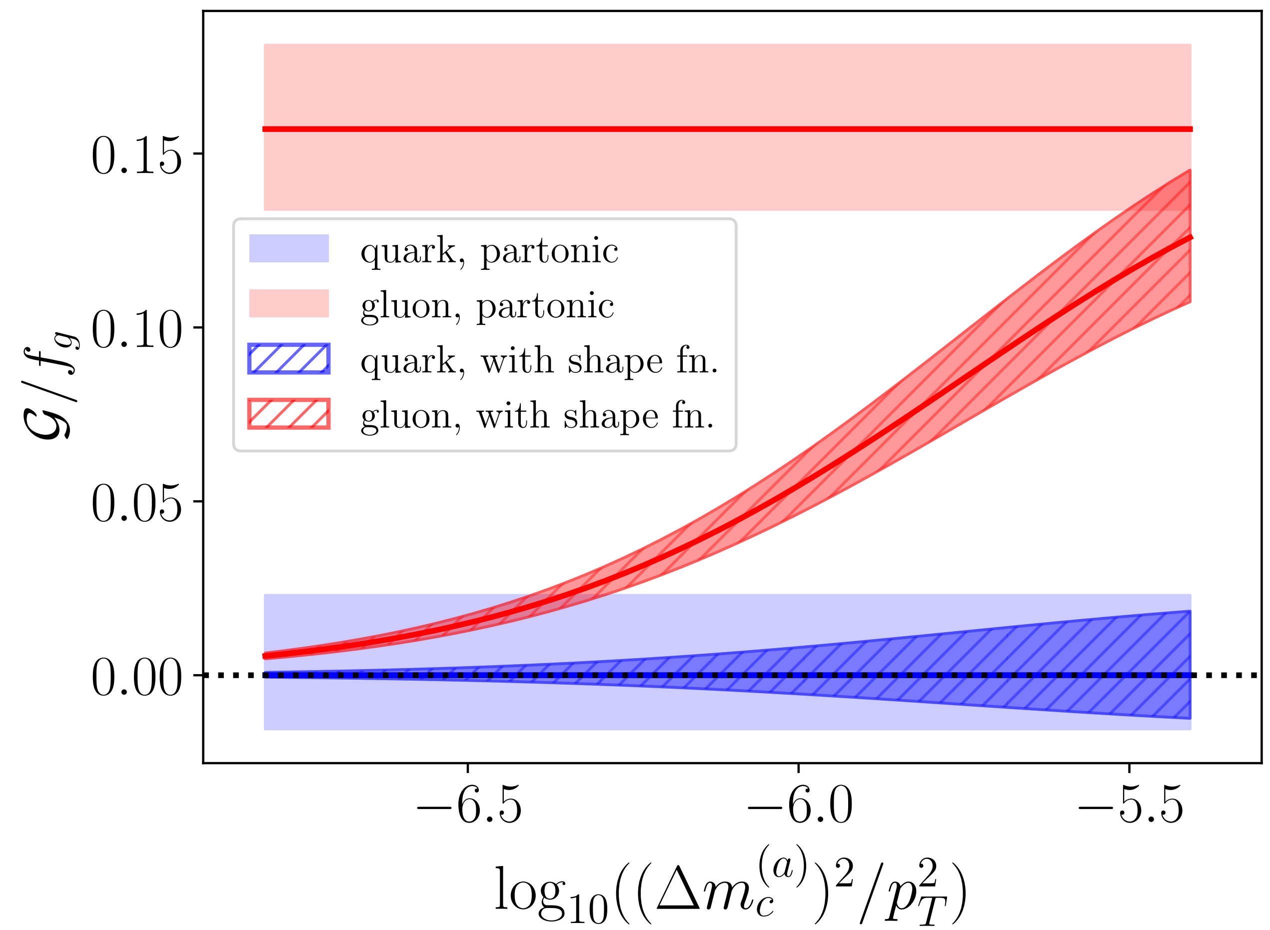
It turns out small jet radius works

Pure Quark and Gluon Observables in Nonperturbative Regime

$$p_T = 800\ \text{GeV},\ \eta_J = 0,\ R = 0.2$$

$$\beta_1 = \beta_2 = 0,\ z^{(a)}_{\text{cut1}} = 0.1,\ z^{(a)}_{\text{cut2}} = 0.4,\ z^{(b)}_{\text{cut1}} = 0.02,\ z^{(b)}_{\text{cut2}} = 0.08$$



