

SCET 2023 – LBNL

Non-Global Logarithms at NLL accuracy

Nicolas Schalch

in collaboration with Thomas Becher & Xiaofeng Xu

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NS Tell me about non-global logarithms in physics



In physics, the term "non-global logarithms" is used in the context of quantum chromodynamics [...]



In QCD calculations, non-global logarithms arise when multiple radiation emissions occur in different regions of space and time, such that they cannot be described by a single, global logarithmic factor. [...]

[...] they can have a significant impact on the precision of theoretical predictions for experimental measurements. To account for these non-global logarithms, physicists have developed a variety of techniques, including resummation methods [...]

Overall, non-global logarithms are an important consideration in high-energy physics, and they play a crucial role in understanding the behavior of subatomic particles and the nature of the strong nuclear force.

[\[OpenAI\]](#)

- Jet cross sections involve angular cuts which constrain radiation within a corner of the phase space. As a consequence, logarithmically enhanced higher-order corrections known as **Non-Global Logarithms (NGLs)** arise.

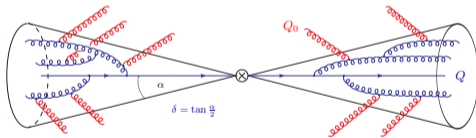
[Dasgupta and Salam, hep-ph/0104277]

- Canonical example for NGLs interjet energy flow, one-loop (global) logarithm is

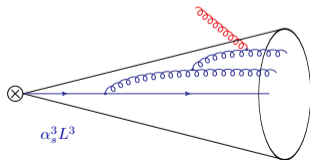
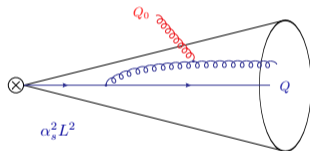
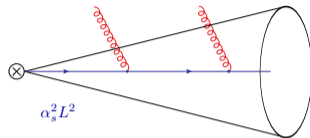
$$\sigma \sim 1 + \frac{g_s^2}{3\pi^2} \left(3 \log \delta + 4 \log \delta \log \frac{Q_0}{Q} + \text{const.} \right)$$

[Sterman and Weinberg, 1977]

Large logarithms $\alpha_s^n L^m$ with $L = \log \frac{Q_0}{Q}$ arise, starting at two-loop order



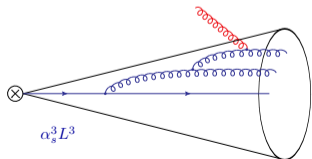
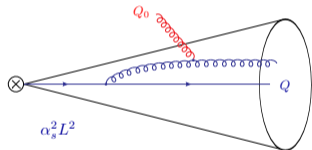
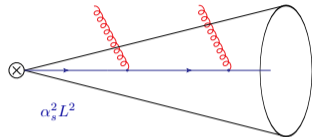
- NGLs arise due to secondary soft gluon emissions inside jets
- Not captured by standard resummation methods
⇒ even leading NGLs $(\alpha_s L)^n$ do not simply exponentiate
- At large N_c leading NGLs can be obtained with a parton shower [Dasgupta and Salam, hep-ph/0104277] or by solving the non-linear BMS integral equation [Banfi et. al., hep-ph/0206076]



- LL in large N_c with general-purpose shower
 - PanScales [PanScales, 2205.02237, 2207.09467]
 - ALARIC [ALARIC, 2208.06057]
- Finite- N_c results for leading NGLs [Weigert, hep-ph/0312050], [Hatta, Ueda, Hagiwara; 1304.6930, 1507.07641, 2011.04154], [De Angelis, Forshaw, Plätzer; 2007.09648]
- First NLL numerical results in the large- N_c limit
 - Extension of BMS framework to NLL [Monni et. al., 2104.06416] and numerical implementation in MC code Gno1e [Monni et. al., 2111.02413]
 - Ingredients for resummation of subleading effects of NGLs using modern EFT techniques [Becher et. al., 1605.02737, 1901.09038, 2112.02108]



Implementation into shower code
This talk: First numerical results



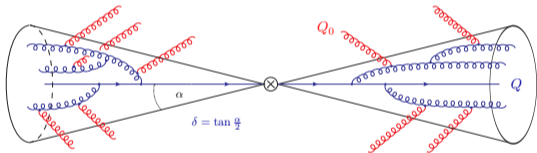
- Cross section for jet production in e^+e^- collisions with veto on radiation factorises into hard \mathcal{H}_m and soft \mathcal{S}_m functions [Becher et. al., 1508.06645]

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

- Factorisation separates contributions from scales Q and Q_0
⇒ natural way to perform **resummation** via **RGEs**

- Hard functions fulfill RG equations

$$\frac{d}{d \log \mu} \mathcal{H}_m(Q, \mu) = - \sum_{l=2}^m \mathcal{H}_l(Q, \mu) \Gamma_{lm}(Q, \mu)$$



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- $\mathcal{H}_m(\{\underline{n}\}, Q, \mu) \sim |\mathcal{M}_m\rangle \langle \mathcal{M}_m|$ describes m hard partons along fixed directions $\{n_1, \dots, n_m\}$

$$\begin{aligned} \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) &= \frac{1}{2Q^2} \sum_{\text{spins}} \prod_{i=1}^m \int \frac{dE_i E_i^{d-3}}{\tilde{c}^\epsilon (2\pi)^2} |\mathcal{M}_m(\{\underline{p}\})\rangle \langle \mathcal{M}_m(\{\underline{p}\})| \\ &\quad \times (2\pi)^d \delta\left(Q - \sum_{i=1}^m E_i\right) \delta^{(d-1)}(\vec{p}_{\text{tot}}) \Theta_{\text{in}}(\{\underline{n}\}) \end{aligned}$$

- $\mathcal{S}_m(\{\underline{n}\}, Q_0, \mu)$ is the squared amplitude with m Wilson lines along fixed directions $\{n_1, \dots, n_m\}$

$$\begin{aligned} \mathcal{S}_m(\{\underline{n}\}, Q_0, \epsilon) &= \int_{X_s} \langle 0 | \mathbf{S}_1^\dagger(n_1) \dots \mathbf{S}_m^\dagger(n_m) | X_s \rangle \langle X_s | \mathbf{S}_1(n_1) \dots \\ &\quad \times \dots \mathbf{S}_m(n_m) | 0 \rangle \theta(Q_0 - E_{\text{out}}) \end{aligned}$$

- Cross section for jet production in e^+e^- collisions with veto on radiation factorises into hard \mathcal{H}_m and soft \mathcal{S}_m functions [Becher et. al., 1508.06645]

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

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⇒ natural way to perform **resummation** via **RGEs**
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$$\frac{d}{d \log \mu} \mathcal{H}_m(Q, \mu) = - \sum_{l=2}^m \mathcal{H}_l(Q, \mu) \Gamma_{lm}(Q, \mu)$$

Procedure to solve the RGEs

- 1 Compute \mathcal{H}_m at hard scale $\mu_h = Q$
 - 2 Evolve \mathcal{H}_m to soft scale $\mu_s = Q_0$
 - 3 Evaluate \mathcal{S}_m at soft scale $\mu_s = Q_0$
- ⇒ Resums large logarithms $\log \frac{Q_0}{Q}$



Clear prescription how to perform resummation at any given accuracy

- LL $\mathcal{H}_2(\mu_h) = \sigma_0 \mathbb{1}$
 $\mathcal{H}_m(\mu_h) = 0$ for $m > 2$
 $\mathcal{S}_m(\mu_s) = \mathbb{1}$
 $\Gamma_{lm}^{(1)}$ one-loop anomalous dimension
- NLL $\mathcal{H}_2(\mu_h) = \sigma_0 |C_V|^2 \mathbb{1}$ one-loop virtual
 $\mathcal{H}_3(\mu_h)$ hard real emission corrections
 $\mathcal{S}_m^{(1)}(\mu_s)$ one-loop soft corrections
 $\Gamma_{lm}^{(2)}$ two-loop anomalous dimension

- One-loop anomalous dimension $\Gamma^{(1)}$

$$\Gamma^{(1)} = \begin{pmatrix} \mathbf{V}_2 & \mathbf{R}_2 & 0 & 0 & \dots \\ 0 & \mathbf{V}_3 & \mathbf{R}_3 & 0 & \dots \\ 0 & 0 & \mathbf{V}_4 & \mathbf{R}_4 & \dots \\ 0 & 0 & 0 & \mathbf{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathbf{R}_m = -4 \sum_{(ij)} \mathbf{T}_{i,L}^a \mathbf{T}_{j,R}^{\tilde{a}} W_{ij}^q \theta_{\text{in}}(n_q)$$

$$\mathbf{V}_m = 2 \sum_{(ij)} (\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} + \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}) \int [d\Omega_q] W_{ij}^q$$
$$- 2 \sum_{(ij)} [\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} - \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}] \times i\pi \Pi_{ij}$$

Glauber/Coulomb contribution

- ! $\Gamma^{(1)}$ infinite dimensional matrix
- No Glauber contribution in e^+e^-
- Singular when soft emission is collinear to hard partons
⇒ Collinear finite by combining real and virtual

- Two-loop anomalous dimension $\Gamma^{(2)}$ has been calculated by considering (double) soft limits of hard functions

$$\Gamma^{(2)} = \begin{pmatrix} \mathbf{v}_2 & \mathbf{r}_2 & \mathbf{d}_2 & 0 & \dots \\ 0 & \mathbf{v}_3 & \mathbf{r}_3 & \mathbf{d}_3 & \dots \\ 0 & 0 & \mathbf{v}_4 & \mathbf{r}_4 & \dots \\ 0 & 0 & 0 & \mathbf{v}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

\mathbf{d}_m : double real emission

\mathbf{r}_m : real-virtual correction

\mathbf{v}_m : double-virtual correction

$$\mathbf{d}_m = \sum_{(ij)} \sum_k i f^{abc} \left(T_{i,L}^a T_{j,L}^b T_{k,R}^c - T_{i,R}^a T_{j,R}^b T_{k,L}^c \right) K_{ijk;qr} \theta_{in}(n_q) \theta_{in}(n_r)$$

$$-2 \sum_{(ij)} T_{i,L}^c T_{j,R}^c K_{ij;qr} \theta_{in}(n_q) \theta_{in}(n_r)$$

$$\mathbf{r}_m = -2 \sum_i \sum_{(jk)} i f^{abc} (T_{i,L}^a T_{j,R}^b T_{k,R}^c - T_{i,R}^a T_{j,L}^b T_{k,L}^c) \int [d^2\Omega_r] K_{ijk;qr} \theta_{in}(n_q)$$

$$- \sum_{(ij)} T_{i,L}^a T_{j,R}^a \left\{ W_{ij}^q \left[4\beta_0 \ln(2W_{ij}^q) + \gamma_1^{\text{cusp}} \right] - 2 \int [d^2\Omega_r] K_{ij;qr} \right\} \theta_{in}(n_q)$$

$$+ 8i\pi \sum_i \sum_{(jk)} i f^{abc} \left(T_{i,L}^a T_{j,R}^b T_{k,R}^c + T_{i,R}^a T_{j,L}^b T_{k,L}^c \right) W_{ij}^q \ln W_{jk}^q \theta_{in}(n_q)$$

$$\mathbf{v}_m = \sum_{(ijk)} i f^{abc} \left(T_{i,L}^a T_{j,L}^b T_{k,L}^c - T_{i,R}^a T_{j,R}^b T_{k,R}^c \right) \int [d^2\Omega_q] \int [d^2\Omega_r] K_{ijk;qr}$$

$$+ \sum_{(ij)} \frac{1}{2} (T_{i,L}^a T_{j,L}^a + T_{i,R}^a T_{j,R}^a) \int [d^2\Omega_q] W_{ij}^q \left[4\beta_0 \ln(2W_{ij}^q) + \gamma_1^{\text{cusp}} \right]$$

$$- i\pi \sum_{(ij)} \frac{1}{2} (T_{i,L}^a T_{j,L}^a - T_{i,R}^a T_{j,R}^a) \Pi_{ij} \gamma_1^{\text{cusp}}$$