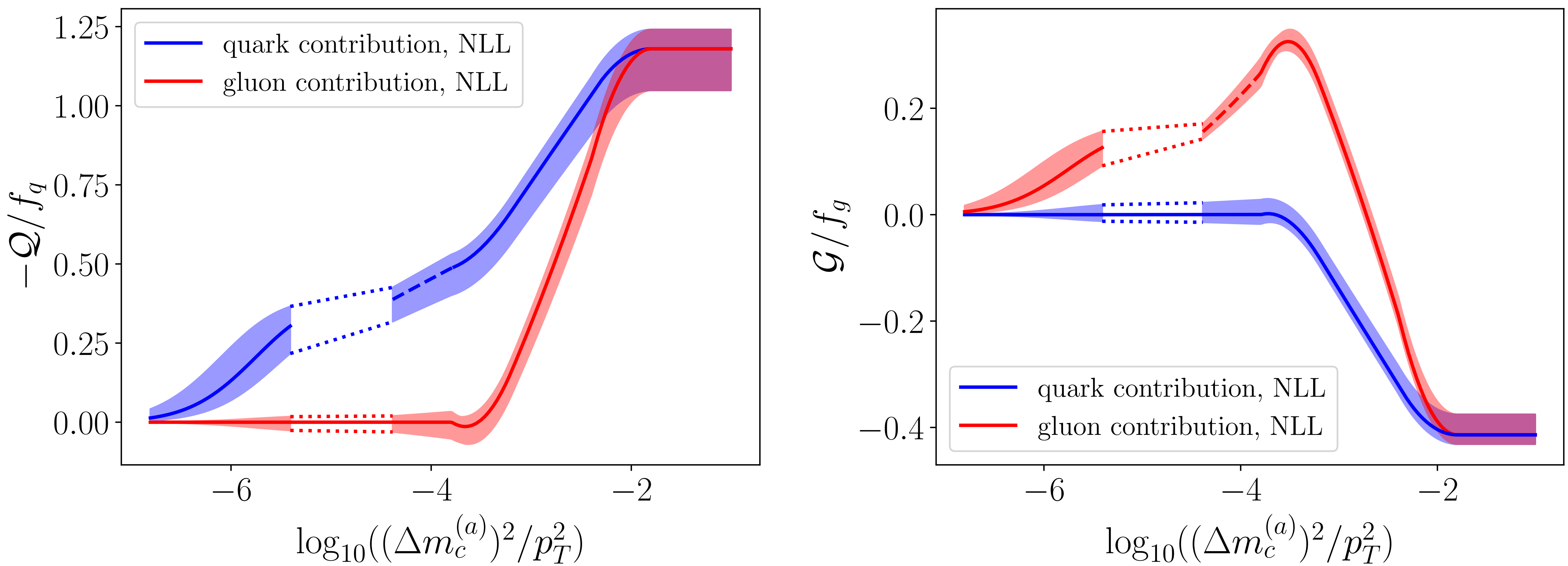


Shape function makes distributions vanishing in small mass limit

Pure Quark and Gluon Observables in Full Regime

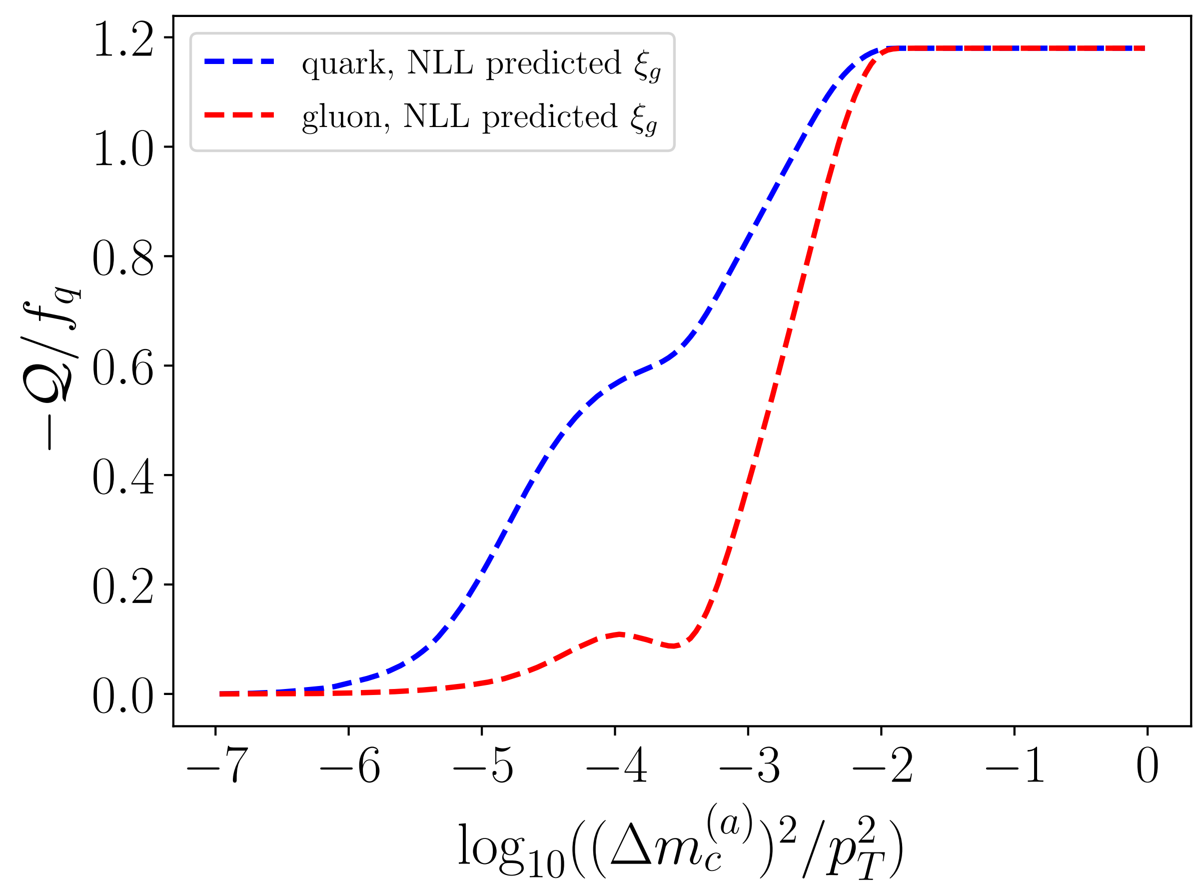
$$p_T = 800 \text{ GeV},\ \eta_J = 0,\ R = 0.2$$

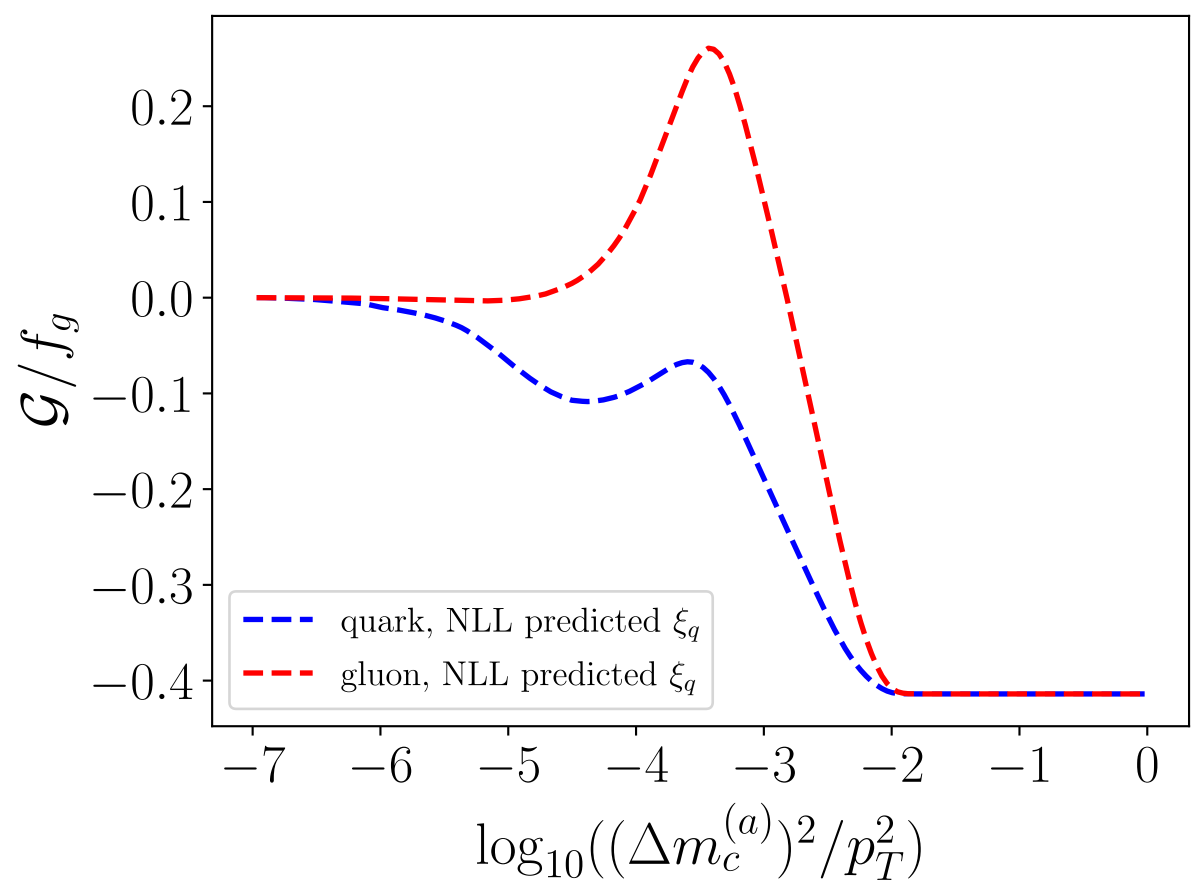
$$\beta_1 = \beta_2 = 0,\ z_{\text{cut1}}^{(a)} = 0.1,\ z_{\text{cut2}}^{(a)} = 0.4,\ z_{\text{cut1}}^{(b)} = 0.02,\ z_{\text{cut2}}^{(b)} = 0.08$$