+ additional terms from converting
to angular integrals in $d = 4$

- Started with a factorisation theorem for jet production with veto on radiation which provides natural way to perform resummation via RG evolution of hard functions \mathcal{H}_m

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$
$$\frac{d}{d \log \mu} \mathcal{H}_m(Q, \mu) = - \sum_{l=2}^m \mathcal{H}_l(Q, \mu) \Gamma_{lm}(Q, \mu)$$

The RG evolution is governed by the anomalous dimension which has been extracted up to two-loops

$$\Gamma = \frac{\alpha_s}{4\pi} \Gamma^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma^{(2)} + \dots \quad \Rightarrow \text{NLL accuracy}$$

❓ How to solve complicated RGEs ❓
 \Rightarrow Monte-Carlo Methods

- In practice coupled RGEs for hard functions \mathcal{H}_m , however these simplify at LL due to the form of $\Gamma^{(1)}$

$$\Gamma^{(1)} = \begin{pmatrix} V_2 & \mathbf{R}_2 & 0 & 0 & \dots \\ 0 & V_3 & \mathbf{R}_3 & 0 & \dots \\ 0 & 0 & V_4 & \mathbf{R}_4 & \dots \\ 0 & 0 & 0 & V_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

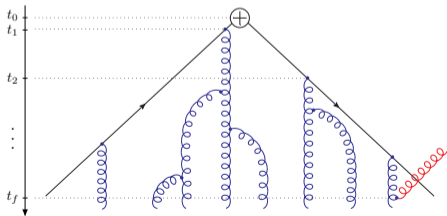
$$\frac{d}{dt} \mathcal{H}_m(t) = \mathcal{H}_m(t) \mathbf{V}_m + \mathcal{H}_{m-1}(t) \mathbf{R}_{m-1}$$

$$\mathcal{H}_m(t) = \mathcal{H}_m(t_0) e^{(t-t_0) \mathbf{V}_m} + \int_{t_0}^t dt' \mathcal{H}_{m-1}(t') \mathbf{R}_{m-1} e^{(t-t') \mathbf{V}_m}$$

- Introduce shower-time t

$$\mu \rightarrow t = \frac{\alpha_s}{4\pi} \log \frac{\mu_h}{\mu_s} + \mathcal{O}(\alpha_s^2)$$

[Balsiger et. al., 1803.07045]



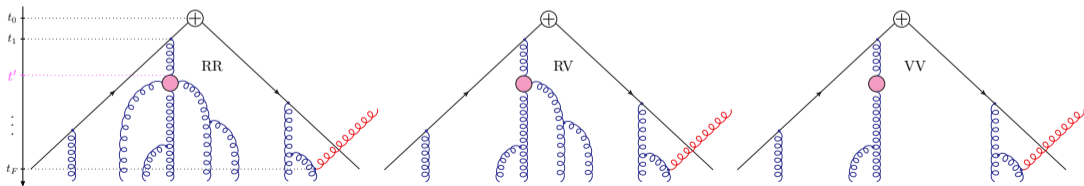
- Include corrections of \mathcal{H}_m due to $\Gamma^{(2)}$ \Rightarrow NLL resummation

$$\begin{aligned} \Delta\mathcal{H}_m(t) &= \mathcal{H}_k(t_0)\Delta\mathcal{U}_{km}(t, t_0) \\ &= \mathcal{H}_k(t_0) \int_{t_0}^t dt' \mathcal{U}_{kl}(t' - t_0) \cdot \frac{\alpha(t')}{4\pi} \left(\Gamma_{ll'}^{(2)} - \frac{\beta_1}{\beta_0} \Gamma_{ll'}^{(1)} \right) \cdot \mathcal{U}_{l'm}(t - t') \end{aligned}$$

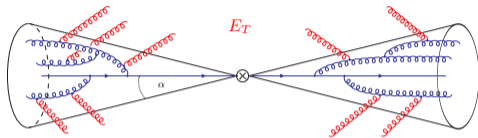
LL evolution

Insertion of $\Gamma^{(2)}$

LL evolution



- Reminder: $e^+e^- \rightarrow 2$ jets

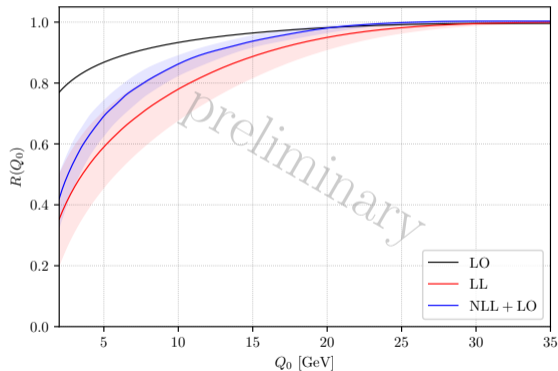


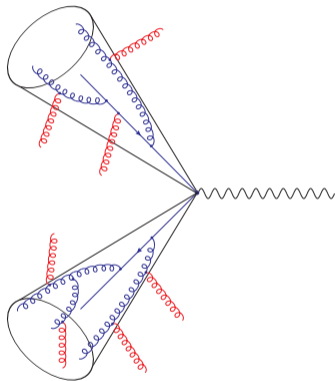
- Gap fraction: fraction of events with transverse energy E_T in gap below Q_0

$$R(Q_0) = \frac{1}{\sigma_{\text{tot}}} \sigma(Q, Q_0) \equiv \int_0^{Q_0} dE_T \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dE_T}$$

- Bands obtained through scale variation
- Qualitative agreement at NLL with Gno1e
[\[Monni et. al. 2111.02413\]](#)

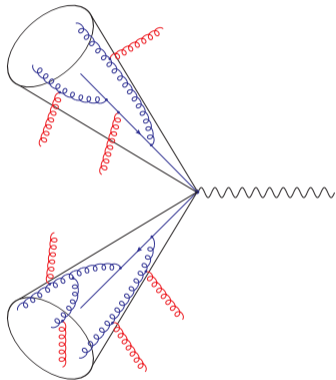
Opening angle $\alpha = \frac{\pi}{3}$, hard scale $Q = M_Z$





Simplest observable: gap fraction in Z – production

- Gap is defined by vetoing hadronic radiation at central rapidities in interval ΔY around the beam
- Cross section factorises in the large- N_c limit exactly in the same way as for e^+e^-
- Same observables used by PanScales collaboration
- First relevant process in a long list



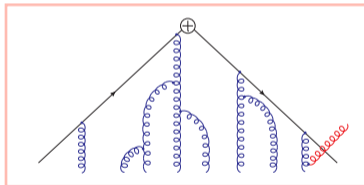
- Required ingredients to resum subleading NGLs

- One-loop virtual $\mathcal{H}_2(\mu_h) = \sigma_0 |C_V|^2 \mathbb{1}$ (✓)
- Hard real emission corrections $\mathcal{H}_3^{(1)}(\mu_h)$ ✗
- One-loop soft corrections $\mathcal{S}_m^{(1)}(\mu_s)$ (✓)
- Two-loop anomalous dimension $\Gamma_{lm}^{(2)}$ (✓)

- The virtual corrections due to $\mathcal{H}_2^{(1)}$ factorise such that we obtain these by multiplying the standard dijet hard function H_2 to the LL result \mathcal{S}_2

$$\frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes U_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle = \sigma_0 H_2(Q, \mu_h) \langle \mathcal{S}_2(\{\bar{n}, n\}, Q_0, \mu_h) \rangle$$

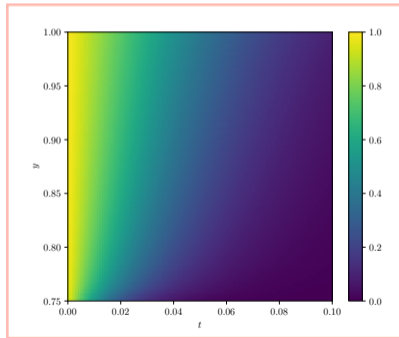
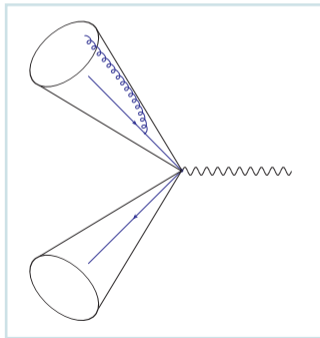
$$C_F \left[-8 \ln^2 \frac{\mu}{Q} - 12 \ln \frac{\mu}{Q} - 16 + \frac{7}{3} \pi^2 \right]$$



Extraction of $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_3^{(1)}$

- To calculate the contribution due to $\mathcal{H}_3^{(1)}$, which depends on $\theta_{qg}(y)$, we convolute the real corrections of $q\bar{q} \rightarrow Z$ with the three particle soft function which is obtained via a LL RG evolution

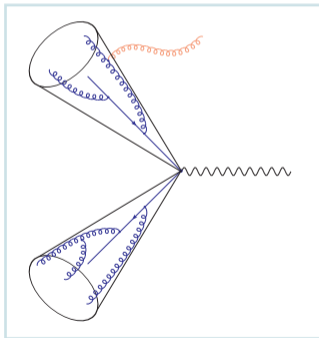
$$\frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(\{n_1, n_2, n_3\}, Q, \mu_h) \otimes U_{3m}(\{\underline{n}\}, \mu_s, \mu_h) \hat{\otimes} 1 \rangle$$



Calculation of $\mathcal{S}_m^{(1)}$ and ΔU_{2m}

- The one-loop corrections to the soft function $\mathcal{S}_m^{(1)}$, which represents the emission of a soft particle into the gap, is directly obtained from our shower

$$\frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle$$

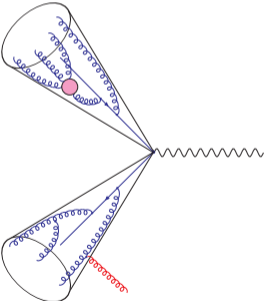


$$\int \frac{d^d k}{(2\pi)^{d-1}} \frac{W_{ij}^k}{E_k^2} \delta(k^2) \theta(k^0) \theta(Q_0 - k_T) \Theta_{\text{out}}(n_k)$$

Calculation of $\mathcal{S}_m^{(1)}$ and ΔU_{2m}

- The contributions due to the insertion of the two-loop anomalous dimension is obtained within our parton shower framework; in practice, we start a LL shower prior to the insertion and then restart a LL shower

$$\sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta U_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle$$