Monte Carlo Studies w/o ISR and MPI

Pythia + JETlib

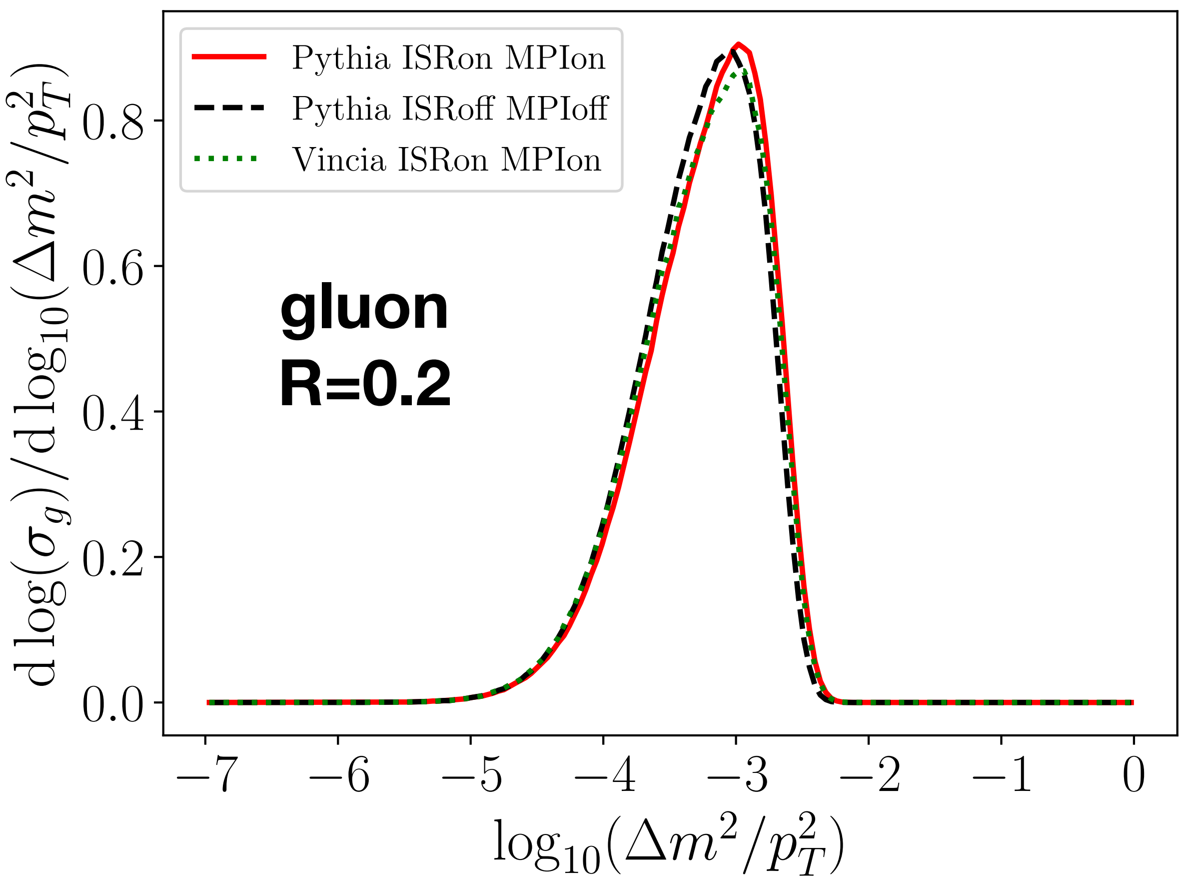


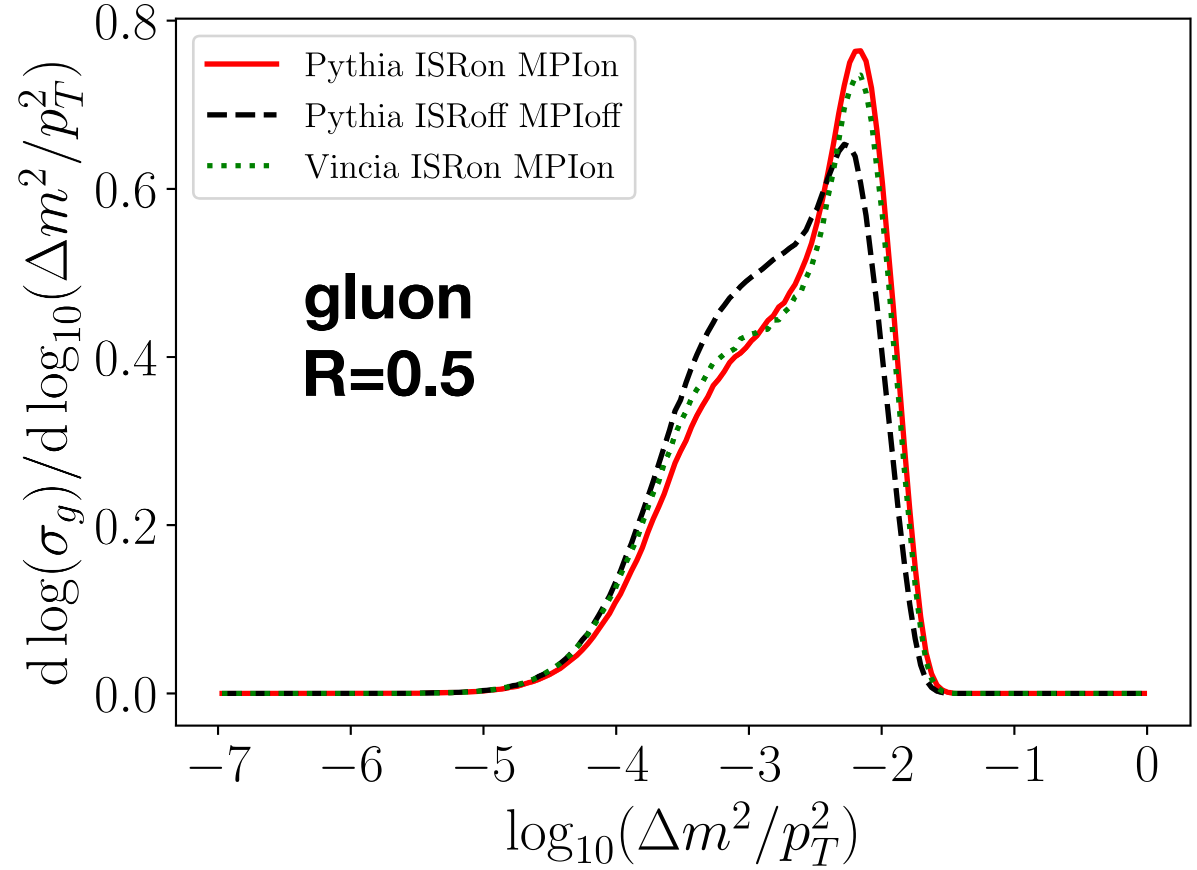


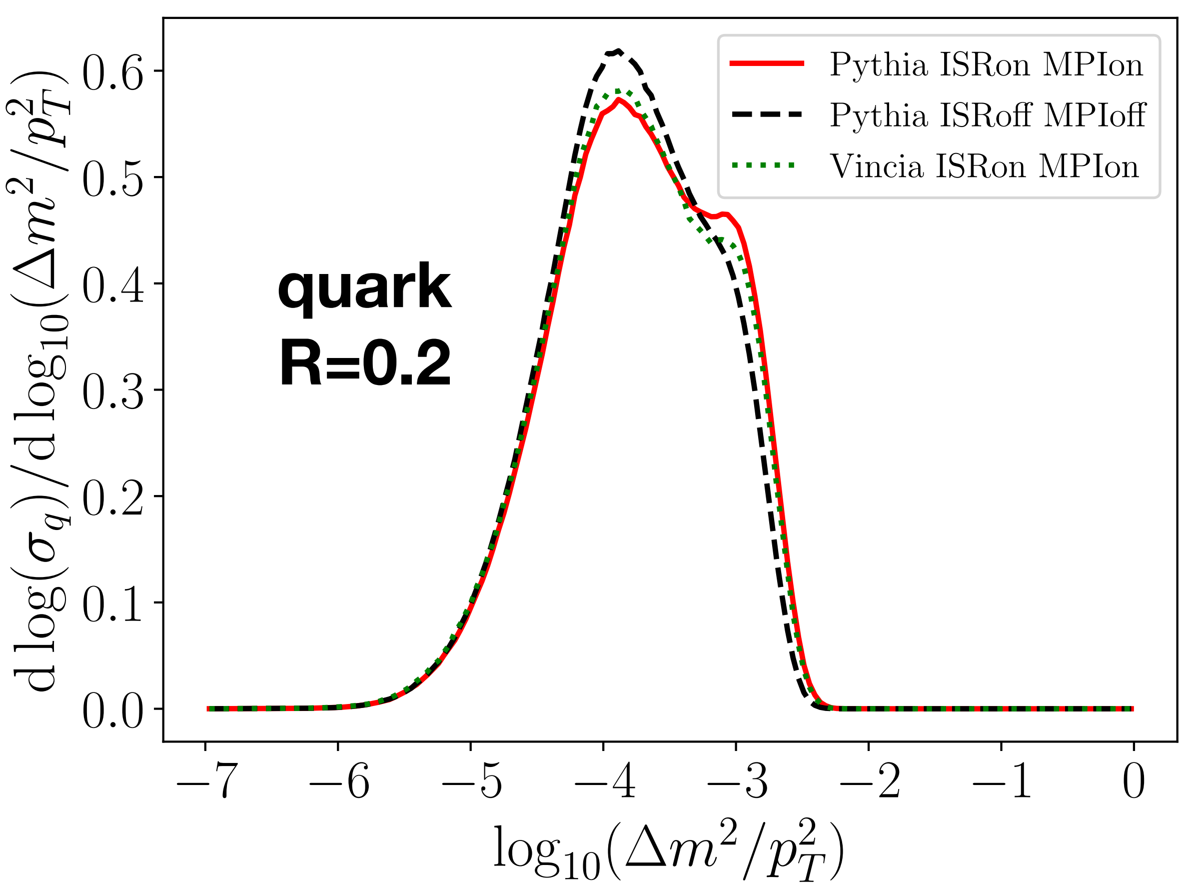
- Gluon observable worse than quark observable