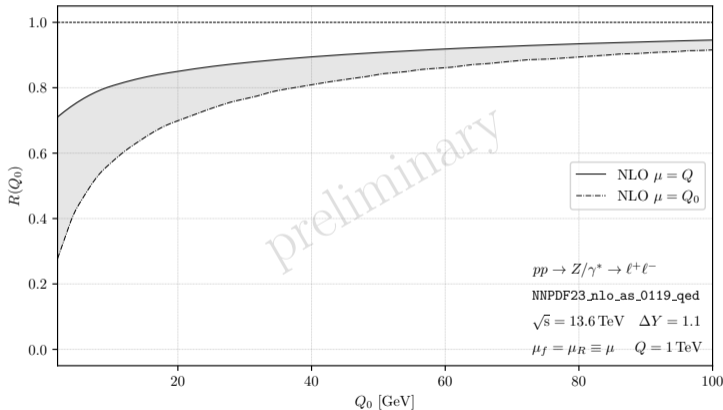
$$\mathbf{d}_m \sim 2 N_c (K_{ij;qr} + K_{ji;qr}) \theta_{in}(n_q) \theta_{in}(n_r)$$

$$\mathbf{r}_m \sim -N_c^2 \int [d^2\Omega_r] (K_{ii;qr} + K_{jj;qr}) \theta_{in}(n_q) + N_c \left\{ W_{ij}^q \left[4\beta_0 \ln(2W_{ij}^q) + \gamma_1^{\text{cusp}} \right] - 2 \int [d^2\Omega_r] (K_{ij;qr} + K_{ji;qr}) \right\} \theta_{in}(n_q)$$

$$\mathbf{v}_m \sim -N_c \int [d^2\Omega_q] W_{ij}^q \left[4\beta_0 \ln(2W_{ij}^q) + \gamma_1^{\text{cusp}} \right]$$

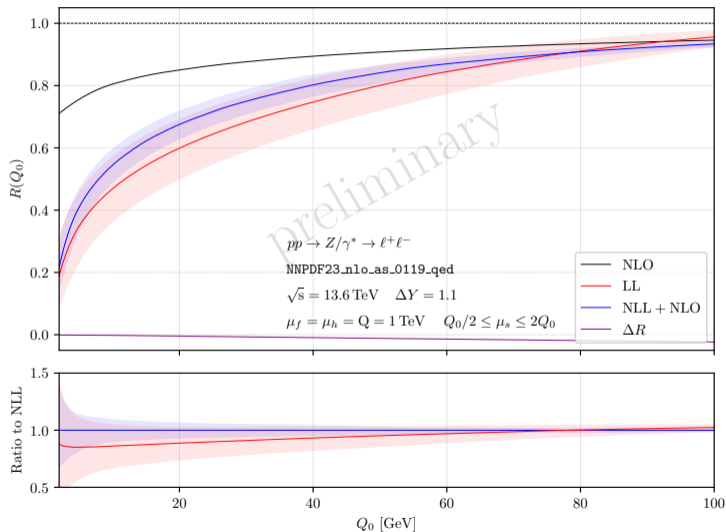
+ additional terms from converting to angular integrals in $d = 4$

Fixed order gap fraction for $pp \rightarrow Z/\gamma^* \rightarrow \ell^+\ell^-$



$\mu = Q$ or $\mu = Q_0$?

Resummed gap fraction for $pp \rightarrow Z/\gamma^* \rightarrow \ell^+\ell^-$



- Bands obtained through 2-point soft scale variation
- Superleading logarithms at finite N_c are a small effect for Z – production [Becher et. al. 2107.01212]

Summary

- Fiducial cuts lead to phase-space constraints
⇒ Intricate pattern of logs: **NGLs** (& SLL)
- Implemented two-loop anomalous dimension in PS
⇒ **NLL** resummation for gap fraction
⇒ First results for Hadron Collider process



BACKUP

- Anomalous dimension Γ arises from soft singularities of hard functions

$$\mathcal{H}_m(Q, \mu) = \sum_{l=2}^m \mathcal{H}_l^{\text{bare}}(Q, \mu) (\mathbf{Z}^{-1})_{lm}(Q, \mu)$$

$$(\mathbf{Z}^{-1}) = \mathbb{1} + \frac{\alpha_s}{4\pi} \frac{1}{2\epsilon} \Gamma^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{8\epsilon^2} \Gamma^{(1)} \otimes \Gamma^{(1)} - \frac{\beta_0}{4\epsilon^2} \Gamma^{(1)} + \frac{1}{4\epsilon} \Gamma^{(2)} \right]$$

$\Gamma^{(1)}$ is $(-2) \times$ soft divergence of the one-loop hard function

Renormalisation $Z_\alpha = 1 - \frac{\beta_0}{\epsilon} \frac{\alpha_s}{4\pi}$

$\Gamma^{(2)}$ is (-4) times single pole of the two-loop hard function

- At $\mathcal{O}(\alpha_s)$ soft singularities arise when either a real or a virtual gluon becomes soft

- To obtain the real contribution take the soft limit of \mathcal{H}_{m+1}

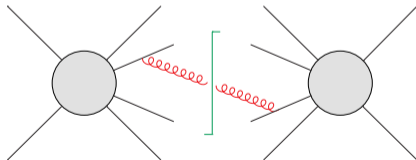
soft dipole W_{ij}^q

$$\begin{aligned}\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) &= -g_S^2 \int_0^\Lambda \frac{dE_q E_q^{d-3}}{\tilde{c}^\epsilon (2\pi)^2} \theta_{\text{in}}(n_q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot n_q n_j \cdot n_q} \mathbf{T}_i^a \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) \mathbf{T}_j^{\bar{a}} \\ &= \frac{2}{\epsilon} \frac{\alpha_S}{4\pi} \theta_{\text{in}}(n_q) \sum_{(ij)} W_{ij}^q \mathbf{T}_i^a \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) \mathbf{T}_j^{\bar{a}}\end{aligned}$$