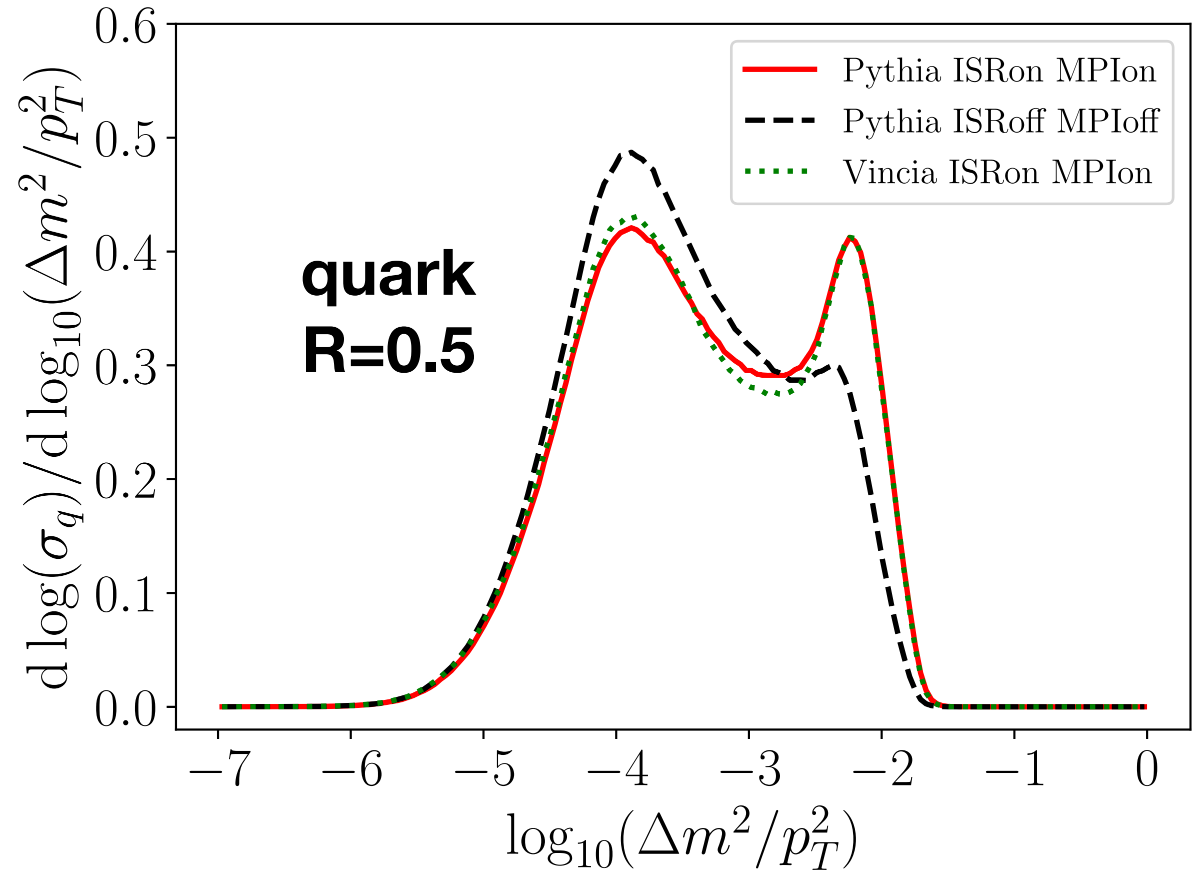
- Monte Carlos may not be tuned well to describe soft radiation:
  (1) determination of linear combination coefficients
  (2) shape of spectrum

Impact of ISR and MPI

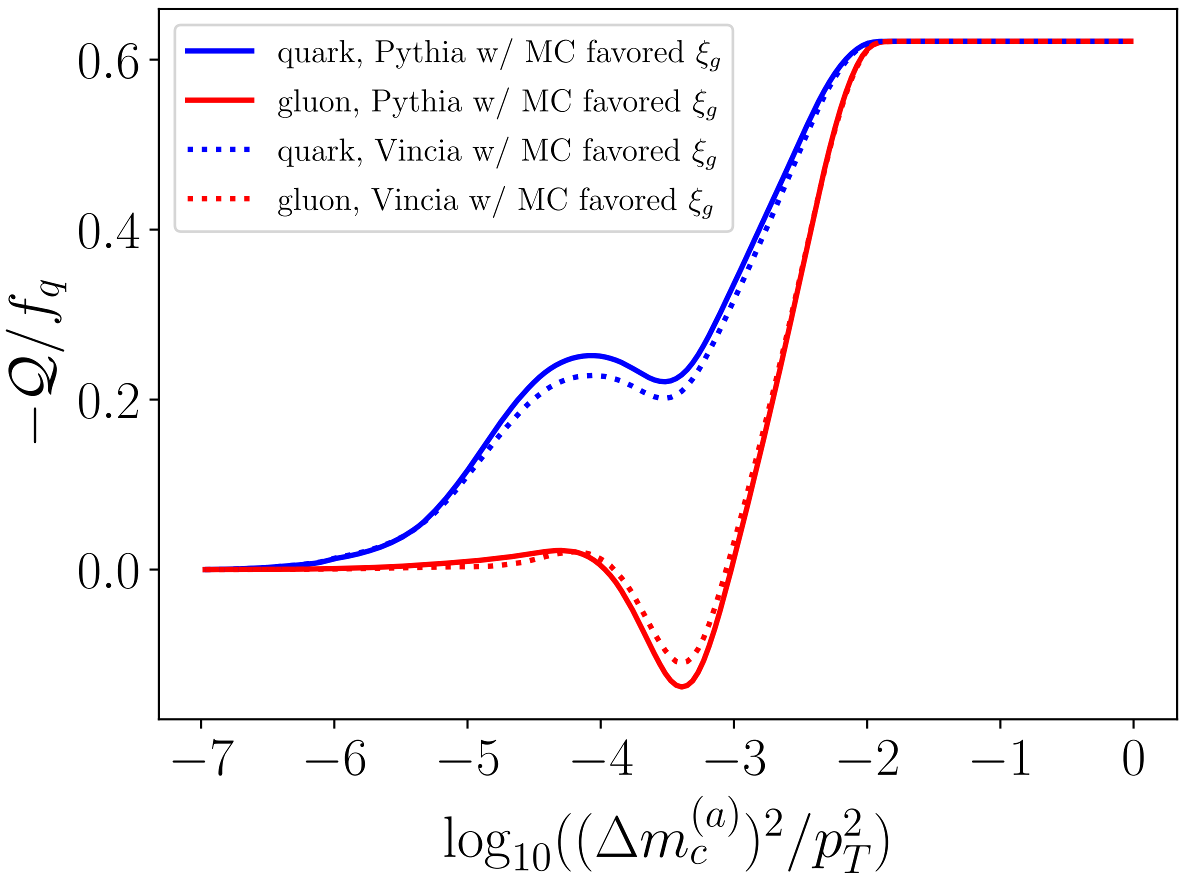


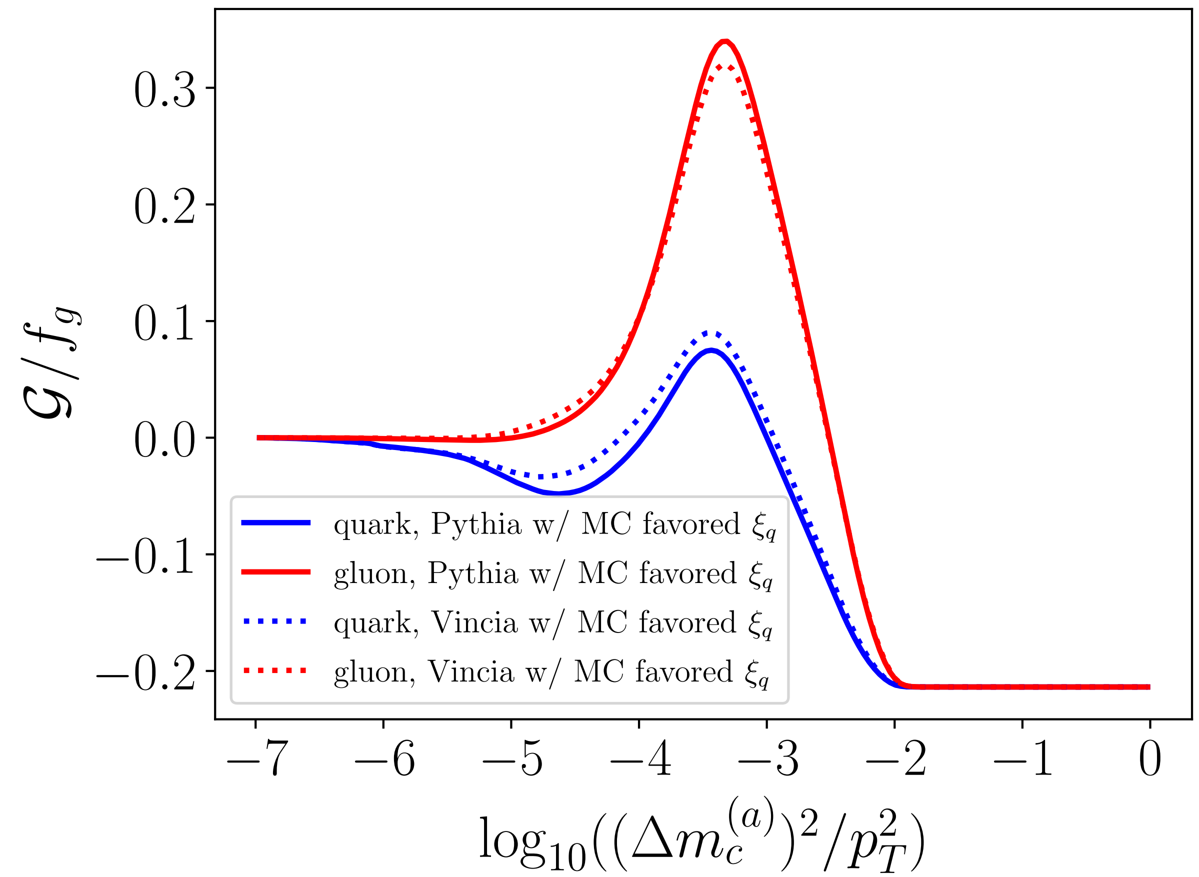






Monte Carlo Studies w ISR and MPI





Most contamination removed, construction with tuned combination coefficient works for pure quark observable