- Directly obtain

$$\mathbf{R}_m = -4 \sum_{(ij)} \mathbf{T}_{i,L}^a \mathbf{T}_{j,R}^{\bar{a}} W_{ij}^q \theta_{\text{in}}(n_q)$$

⇒ generates additional gluon inside jet

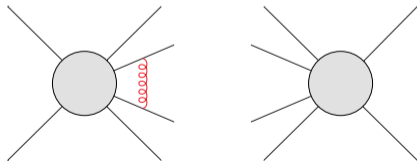


- To obtain the virtual contribution perform q^0 integral using residues

$$\begin{aligned}\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) &= \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0] [-n_j \cdot q + i0]} \mathbf{T}_i \cdot \mathbf{T}_j \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.} \\ &= -\frac{\alpha_S}{4\pi} \frac{1}{\epsilon} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[\int [d\Omega_q] W_{ij}^q + i\pi \right] \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}\end{aligned}$$

- Directly obtain

$$\begin{aligned}V_m &= 2 \sum_{(ij)} (\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} + \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}) \int [d\Omega_q] W_{ij}^q \\ &\quad - 2 \sum_{(ij)} [\mathbf{T}_{i,L} \cdot \mathbf{T}_{j,L} - \mathbf{T}_{i,R} \cdot \mathbf{T}_{j,R}] \times i\pi \Pi_{ij}\end{aligned}$$



- Three-leg correlations (finite in all collinear limits \rightarrow pseudo singular)

$$K_{ijk;qr} = 8 \left(W_{ik}^q W_{jk}^r - W_{ik}^q W_{jq}^r - W_{ir}^q W_{jk}^r + W_{ij}^q W_{jq}^r \right) \ln \left(\frac{n_{kq}}{n_{kr}} \right)$$

- Two-leg correlations (diverges for $q \parallel r$)

$$K_{ij;qr} = C_A K_{ij;qr}^{(a)} + [n_F T_F - 2C_A] K_{ij;qr}^{(b)} + [C_A - 2n_F T_F + n_S T_S] K_{ij;qr}^{(c)}$$

$$K_{ij;qr}^{(a)} = \frac{4n_{ij}}{n_{iq} n_{qr} n_{jr}} \left[1 + \frac{n_{ij} n_{qr}}{n_{iq} n_{jr} - n_{ir} n_{jq}} \right] \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}}$$

$$K_{ij;qr}^{(b)} = \frac{8n_{ij}}{n_{qr} (n_{iq} n_{jr} - n_{ir} n_{jq})} \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}}$$

$$K_{ij;qr}^{(c)} = \frac{4}{n_{qr}^2} \left(\frac{n_{iq} n_{jr} + n_{ir} n_{jq}}{n_{iq} n_{jr} - n_{ir} n_{jq}} \ln \frac{n_{iq} n_{jr}}{n_{ir} n_{jq}} - 2 \right)$$

! RGEs not yet in a suitable form for implementation in a MC framework

- Change variables from $\mu \rightarrow t = \frac{\alpha_s}{4\pi} \log \frac{\mu_h}{\mu_s} + \mathcal{O}(\alpha_s^2)$

$$\frac{d}{dt} \langle \mathcal{H}(t) | = \langle \mathcal{H}(t) | \hat{\Gamma}(t) \quad \rightarrow \text{formal solution} \quad \langle \mathcal{H}(t) | = \langle \mathcal{H}(0) | \mathbb{P} \exp \left[\int_0^t dt' \hat{\Gamma}(t') \right]$$

- Expand anomalous dimension perturbatively $\hat{\Gamma}(t) = \Gamma^{(1)}(t) + \frac{\alpha_s}{4\pi} \Delta\Gamma(t) + \mathcal{O}(\alpha_s^2) \rightarrow$ Interaction picture

$$\frac{d}{dt} \langle \mathcal{H}^I(t) | = \langle \mathcal{H}^I(t) | e^{t\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t) \right] e^{-t\Gamma^{(1)}}$$

- Solve RG evolution iteratively including subleading contributions due to $\Delta\Gamma$

$$\sigma \sim \langle \mathcal{H}(t) | \mathcal{S}(t) \rangle = \langle \mathcal{H}(0) | \left[e^{t\Gamma^{(1)}} + \int_0^t dt' e^{t'\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t') \right] e^{(t-t')\Gamma^{(1)}} \right] | \mathcal{S}(t) \rangle \rightarrow \text{suitable for MC}$$

- Consider the first order ODE as toy RGE

$$\frac{d}{dt}H(t) = -(\Gamma + \Delta\Gamma)H(t)$$

- According to our discussion the solution at LL accuracy

$$\Gamma \cdot H(t) = H(0) \Gamma e^{-t\Gamma}$$

- Interpretation of $\int_0^t dt' \Gamma e^{-t'\Gamma} = e^{-t\Gamma}$ as a probability

$$e^{-t\Gamma} = z \quad \Leftrightarrow \quad t = -\frac{1}{\Gamma} \log(z)$$

Pseudo Code

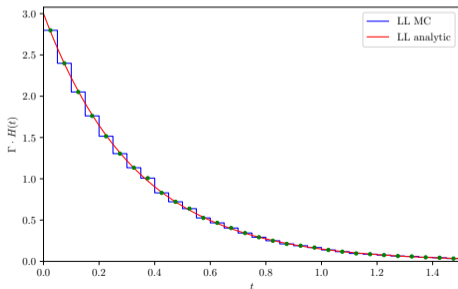
```
# generate n random numbers z in [0,1]
zval = random(n)
# calculate corresponding t values
tval = - 1 / Γ * log(zval)
# bin t values in a histogram
hist = bin(tval)
```

- Analytical result is, $\Delta\Gamma = \alpha t$

$$\Gamma \cdot H(t) = H(0) \Gamma e^{-t\Gamma} \left[1 - \frac{1}{2} \alpha t^2 + \mathcal{O}(\alpha^2) \right]$$