Once again gluon observable not effective in MC, not a problem of ISR/MPI
Mass dependence of quark jet spectrum in Monte Carlo different from factorization

Conclusions

- Construct pure quark/gluon observables with collinear drop
    - Linear combination of cumulative jet mass observables w/ different CD parameters $z_{\mathrm{cut}\,i}$
    - Rescaling jet mass —> robust against nonperturbative correction
- Monte Carlo studies
    - Small jet radius removes ISR/MPI effects
    - Pure quark observable works, but gluon one does not
- Calculate (collinear-)soft function nonperturbatively? Quantum computing w/ Hamiltonian?

19 plaquettes for SU(2): 2303.14264

Jet Grooming: Soft Drop

- **Soft Drop w/ parameters** $(z_{\rm cut}, \beta)$

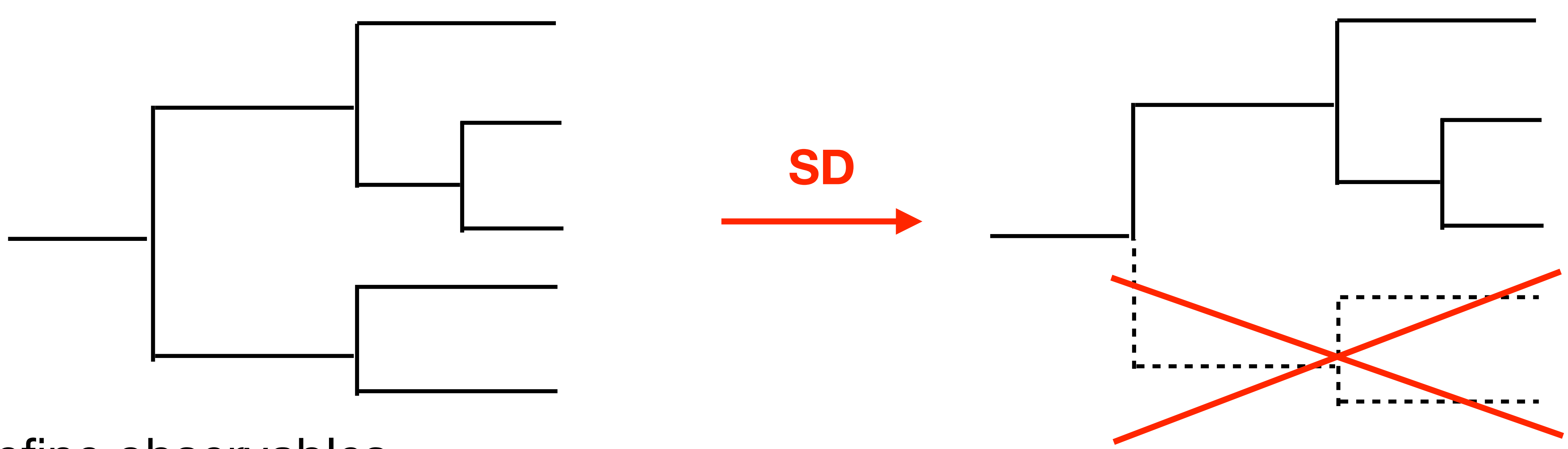
M.Dasgupta,A.Fregoso,S.Marzani
G.P. Salam arXiv:1307.0007

A.J.Larkoski,S.Marzani,G.Soyez
J.Thaler arXiv:1402.2657

- Start with jet defined by anti-kT with radius R
- Re-cluster the jet in Cambridge-Aachen algorithm: first combine pairs w/ smallest

$$\Delta R_{ij} = \frac{2p_i \cdot p_j}{p_{Ti}\, p_{Tj}} = (\theta_i - \theta_j)\cosh y_J \approx \theta_{ij} \cosh \eta_J$$

- Obtain a tree, consistent with LL branching history: large angle radiated first
- Keep removing the softer branch until $\frac{\min(p_{Ti}, p_{Tj})}{p_{Ti} + p_{Tj}} > z_{\rm cut}\left(\frac{\Delta R_{ij}}{R_0}\right)^{\beta} \approx \tilde{z}_{\rm cut}\, \theta_{ij}^{\beta}$



- Define observables

Jet Mass in Soft Drop

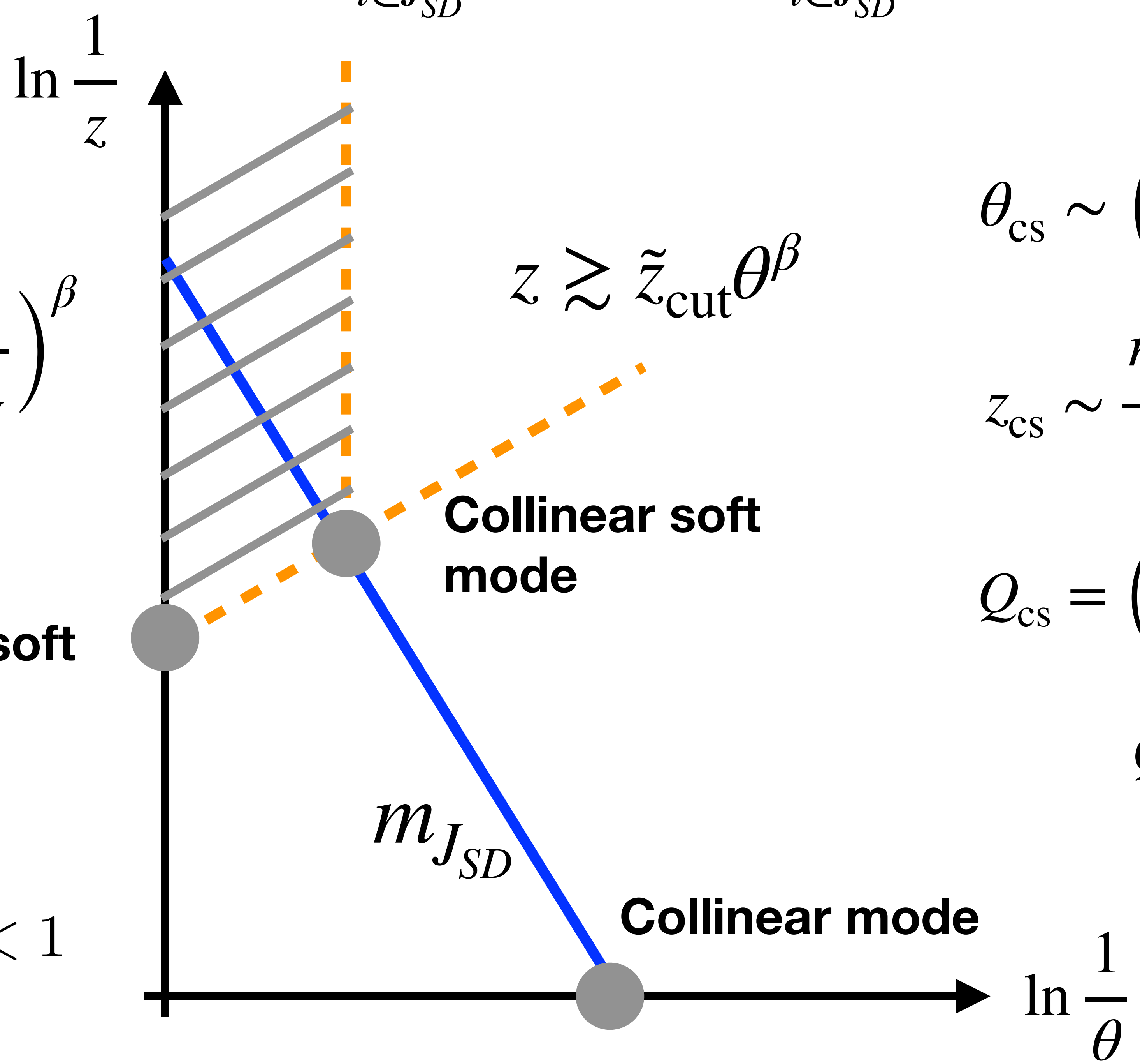
- **Jet mass in SD:** $m^2_{J_{SD}} = \Big(\sum_{i\in J_{SD}} p_i^\mu\Big)^2 = P_J^- \Big(\sum_{i\in J_{SD}} p_i^+\Big)$

To contribute, must pass SD criterion

$$\theta_{gs} = \frac{R}{\cosh \eta_J}$$

$$z_{gs} = \tilde{z}_{cut}\left(\frac{R}{\cosh \eta_J}\right)^\beta$$

$$Q_{gs} = p_T R z_{gs}$$



$$\theta_{cs} \sim \left(\frac{m^2_{J_{SD}}}{Q Q_{cut}}\right)^{\frac{1}{2+\beta}}$$

$$z_{cs} \sim \frac{m^2_{J_{SD}}}{Q^2}\frac{1}{\theta^2_{cs}}$$

$$Q_{cs} = \left(\frac{m^2_{J_{SD}}}{Q}\right)^{\frac{1+\beta}{2+\beta}} Q_{cut}^{\frac{1}{2+\beta}}$$

$$Q_{cut} = 2^\beta Q \tilde{z}_{cut}$$

$$\frac{m^2_{J_{SD}}}{(p_T R)^2} \ll z_{cut} \ll 1$$

$$\frac{d\sigma}{dm_{J_{SD}}} \sim H \times S_G \times J \otimes S_C$$

C.Frye,A.J.Larkoski,M.D.Schwartz, K.Yan arXiv:1603.09338

Global Soft Functions

- **One loop in dim. reg.**

$$S_{G_j}(Q_{\mathrm{gs}1},\beta_1,\epsilon) = 1 + \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^+k^-} 2\pi\delta^+(k^2)\, \overline{\Theta}^{(\mathrm{gs})}_{\mathrm{SD}1}\, \Theta_{\mathrm{alg}}$$

$$S_{\overline{G}_j}(Q_{\mathrm{gs}2},\beta_2,\epsilon) = 1 + \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^+k^-} 2\pi\delta^+(k^2) \Big(-\overline{\Theta}^{(\mathrm{gs})}_{\mathrm{SD}2}\Big) \Theta_{\mathrm{alg}}$$

$$\overline{\Theta}^{(\mathrm{gs})}_{\mathrm{SD}\,i} = \theta\bigg(Q\tilde{z}_{\mathrm{cut}\,i} \Big(\frac{2k^+}{k^-}\Big)^{\frac{\beta_i}{2}} - k^+ - k^- \bigg) \qquad \Theta_{\mathrm{alg}} = \theta\Big(R^2 - 4\cosh^2\!\eta_J \frac{k^+}{k^-}\Big)$$

Failing soft drop   Inside jet

- **Renormalization in MSbar**

$$S^{\mathrm{ren}}_{G_j}(Q_{\mathrm{gs}1},\beta_1,\mu) = 1 + \frac{\alpha_s(\mu)C_j}{\pi(1+\beta_1)}\Big(\ln^2\frac{\mu}{Q_{\mathrm{gs}1}} - \frac{\pi^2}{24}\Big)$$

$$S^{\mathrm{ren}}_{\overline{G}_j}(Q_{\mathrm{gs}2},\beta_2,\mu) = 1 - \frac{\alpha_s(\mu)C_j}{\pi(1+\beta_2)}\Big(\ln^2\frac{\mu}{Q_{\mathrm{gs}2}} - \frac{\pi^2}{24}\Big)$$

Global Soft Functions

- **One loop in dim. reg.**

$$S_{G_j}(Q_{\rm gs1},\beta_1,\epsilon) = 1 + \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\int \frac{{\rm d}^d k}{(2\pi)^d}\frac{1}{k^+k^-}2\pi\delta^+(k^2)\,\overline{\Theta}^{\rm (gs)}_{\rm SD1}\,\Theta_{\rm alg}$$

$$S_{\overline{G}_j}(Q_{\rm gs2},\beta_2,\epsilon) = 1 + \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\int \frac{{\rm d}^d k}{(2\pi)^d}\frac{1}{k^+k^-}2\pi\delta^+(k^2)\big(-\overline{\Theta}^{\rm (gs)}_{\rm SD2}\big)\Theta_{\rm alg}$$

- **RGE of global soft functions**

$$\frac{\rm d}{{\rm d}\ln\mu}\ln S^{\rm ren}_{G_j}(Q_{\rm gs1},\beta_1,\mu) = \frac{2C_j}{1+\beta_1}\Gamma_{\rm cusp}\ln\frac{\mu}{Q_{\rm gs1}} + \gamma_{S_{G_j}}$$

$$\frac{\rm d}{{\rm d}\ln\mu}\ln S^{\rm ren}_{\overline{G}_j}(Q_{\rm gs2},\beta_2,\mu) = -\frac{2C_j}{1+\beta_2}\Gamma_{\rm cusp}\ln\frac{\mu}{Q_{\rm gs2}} - \gamma_{\overline{S}_{G_j}}$$