- Numerical results using

$$\Gamma = 3, \quad \alpha = \frac{1}{2}, \quad H(0) = 1$$



- Consider the first order ODE as toy RGE

$$\frac{d}{dt}H(t) = -(\Gamma + \Delta\Gamma)H(t)$$

- Focus on the solution at **NLL** accuracy

$$\Gamma \cdot H(t) = H(0) \int_0^t dt' \Gamma e^{-t'\Gamma} \cdot \frac{\Delta\Gamma}{\Gamma} \cdot \Gamma e^{-(t-t')\Gamma}$$

- Use probabilistic interpretation for $\Gamma e^{-t\Gamma}$

Pseudo Code

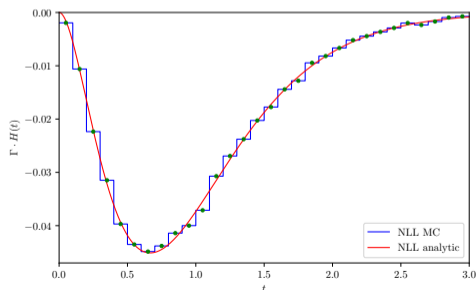
```
# generate random time step  $\Delta t_1$   
 $\Delta t_1 = -1 / \Gamma * \log(z)$   
# calculate weight at  $\Delta t_1$   
weight =  $\Delta\Gamma(\Delta t_1) / \Gamma$   
# generate random time step  $\Delta t_2$  and bin weight  
hist = bin( $\Delta t_1 + \Delta t_2$ , weight)
```

- Analytical result is, $\Delta\Gamma = \alpha t$

$$\Gamma \cdot H(t) = H(0) \Gamma e^{-t\Gamma} \left[1 - \frac{1}{2}\alpha t^2 + \mathcal{O}(\alpha^2) \right]$$

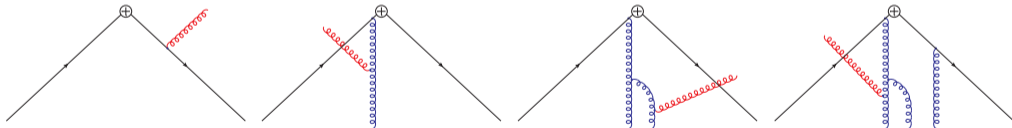
- Numerical results using

$$\Gamma = 3, \quad \alpha = \frac{1}{2}, \quad H(0) = 1$$



- LL Shower in the large N_c limit is equivalent to [Dasgupta and Salam, hep-ph/0104277]

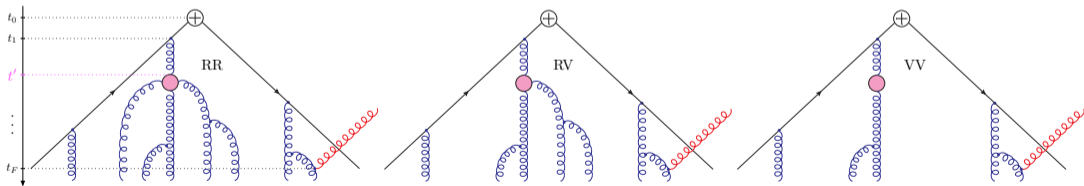
$$\begin{aligned}\sigma_{\text{LL}}(Q, Q_0) &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)} \otimes U_{2m} \hat{\otimes} \mathcal{S}_m^{(0)} \rangle \\ &= \langle \mathcal{H}_2^{(0)} + \int [d\Omega_3] \mathcal{H}_3^{\text{LL}} + \int [d\Omega_3] [d\Omega_4] \mathcal{H}_4^{\text{LL}} + \int [d\Omega_3] [d\Omega_4] [d\Omega_5] \mathcal{H}_5^{\text{LL}} + \dots \rangle\end{aligned}$$



- Include corrections of \mathcal{H}_m due to $\Gamma^{(2)} \Rightarrow$ NLL resummation

$$\begin{aligned} \Delta \mathcal{H}_m(t) &= \mathcal{H}_k(t_0) \Delta U_{km}(t, t_0) \\ &= \mathcal{H}_k(t_0) \int_{t_0}^t dt' \mathbf{U}_{kl}(t' - t_0) \cdot \frac{\alpha(t')}{4\pi} \left(\Gamma_{ll'}^{(2)} - \frac{\beta_1}{\beta_0} \Gamma_{ll'}^{(1)} \right) \cdot \mathbf{U}_{l'm}(t - t') \end{aligned}$$

where $U_{ab}(t_2 - t_1)$ is the LL evolution

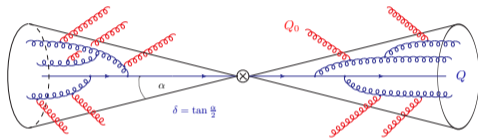


- o Cross section has multiple contributions at NLL

$$\begin{aligned} \frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\ &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \end{aligned}$$

[Becher et. al., 1901.09038]

- o Reminder: $e^+e^- \rightarrow 2\text{jets}$

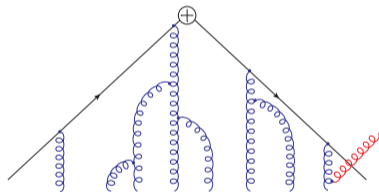


- o Cross section has multiple contributions at NLL

$$\begin{aligned}\frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\ &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle\end{aligned}$$

[Becher et. al., 1901.09038]

1 LL evolution

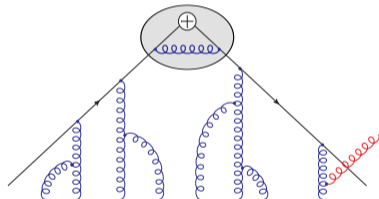


- Cross section has multiple contributions at NLL

$$\begin{aligned}\frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\ &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle\end{aligned}$$