Collinear Soft Functions

- **One loop in dim. reg.**

$$\hat{S}_{C_j}(\ell_1^+, \beta_1, \mu): \quad \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2\pi\delta^+(k^2)}{k^+ k^-} \Big(\delta(k^+ - \ell_1^+) - \delta(\ell_1^+)\Big) \Theta_{\mathrm{SD1}}^{(\mathrm{cs})}$$

$$\hat{D}_{C_j}(\ell_2^+, \beta_2, \mu): \quad \frac{4g^2 C_j \mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2\pi\delta^+(k^2)}{k^+ k^-} \Big(\delta(k^+ - \ell_2^+) - \delta(\ell_2^+)\Big) \big(- \Theta_{\mathrm{SD2}}^{(\mathrm{cs})}\big)$$

- **Renormalization in MSbar in Laplace space**

$$\tilde{f}(y) = \int_0^\infty \mathrm{d}\Delta m^2 \, \exp\big(- y e^{-\gamma_E} \Delta m^2 \big) f(\Delta m^2)$$

$$\tilde{\hat{S}}_{C_j}\Big(yQQ_{\mathrm{cut1}}^{\frac{-1}{1+\beta_1}}, \beta_1, \mu\Big) = 1 + \frac{\alpha_s C_j}{\pi} \frac{2+\beta_1}{1+\beta_1} \left(- \ln^2 \frac{\mu \, y^{\frac{1+\beta_1}{2+\beta_1}} Q^{\frac{1+\beta_1}{2+\beta_1}}}{Q_{\mathrm{cut1}}^{\frac{1}{2+\beta_1}}} + \frac{\pi^2}{24} \right)$$

$$\tilde{\hat{D}}_{C_j}\Big(yQQ_{\mathrm{cut2}}^{\frac{-1}{1+\beta_2}}, \beta_2, \mu\Big) = 1 - \frac{\alpha_s C_j}{\pi} \frac{2+\beta_2}{1+\beta_2} \left(- \ln^2 \frac{\mu \, y^{\frac{1+\beta_2}{2+\beta_2}} Q^{\frac{1+\beta_2}{2+\beta_2}}}{Q_{\mathrm{cut2}}^{\frac{1}{2+\beta_2}}} + \frac{\pi^2}{24} \right)$$

Collinear Soft Functions

- **One loop in dim. reg.**

$$\hat{S}_{C_j}(\ell_1^+,\beta_1,\mu):\quad \frac{4g^2C_j\mu^{2\epsilon}e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2\pi\delta^+(k^2)}{k^+k^-}\Big(\delta(k^+-\ell_1^+)-\delta(\ell_1^+)\Big)\Theta_{\mathrm{SD1}}^{(\mathrm{cs})}$$

$$\hat{D}_{C_j}(\ell_2^+,\beta_2,\mu):\quad \frac{4g^2C_j\mu^{2\epsilon}e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2\pi\delta^+(k^2)}{k^+k^-}\Big(\delta(k^+-\ell_2^+)-\delta(\ell_2^+)\Big)\Big(-\Theta_{\mathrm{SD2}}^{(\mathrm{cs})}\Big)$$

- **RGE of collinear soft functions**

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln\tilde{\hat{S}}_{C_j}\Big(yQQ_{\mathrm{cut1}}^{\frac{-1}{1+\beta_1}},\beta_1,\mu\Big)=2C_j\Gamma_{\mathrm{cusp}}[\alpha_s]\,\ln\frac{Q_{\mathrm{cut1}}^{\frac{1}{1+\beta_1}}}{\mu^{\frac{2+\beta_1}{1+\beta_1}}Qy}+\gamma_{S_{C_j}}[\alpha_s]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln\tilde{\hat{D}}_{C_j}\Big(yQQ_{\mathrm{cut2}}^{\frac{-1}{1+\beta_2}},\beta_2,\mu\Big)=-2C_j\Gamma_{\mathrm{cusp}}[\alpha_s]\,\ln\frac{Q_{\mathrm{cut2}}^{\frac{1}{1+\beta_2}}}{\mu^{\frac{2+\beta_2}{1+\beta_2}}Qy}+\gamma_{D_{C_j}}[\alpha_s]$$

Cumulative Jet Mass in Collinear Drop

- **Cumulative jet mass distribution** $$\Sigma(\Delta m_c^2) = \frac{1}{\sigma}\int_0^{\Delta m_c^2} \mathrm{d}\Delta m^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Delta m^2}$$
- **In perturbative region** $$\hat{\Sigma}(\Delta m_c) = \sum_{j=q,g} f_j\, \hat{\Sigma}_j(\Delta m_c)$$
- **At NLL** $f_j = H_j(p_T, \eta_J, R)/\sigma_0$

Y.T.Chien, I.W.Stewart, 1907.11107

$$\hat{\Sigma}_j^{\mathrm{NLL}} = \exp\left[\frac{2C_j}{1+\beta_1}K(\mu_{\mathrm{gs}1},\mu) - \frac{2C_j}{1+\beta_2}K(\mu_{\mathrm{gs}2},\mu) - 2C_j\frac{2+\beta_1}{1+\beta_1}K(\mu_{\mathrm{cs}1},\mu) + 2C_j\frac{2+\beta_2}{1+\beta_2}K(\mu_{\mathrm{cs}2},\mu)\right]$$
$$\times\left(\frac{\mu_{\mathrm{gs}1}}{Q_{\mathrm{gs}1}}\right)^{\frac{2C_j}{1+\beta_1}\omega(\mu_{\mathrm{gs}1},\mu)}\left(\frac{\mu_{\mathrm{gs}2}}{Q_{\mathrm{gs}2}}\right)^{\frac{-2C_j}{1+\beta_2}\omega(\mu_{\mathrm{gs}2},\mu)}\left(\frac{Q_{\mathrm{cut}1}^{\frac{1}{1+\beta_1}}}{Q\mu_{\mathrm{cs}1}^{\frac{2+\beta_1}{1+\beta_1}}}\right)^{2C_j\omega(\mu_{\mathrm{cs}1},\mu)}\left(\frac{Q_{\mathrm{cut}2}^{\frac{1}{1+\beta_2}}}{Q\mu_{\mathrm{cs}2}^{\frac{2+\beta_2}{1+\beta_2}}}\right)^{-2C_j\omega(\mu_{\mathrm{cs}2},\mu)}\frac{(e^{-\gamma_E}\Delta m_c^2)^\eta}{\Gamma(1+\eta)}\Bigg|_{\eta=2C_j\omega(\mu_{\mathrm{cs}1},\mu_{\mathrm{cs}2})}$$

Sudakov factors with both negative and positive signs

$$K(\mu_1,\mu_2) = \int_{\alpha_s(\mu_1)}^{\alpha_s(\mu_2)} d\alpha \frac{\Gamma_{\mathrm{cusp}}(\alpha)}{\beta(\alpha)}\int_{\alpha_s(\mu_1)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')} \qquad \omega(\mu_1,\mu_2) = \int_{\alpha_s(\mu_1)}^{\alpha_s(\mu_2)} d\alpha\frac{\Gamma_{\mathrm{cusp}}(\alpha)}{\beta(\alpha)}$$