[Becher et. al., 1901.09038]

2 Virtual corrections, LL evolution

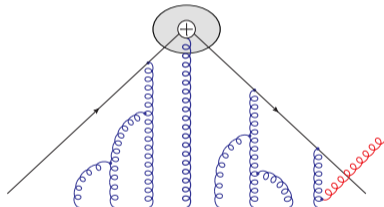


- o Cross section has multiple contributions at NLL

$$\begin{aligned}\frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\ &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle\end{aligned}$$

[Becher et. al., 1901.09038]

3 Real corrections, LL evolution

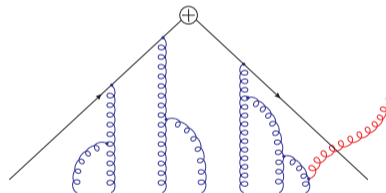


- Cross section has multiple contributions at NLL

$$\begin{aligned}\frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\ &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\ &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle\end{aligned}$$

[Becher et. al., 1901.09038]

4 Soft corrections, LL evolution

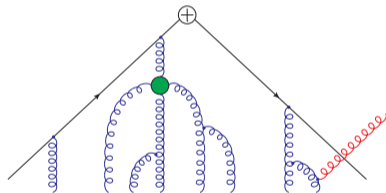


- o Cross section has multiple contributions at NLL

$$\begin{aligned}
 \frac{\sigma^{\text{NLL}}(Q, Q_0)}{\sigma_0} &= \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\
 &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\
 &+ \frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle \mathcal{H}_3^{(1)}(Q, \mu_h) \otimes \mathbf{U}_{3m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle \\
 &+ \frac{\alpha_s(\mu_s)}{4\pi} \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} \mathcal{S}_m^{(1)}(Q_0, \mu_s) \rangle \\
 &+ \sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q, \mu_h) \otimes \Delta \mathbf{U}_{2m}(\mu_s, \mu_h) \hat{\otimes} 1 \rangle
 \end{aligned}$$

[Becher et. al., 1901.09038]

5 Two-loop running, NLL evolution



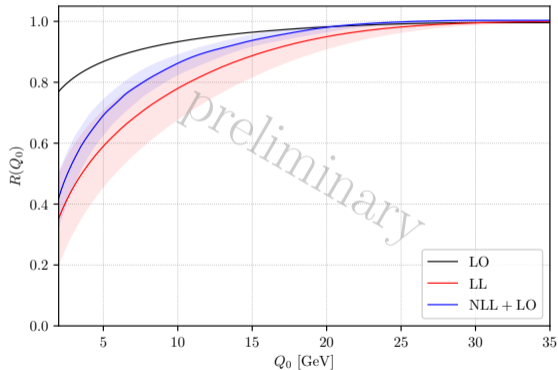
- Gap fraction, events with energy E_s below Q_0

$$R(Q_0) = \frac{1}{\sigma_{\text{tot}}} \sigma(Q, Q_0) \equiv \int_0^{Q_0} dE_s \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dE_s}$$

Opening angle $\alpha = \frac{\pi}{3}$

Hard scale $Q = M_Z$

- Bands obtained through scale variation
- Qualitative agreement at NLL with Gno1e
[\[Monni et. al. 2111.02413\]](#)



- The differential cross section for Z – Boson production is written as a convolution between the partonic cross section $d\hat{\sigma}_{ij}$ and the PDFs f_l

$$\frac{d\sigma}{dQ^2} = \frac{4\pi^2\alpha}{N_c s} \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) \frac{d\hat{\sigma}_{ij}}{dQ^2}$$

$$\frac{d\hat{\sigma}_{ij}}{dQ^2} = \int d\Pi_f |\mathcal{M}_{ij}|^2 \delta\left(z - \frac{Q^2}{\hat{s}}\right) = \hat{\Sigma}_{ij}^{(0)} + \left(\frac{\alpha_S}{4\pi}\right) \hat{\Sigma}_{ij}^{(1)} + \mathcal{O}(\alpha_S^2)$$

- The LO partonic cross section is expressed in terms of a delta-distribution in $z = \frac{Q^2}{\hat{s}}$

$$\frac{\hat{\Sigma}_{q\bar{q}}^{(0)}}{e_q^2} \sim \int d\Pi_f \left| \begin{array}{c} \text{Diagram: two incoming lines merging into a vertex, which then emits a wavy line (Z boson) to a vertical line representing the final state.} \end{array} \right|^2 \sim \delta(1-z)$$

- At NLO we obtain virtual and real contributions \Rightarrow dimensional regularisation in $d = 4 - 2\epsilon$ to make both UV and IR divergences explicit, e.g. the $q\bar{q}$ - channel yields

$$\frac{\hat{\Sigma}_{q\bar{q}}^{(1)}}{e_q^2} \sim \int d\Pi_f \left[\text{Diagram 1} \right]^2 + \int d\Pi_f \left[\text{Diagram 2} \right]^2 + \left[\text{Diagram 3} \right]^2$$

$$\sim C_F \left\{ (\delta(y) + \delta(1-y)) \left[\delta(1-z)(2\zeta_2 - 4) + 4 \left[\frac{\log(1-z)}{1-z} \right]_+ - 2(1+z) \log(1-z) - \frac{(1+z^2)}{1-z} \log(1-z) + 1-z \right] \right.$$

$$\left. + \left[(1+z^2) \left[\frac{1}{1-z} \right]_+ + \left(\left[\frac{1}{y} \right]_+ + \left[\frac{1}{1-y} \right]_+ \right) - 2(1-z) \right] + \mathcal{O}(\epsilon) \right\}$$

! New channel opens up: $\hat{\Sigma}_{qg}^{(1)}$ also needs to be taken into account

